2D Transforms

Lecture 4

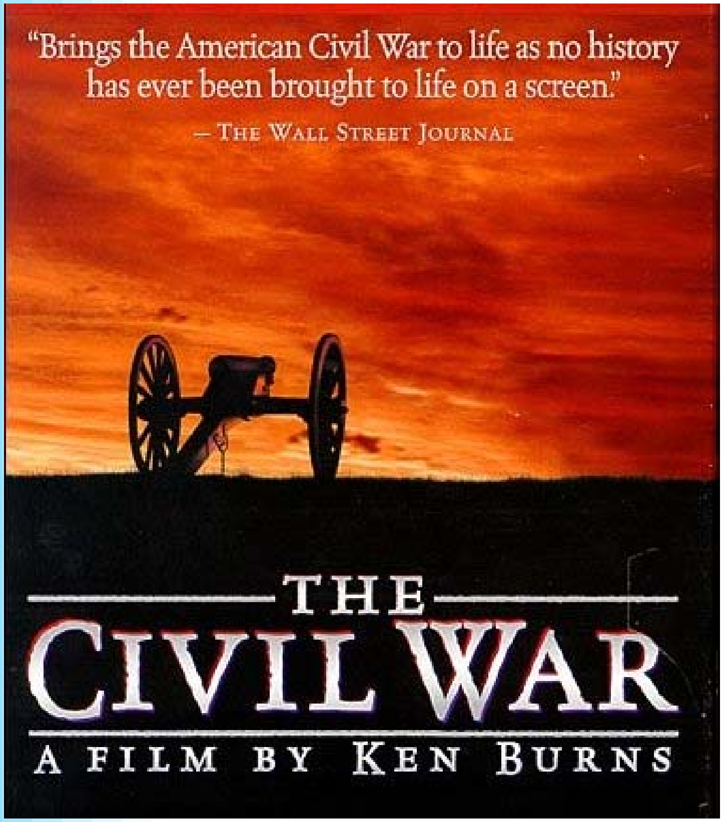
CISC440/640

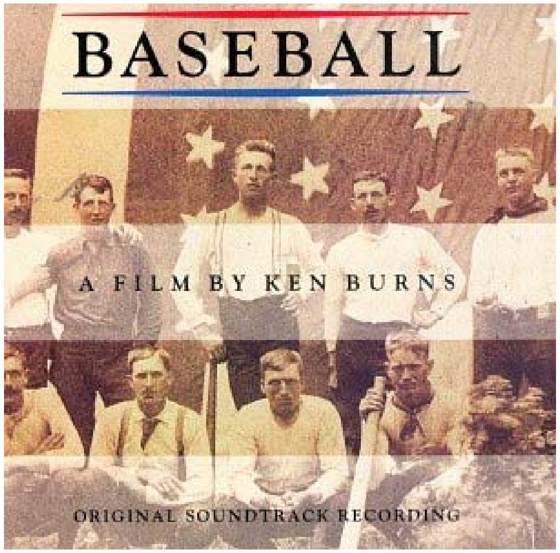
Spring 2015

Department of Computer and Information Science

Where are we going?

Assignment #1 part 2:





The
"Ken Burns"
Effect

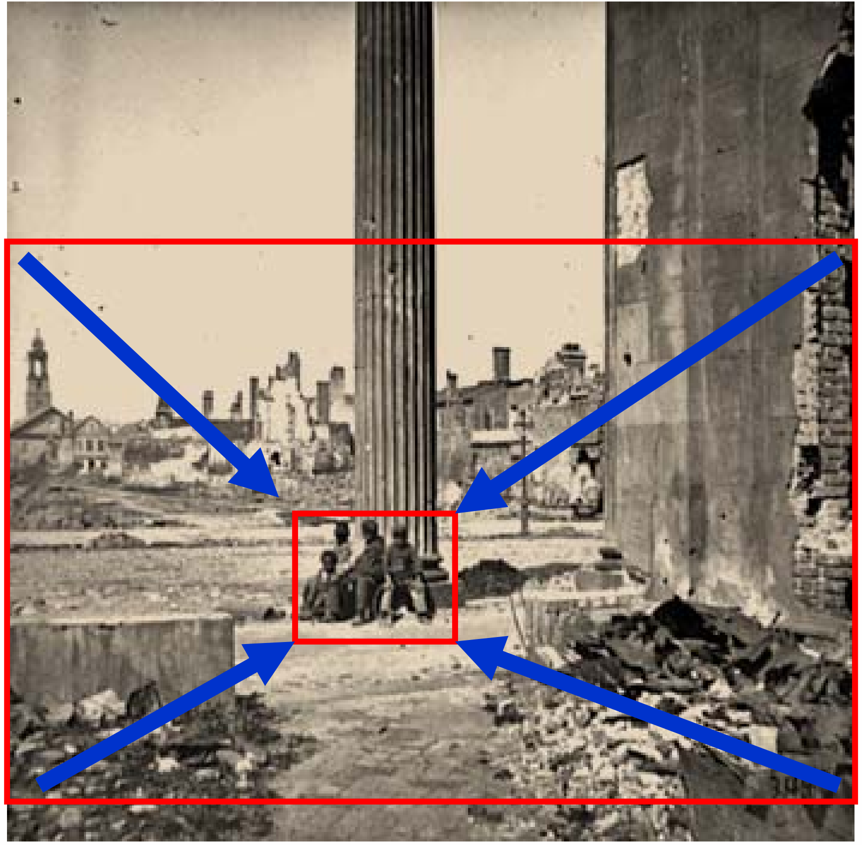
Where are we going?

A preview of assignment #1:

"Panning Effect"

Where are we going?

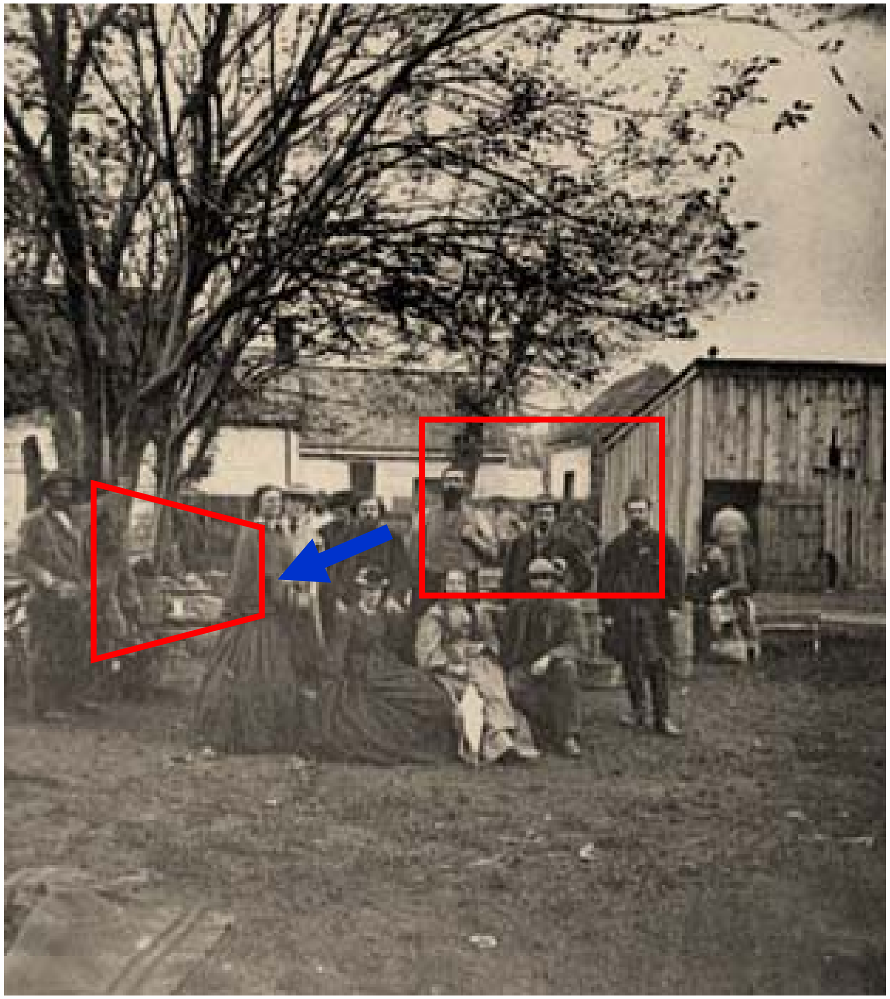
A preview of assignment #1:



“Zooming Effect”

Where are we going?

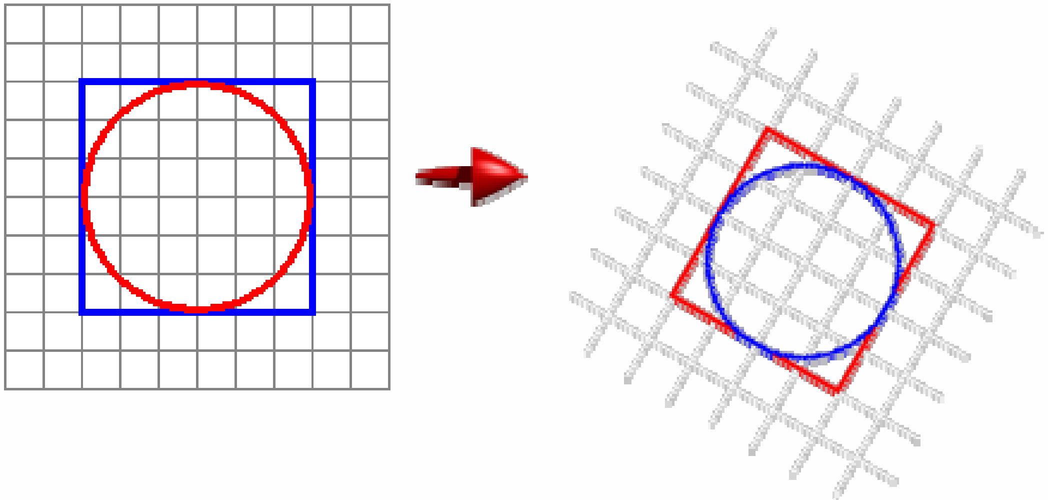
A preview of assignment #1:

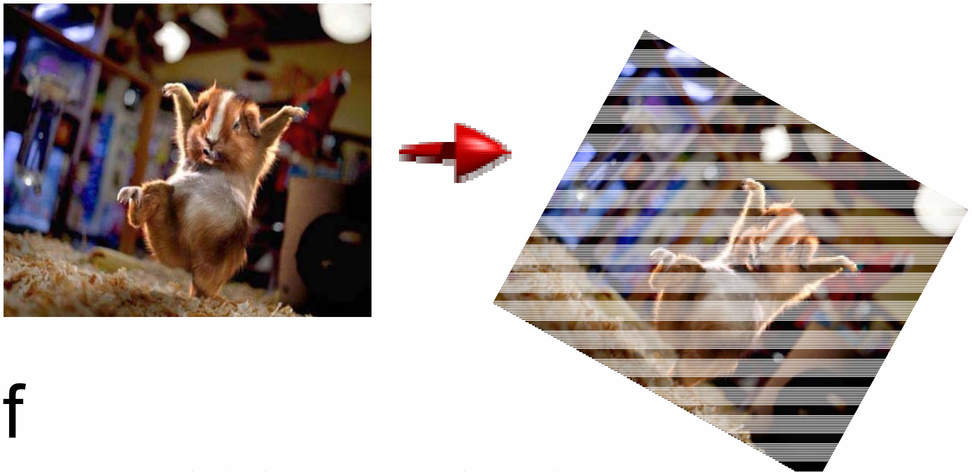


"Perspective Effect"

Two-Dimensional Geometric Transforms

- Functions for mapping points from one place to another
- Geometric transforms can be applied to
  - drawing primitives (lines, conics, triangles)
  - pixel coordinates of an image





Shirley Book, chapter 6

Translation

- Translations have the following form

$$x' = x + t_x$$
$$y' = y + t_y$$

- For every translation ***there exists an inverse function*** which undoes the translation. In our case the inverse looks like:

$$x = x' - t_x$$
$$y = y' - t_y$$

- There also exists a special translation, called the ***identity***, that leaves every point unchanged.

$$x' = x + 0$$
$$y' = y + 0$$

2D Rotations

- **Another group of 2-transforms are the rotations about the origin.**

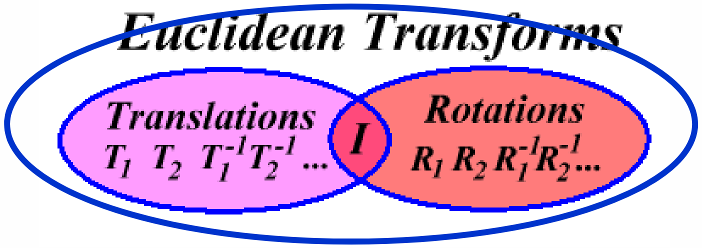
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$R_{\theta=0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Euclidean Transforms

- **The combination of translations and rotation functions defines the Euclidean Group**



- **Properties of Euclidean Transformations:**
  - **They preserve distances**
  - **They preserve angles**
  - **How do you represent these functions?**

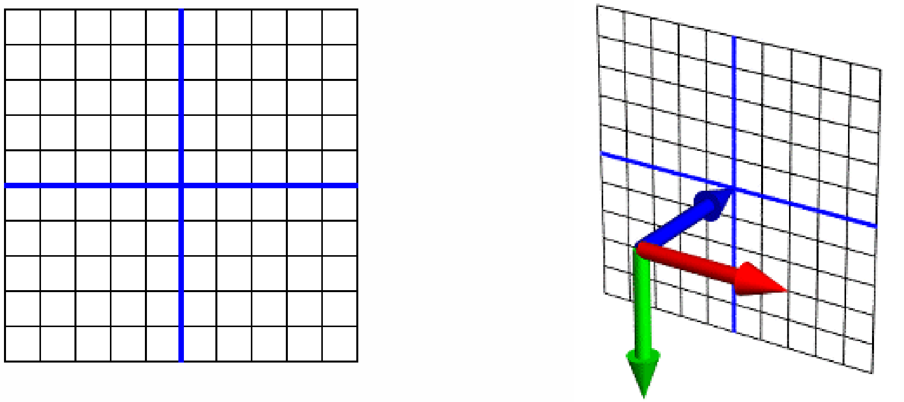
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Problems with this Form

- Must consider Translation and Rotation separately
- Computing the inverse transform involves multiple steps
- Order matters between the R and T parts

$$R(T(\bar{x})) \neq T(R(\bar{x}))$$

- *These problem can be remedied by considering our 2 dimensional plane as a subspace within 3D.*



Choosing a Subspace

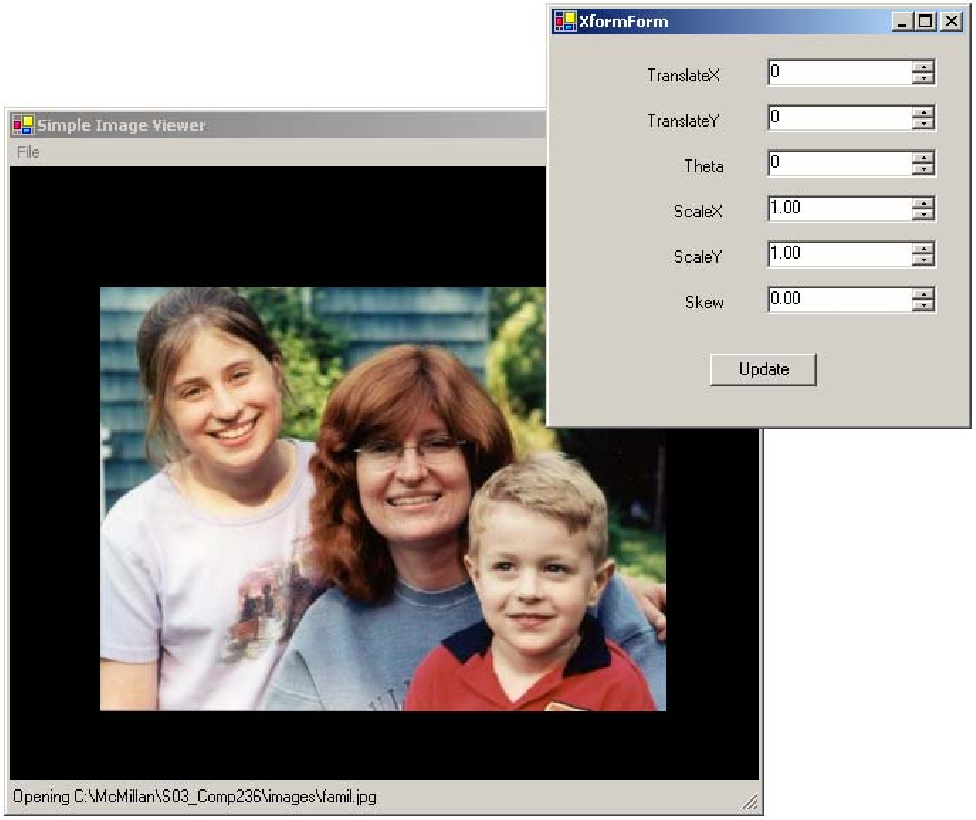
- We could use **any** planar subspace as long as it does not contain the origin
- WLOG assume the our 2D space of points lies on the 3D plane *z = 1* . Now we can express all Euclidean Transforms in matrix form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- This gives a three parameter group of Transformations.

Playing with Euclidean Transforms

- In what order are the translation and rotation performed?
- Will this family of transforms always generate points on our chosen 3-D plane? Why?

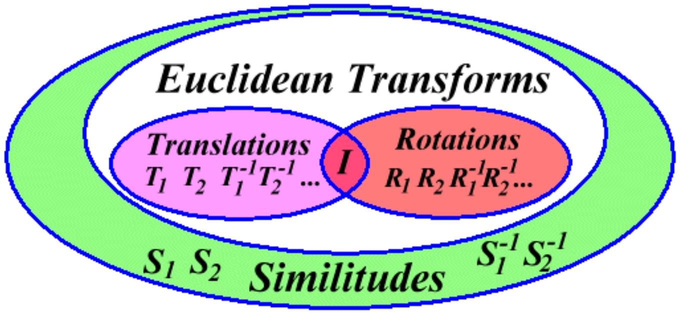


TranslateX	0
TranslateY	0
Theta	0
ScaleX	1.00
ScaleY	1.00
Skew	0.00

Opening C:\McMillan\S03_Comp236\images\famil.jpg

Similitude Transforms

- We can define a 4-parameter superset of Euclidean Transforms with additional capabilities

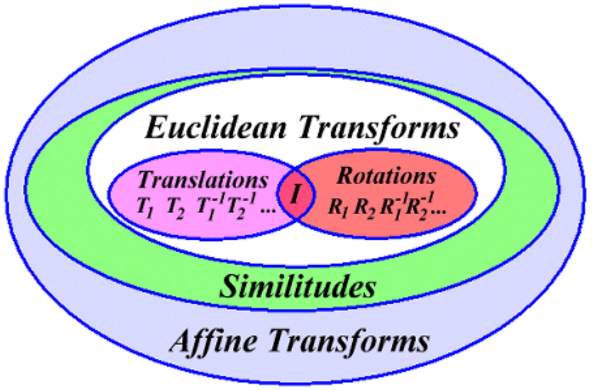


- **Properties of SimiItudes:**
  - Distances between points are changed by a fixed ratio
  - Angles are preserved
  - Maintains a “Similar” shape (similar triangles, circles map to circles, etc.)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta & t_x \\ \pm\sigma\sin\theta & \pm\sigma\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine Transformations

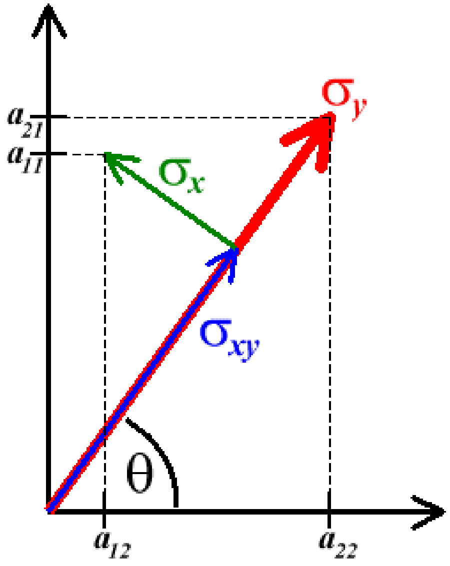
- A 6-parameter group of transforms



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_{xy}\sin\theta + \sigma_x\cos\theta & \sigma_{xy}\cos\theta - \sigma_x\sin\theta & t_x \\ \sigma_y\sin\theta & \sigma_y\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

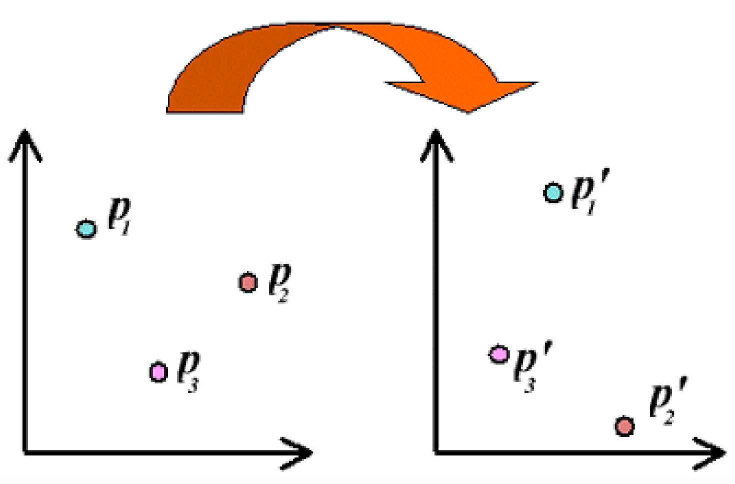
Affine Properties

- To the right is a simple illustration of how we can map our parameters into a six arbitrary numbers
- Properties of Affine Transforms:
  - They preserve our selected plane (sometimes called the *Affine plane*)
  - They preserve parallel lines



Determining Affine Transforms

- The coordinates of three corresponding points (non-colinear) uniquely determine an Affine Transform



If we know where we would like at least three points to map to, we can solve for an Affine transform that will give this mapping.

Solution Method

We set up 6 linear equations in terms of our 6 unknown values. In this case, we know the coordinates before and after the mapping, and we wish to solve for the entries in our Affine transform matrix.

$$\underbrace{\begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \\ x_3' \\ y_3' \end{bmatrix}}_{\mathbf{x}'} = \underbrace{\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \end{bmatrix}}_{\mathbf{X}} \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}}_{\mathbf{a}}$$

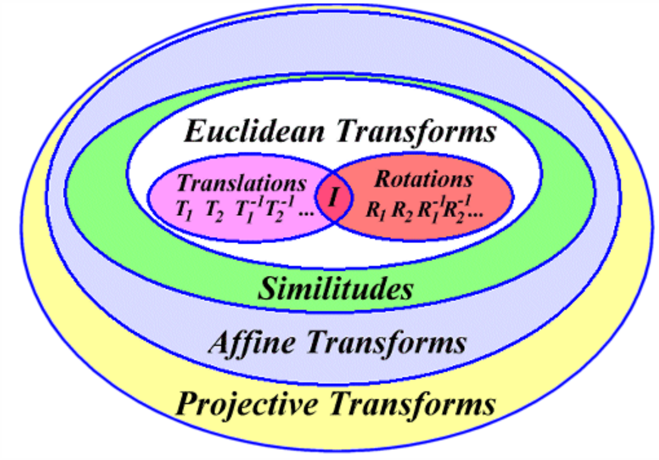
This gives the following solution:

$$\mathbf{X}^{-1}\mathbf{x}' = \mathbf{a}$$

Projective Transformations

- The most general linear transformation that we can apply to 2-D points

- There is something different about this group of transformations. The result will not necessarily lie on our selected plane. Since our world (to this point) is 2D we need some way to deal with this.



$$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projective Space

- The mapping of points from an *N*-D space to a *M*-D subspace (*M* < *N*)
- We need a rule for mapping points resulting from this transform back onto our plane z = 1. We will identify all points on lines through the origin of the 3-D space, and prefer the one that intersects our selected plane.

Thus, $x' \equiv \alpha x'$

Since the origin lies on all of these lines (and thus cannot be uniquely specified) we will disallow it. This is no big loss since it wasn't on our selected plane anyway (This is the real reason that we chose a plane not containing the origin).

If we want to find the coordinate of any point in our selected plane we need only scale it such that it's third coordinate, *w*, is 1.

Projective Transforms

- Since all of the resulting points are defined to within a non-zero scale factor. We can also scale the transformation by an arbitrary number and it will still give the same result.

$$\alpha \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \equiv \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- We might as well choose α so that one of the parameters of our matrix is 1 (i.e. $p_{33} = 1$).

Degrees of Freedom

- A projective transform has 8 free-parameters

$$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Which can be expressed as the following linear rational equation:

$$x' = \frac{p_{11}x + p_{12}y + p_{13}}{p_{31}x + p_{32}y + 1} \qquad y' = \frac{p_{21}x + p_{22}y + p_{23}}{p_{31}x + p_{32}y + 1}$$

Rearranging terms gives a linear expression in the coefficients:

$$x' = p_{11}x + p_{12}y + p_{13} - p_{31}xx' - p_{32}yx'$$

$$y' = p_{21}x + p_{22}y + p_{23} - p_{31}xy' - p_{32}yy'$$

Specifying a projective transform

- A projective transform can be uniquely defined by how it maps 4 points

$$\begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x'_1x_1 & -x'_1y_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -y'_1x_1 & -y'_1y_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x'_2x_2 & -x'_2y_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -y'_2x_2 & -y'_2y_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x'_3x_3 & -x'_3y_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -y'_3x_3 & -y'_3y_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x'_4x_4 & -x'_4y_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -y'_4x_4 & -y'_4y_4 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{31} \\ p_{32} \end{bmatrix}$$

Projective Examples

For part 2

- Find some neat high resolution photographs
  - Old home photos
  - Historic photos
  - Some place unique
  - Panorama's
- Scan them at a high resolution
  - At least 2048 by 1536
- Learn a tool for changing image formats