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On Space-Time Block Codes from Complex Orthogonal Designs ⋆

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Abstract. Space-time block codes from orthogonal designs recently proposed by Alamouti, and Tarokh-Jafarkhani-Calderbank have attracted considerable attention due to the fast maximum-likelihood (ML) decoding and the full diversity. There are two classes of space-time block codes from orthogonal designs. One class consists of those from real orthogonal designs for real signal constellations which have been well developed in the mathematics literature. The other class consists of those from complex orthogonal designs for complex constellations for high data rates, which are not well developed as the real orthogonal designs. Since orthogonal designs can be traced back to decades, if not centuries, ago and have recently invoked considerable interests in multi-antenna wireless communications, one of the goals of this paper is to provide a tutorial on both historical and most recent results on complex orthogonal designs. For space-time block codes from both real and (generalized) complex orthogonal designs (GCODs) with or without linear processing, Tarokh, Jafarkhani and Calderbank showed that their rates cannot be greater than 1. While the maximum rate 1 can be reached for real orthogonal designs for any number of transmit antennas from the Hurwitz–Radon constructive theory, Liang and Xia recently showed that rate 1 for the GCODs (square or non-square size) with linear processing is not reachable for more than two transmit antennas. For GCODs of square size, the designs with the maximum rates have been known, which are related to the Hurwitz theorem. In this paper, We briefly review these results and give a simple and intuitive interpretation of the realization. For GCODs without linear processing (square or non-square size), we prove that the rates cannot be greater than $3/4$ for more than two transmit antennas.

Keywords: diversity, (generalized) complex orthogonal designs, Hurwitz theorem, space-time block codes, wireless communications.

1. Introduction

In a high data rate wireless communication system, bandwidth limitation and channel fading are two major obstacles to achieve the reliable communication. Teletar [1] and Foschini and Gans [2] have recently shown that there is a huge potential capacity gain of multiple antenna systems compared to single antenna systems. They showed that the capacity of a multiple antenna system grows at least linearly with the number of transmit antennas, provided that the number of receive antennas is greater than or equal to the number of transmit antennas. To approach the potential huge capacity of multiple antenna systems, new coding and modulation, which is called *space-time coding*, has attracted considerable attention lately, see for example, [3–17].

The fundamental performance criteria of space-time codes were derived by Guey, Fitz, Bell and Kuo [3], Tarokh, Seshadri and Calderbank [4], and later extended by Hammons and

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El Gamal [8] for PSK modulations. In [4], Tarokh, Seshadri and Calderbank also presented a few space-time trellis codes for 2–4 transmit antennas which perform well in slow-fading channels and come close to the outage capacity promised by Teletar [1] and Foschini and Gans [2]. However, the maximum-likelihood (ML) decoding complexity of the space-time trellis codes is high.

Later, Alamouti in [5] introduced a simple transmit scheme for two transmit antennas which achieves full diversity and has a fast ML decoding at the receiver. Motivated by the Alamouti's scheme, Tarokh, Jafarkhani and Calderbank in [6] proposed a general scheme, *space-time block codes*, from orthogonal designs for any number of transmit antennas, which has the full diversity and a fast ML decoding of space-time block codes. In particular, the transmitted symbols can be decoded separately, not jointly. Thus, the decoding complexity increases linearly, not exponentially, with the code size.

There are two classes of space-time block codes from orthogonal designs. One class consists of those from real orthogonal designs for real constellations such as PAM. These codes have been well developed. There are systematic constructions with optimal symbol transmission rate 1 for any number of transmit antennas [6], which are based on the Hurwitz–Radon constructive theory [20, 24]. Ganesan and Stoica later revisited this scheme from a maximum SNR approach [9]. Real orthogonal designs have been motivated for the compositions of quadratic forms started in the 1700s [24, 25]. The other class consists of those from complex orthogonal designs for complex constellations such as QAM and PSK. Unlike space-time block codes from real orthogonal designs, these codes or complex orthogonal designs or Hermitian compositions of quadratic forms [25] are not well understood. In this paper, we focus on the discussion of space-time block codes from complex orthogonal designs, while they are important to achieve high data rates using QAM signal constellations in broadband wireless communications.

A *complex orthogonal design* (COD) in variables $x_1, x_2, \ldots, x_n$ is an $n \times n$ matrix $\mathcal{O}$ such that: (i) the entries of $\mathcal{O}$ are 0, $\pm x_1, \pm x_2, \ldots, \pm x_n$, or their conjugates $\pm x_1^*, \pm x_2^*, \ldots, \pm x_n^*$, or multiples of them by $\mathbf{i}$ where $\mathbf{i} = \sqrt{-1}$; and (ii) $\mathcal{O}^{\mathcal{H}}\mathcal{O} = (|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2)I_n$, where the superscript $\mathcal{H}$ stands for the complex conjugate and transpose of a matrix, and I_n is the $n \times n$ identity matrix. Tarokh, Jafarkhani and Calderbank showed in [6] that space-time block codes constructed in this way exist only for two transmit antennas. Then they tried to relax the definition of complex orthogonal designs to allow linear processing at the transmitter, i.e., the entries of $\mathcal{O}$ may be complex linear combinations of $x_1, x_1^*, x_2, x_2^*, \ldots, x_n, x_n^*$. However, they also proved that this extension fails to provide new designs. Later, Ganesan and Stoica [9] revisited this result by connecting this problem to the amicable design theory [22, 24].

Tarokh, Jafarkhani and Calderbank [6] observed that it is not necessary for the complex orthogonal designs (with or without linear processing) to be square matrices in order to construct space-time block codes. Space-time block codes allow non-square designs. Subsequently, they introduced the definition of *generalized complex orthogonal designs* (GCODs). Furthermore, they proposed *generalized complex orthogonal designs with linear processing* (GCODs with linear processing for short). The detailed definitions are reviewed later. With these new definitions, there are space-time block codes from GCODs that can be used for any number of transmit antennas. However, for more than six transmit antennas, the known space-time block codes from GCODs with linear processing have symbol transmission rate only $1/2$, far from the maximum symbol transmission rate 1 of those codes from real orthogonal designs for real constellations. The existing space-time block codes from (generalized) complex orthogonal designs with or without linear processing can be summarized as follows:

- For 2 transmit antennas, space-time block code exists with the maximum symbol transmission rate 1 from COD (Alamouti's scheme [5]);
- For 3 and 4 transmit antennas, space-time block codes exist with symbol transmission rate $3/4$ from GCODs with linear processing [6] or from GCODs without linear processing [9–11];
- For 5 and 6 transmit antennas, space-time block codes exist with symbol transmission rates $7/11$ and $3/5$, respectively, from GCODs with linear processing [15];
- For any number of transmit antennas, space-time block codes exist with symbol transmission rate $1/2$ from GCODs with linear processing (Tarokh, Jafarkhani and Calderbank [6]).

For two transmit antennas, the Alamouti's scheme achieves the maximum symbol transmission rate 1. However, for more than two transmit antennas, it is not clear what is the maximum symbol transmission rate of space-time block codes from generalized complex orthogonal designs with or without linear processing.

Tarokh, Jafarkhani and Calderbank first mentioned in [6] that the symbol transmission rate of space-time block codes from GCODs with or without linear processing cannot be greater than 1 for any number of transmit antennas. Surprisingly, Liang and Xia later proved in [16] that this symbol transmission rate cannot be 1 for more than two transmit antennas, contrast to the space-time block codes from real orthogonal designs for real constellations in which the symbol transmission rate can achieve the maximum rate 1 for any number of transmit antennas. More precisely, for more than two transmit antennas they proved that $k \leq p - 1$, where k is the number of information symbols in each codeword, and p is the time delay. The symbol transmission rate is defined as $R = k/p$, which means that each codeword of time delay p carries k information symbols. Therefore, the symbol transmission rate $R \leq (p-1)/p < 1$.

In this paper we show that for GCODs without linear processing, the symbol transmission rate cannot be greater than $3/4$ for more than two transmit antennas.

The paper is organized as follows. In Section 2, we will briefly review the theory of space-time block codes and design criteria. In Section 3, we focus on the discussion of GCODs without linear processing. For GCODs of square size, the maximum symbol transmission rate has been characterized completely, which is related to the Hurwitz theorem [18, 19, 23, 25, 10] or the amicable design theory [22, 24, 9]. For GCODs of non-square size, we show that the symbol transmission rate cannot be greater than $3/4$ for more than two transmit antennas. In Section 4, we discuss GCODs with linear processing. Finally, we conclude this paper with some comments and open problems in Section 5.

2. Space-Time Block Codes and Design Criteria

In this section, we briefly review the theory of space-time block codes and diversity criterion. More details can be seen in [6].

We consider a wireless communication system with n transmit antennas and m receive antennas. The channel is assumed to be a *quasi-static and flat Rayleigh* fading channel. A space-time block code is a collection of some matrices. Each matrix is of size $p \times n$ as $\mathbf{c} = \{c_t^i : t = 1, 2, \ldots, p;\ i = 1, 2, \ldots, n\}$. Here, p represents the number of time slots, or *time delay*, for transmitting one codeword. For some information symbols $x_1, x_2, \ldots, x_k$ which are selected from an arbitrary constellation, the entries of the matrix $\mathbf{c}$ are complex linear combinations of $x_1, x_2, \ldots, x_k$ or their conjugates $x_1^*, x_2^*, \ldots, x_k^*$. At time slot t, $t = 1, 2, \ldots, p$, the tth row of the matrix $\mathbf{c}$ is transmitted, i.e., $c_t^1, c_t^2, \ldots, c_t^n$ are transmitted simultaneously from the n transmit antennas. The symbol transmission rate is defined as $R = k/p$, which means that there are k information symbols transmitted in one block with time delay p.

The whole system can be modeled as

$$Y = \mathbf{c}A + N, \tag{2.1}$$

where $Y = \{y_t^j : t = 1, 2, \ldots, p;\ j = 1, 2, \ldots, m\}$ is the received symbol matrix of size $p \times m$ whose entry y_t^j is the signal received at antenna j at time t; $A = \{\alpha_{i,j}\}$ is the channel coefficient matrix of size $n \times m$ whose entry $\alpha_{i,j}$ is the channel coefficient from transmit antenna i to receive antenna j; and $N = \{\eta_t^j\}$ is the noise matrix of size $p \times m$ whose entry η_t^j is the AWGN noise sample at receive antenna j at time t. The noise samples are independent samples of a zero-mean complex Gaussian random variable with variance 1/(2SNR) per dimension. The fading channel is *quasi-static* in the sense that the channel coefficients do not change during one codeword transmission, and change independently from one codeword transmission to the next.

Assume that perfect channel state information is available at the receiver, then the ML decoding at the receiver is

$$\begin{aligned} \min_{\mathbf{c}} \|Y - \mathbf{c}A\|_F^2 &= \min_{\mathbf{c}} \operatorname{tr}\left[(Y - \mathbf{c}A)^{\mathcal{H}}(Y - \mathbf{c}A)\right] \\ &= \min_{\mathbf{c}} \left[\operatorname{tr}(Y^{\mathcal{H}}Y) - \operatorname{tr}(Y^{\mathcal{H}}\mathbf{c}A + A^{\mathcal{H}}\mathbf{c}^{\mathcal{H}}Y) + \operatorname{tr}(\mathbf{c}^{\mathcal{H}}\mathbf{c}AA^{\mathcal{H}})\right], \end{aligned} \tag{2.2}$$

where $\operatorname{tr}(V)$ is the trace of matrix V, and $\|V\|_F$ is the Frobenius norm[1] of matrix V. Notice that $\operatorname{tr}(Y^{\mathcal{H}}\mathbf{c}A + A^{\mathcal{H}}\mathbf{c}^{\mathcal{H}}Y)$ is the linear combination of the first order of $x_1, x_2, \ldots, x_k$ or their conjugates $x_1^*, x_2^*, \ldots, x_k^*$, and $\operatorname{tr}(\mathbf{c}^{\mathcal{H}}\mathbf{c}AA^{\mathcal{H}})$ is the linear combination of the second order of them. Thus, if there are no terms of $x_i x_j$, $x_i x_j^*$ and $x_i^* x_j^*$ with $i \neq j$ in $\operatorname{tr}(\mathbf{c}^{\mathcal{H}}\mathbf{c}AA^{\mathcal{H}})$, for example $\mathbf{c}^{\mathcal{H}}\mathbf{c} = (|x_1|^2 + |x_2|^2 + \ldots + |x_k|^2)I_n$, then the decision metric in (2.2) can be written as the sum of several functions whose variables depend on each x_i, i.e.,

$$\|Y - \mathbf{c}A\|_F^2 = \sum_{i=1}^{k} f_i(x_i).$$

Therefore, the minimization can be done separately on each x_i, not jointly. This leads to the fast ML decoding of the space-time block codes from orthogonal designs.

Suppose that codeword $\mathbf{c}$ be transmitted and the receiver erroneously in favor of codeword $\mathbf{e}$. Then the pairwise error probability is given by

$$\begin{aligned} P(\mathbf{c} \to \mathbf{e}|A) &= Q\left(\sqrt{\frac{\text{SNR}}{2}}\, ||(\mathbf{c} - \mathbf{e})A||_F\right) \\ &\leq \frac{1}{2} \exp\left\{-\frac{\text{SNR}}{4}\, \|(\mathbf{c} - \mathbf{e})A\|_F^2\right\}. \end{aligned} \tag{2.3}$$

[1] The Frobenius norm of V satisfies

$$\|V\|_F^2 = \operatorname{tr}(V^{\mathcal{H}}V) = \operatorname{tr}(VV^{\mathcal{H}}) = \sum_{i,j} |v_{i,j}|^2.$$

For the quasi-static and flat Rayleigh fading channel, (2.3) can be further written as [3, 4]

$$\begin{aligned} P(\mathbf{c} \to \mathbf{e}) &\leq \frac{1}{2}\left[\prod_{i=1}^{r}\left(1+\lambda_i\frac{\mathrm{SNR}}{4}\right)\right]^{-m} \\ &\leq \frac{1}{2}\left(\prod_{i=1}^{r}\lambda_i\right)^{-m}\cdot\left(\frac{\mathrm{SNR}}{4}\right)^{-rm}, \end{aligned} \tag{2.4}$$

where $r = \mathrm{rank}(\mathbf{c}-\mathbf{e})$, and $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the nonzero eigenvalues of $(\mathbf{c}-\mathbf{e})(\mathbf{c}-\mathbf{e})^{\mathcal{H}}$. For high SNR, the upper bound in (2.4) is dominated by the term $(\mathrm{SNR}/4)^{-rm}$. Thus the rank r should be as large as possible. This leads to the *rank criterion* or *diversity criterion*: in order to achieve the maximum diversity, the difference matrix $\mathbf{c} - \mathbf{e}$ has to be of full rank for any pair of distinct codewords $\mathbf{c}$ and $\mathbf{e}$ [3, 4].

Therefore, a "good" space-time block code should possess two properties: (i) the difference matrix between two distinct codewords should be of full rank, i.e., this code achieves the maximum diversity; and (ii) there is a fast ML decoding algorithm. The space-time block codes from orthogonal designs do have these two properties. The special structure of orthogonal designs not only guarantees the maximum diversity, but also provides a fast ML decoding. The transmitted symbols can be decoded separately, not jointly. Thus the decoding complexity increases linearly, not exponentially, with the code size.

Note that, the special structure of orthogonal designs is sufficient, but not necessary, to construct space-time block codes having fast ML decoding and achieving maximum diversity. In fact, if there exist some $n \times n$ positive definite matrices $D_1, D_2, \ldots, D_k$ such that

$$\mathbf{c}^{\mathcal{H}}\mathbf{c} = |x_1|^2 D_1 + |x_2|^2 D_2 + \cdots + |x_k|^2 D_k\,, \tag{2.5}$$

then the space-time block codes from (2.5) possess the two properties. We observe that the difference matrix $\mathbf{c} - \mathbf{e}$ satisfies

$$(\mathbf{c}-\mathbf{e})^{\mathcal{H}}(\mathbf{c}-\mathbf{e}) = |x_1-\tilde{x}_1|^2 D_1 + |x_2-\tilde{x}_2|^2 D_2 + \cdots + |x_k-\tilde{x}_k|^2 D_k\,.$$

Thus, the positive definiteness of the matrices $D_1, D_2, \ldots, D_k$ guarantees that the difference matrix of two distinct codewords is of full rank, and the fact of no terms of $x_i x_j$, $x_i x_j^*$ and $x_i^* x_j^*$ with $i \neq j$ in (2.5) implies that the ML decision metric in (2.2) can be minimized separately on each x_i. Space-time block codes from orthogonal designs can be considered as some special cases of those from (2.5) when $D_1, D_2, \ldots, D_k$ are some diagonal matrices. Even for these special cases, the problem of the maximum symbol transmission rate has not been well understood yet.

3. Generalized Complex Orthogonal Designs (GCODs)

In this section, we focus on the discussion of GCODs without linear processing. We will discuss GCODs with linear processing in next section. For GCODs of square size, the problem of the maximum symbol transmission rate has been solved completely, which is related to the Hurwitz theorem [18, 19, 23, 25, 10] or the amicable design theory [22, 24, 9]. We briefly review the results and give a simple and intuitive interpretation of the realization. For GCODs of non-square size, we prove that the maximum symbol transmission rate cannot be greater than $3/4$ for more than two transmit antennas.

DEFINITION 3.1. *A generalized complex orthogonal design (GCOD for short) in variables* $x_1, x_2, \ldots, x_k$ *is a* $p \times n$ *matrix* G *such that:*

(i) *The entries of* G *are* $0, \pm x_1, \pm x_2, \ldots, \pm x_k$*, or their conjugates* $\pm x_1^*, \pm x_2^*, \ldots, \pm x_k^*$*, or multiples of them by* $\mathbf{i}$ *where* $\mathbf{i} = \sqrt{-1}$;[2]

(ii) $G^{\mathcal{H}} G = (|x_1|^2 + |x_2|^2 + \ldots + |x_k|^2) I_n$*, where* $G^{\mathcal{H}}$ *is the complex conjugate and transpose of* G*. We will use this notation in the rest of this paper.*

The rate of G *is defined as* $R = k/p$*. If* $p = n = k$*, then* G *is a classical complex orthogonal design (COD for short).*

In Definition 3.1, n is related to the number of transmit antennas, p is related to the time delay in each codeword, and the variables $x_1, x_2, \ldots, x_k$ can be arbitrary constellation symbols. The relationship of n, k and p will be discussed later. In particular, for a fixed n, there is an upper bound on the rate k/p. It is worth noting that for a space-time block code from a GCOD, the difference matrix ΔG between two distinct codewords is also a GCOD of the same structure, i.e., $(\Delta G)^{\mathcal{H}}(\Delta G) = (|\Delta x_1|^2 + \cdots + |\Delta x_k|^2) I_n$, which implies that ΔG has full rank unless $\Delta x_1 = \cdots = \Delta x_k = 0$. Thus, from Definition 3.1 (ii), space-time block codes from GCOD achieve the full diversity.

The first space-time block code from GCOD was proposed by Alamouti [5] for two transmit antennas. It is, in fact, a 2×2 COD in two variables x_1, x_2:

$$G_2(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}.$$

Clearly, the rate of G_2 is 1. From later discussion, we know that for space-time block codes from GCODs, the rate 1 is achievable only for two transmit antennas.

For three and four transmit antennas, space-time block codes from GCODs with rate $R = 3/4$ are given by [9–11]:

$$G_3(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 \\ -x_2^* & x_1^* & 0 \\ -x_3^* & 0 & x_1^* \\ 0 & -x_3^* & x_2^* \end{bmatrix}, \quad G_4(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_3^* & x_2^* & x_1 \end{bmatrix}. \quad (3.1)$$

In fact, G_3 is obtained by taking the first three columns of G_4. As a remark, based on the amicable designs [22, 24], Tarokh, Jafarkhani and Calderbank [6] had earlier presented two designs with rate $R = 3/4$ from GCODs with linear processing which are equivalent to the above two designs by applying some unitary operations and changing variables.

For a fixed n, it is desired to have the rate of GCODs as large as possible. It was shown in [6] that this rate cannot exceed 1, i.e., $R \leq 1$. Later, Liang and Xia in [16] proved that this rate cannot be 1, i.e., $R < 1$, for more than two transmit antennas, which is surprisingly different from the real orthogonal designs. However, what is the maximum rate of GCODs for $n > 2$ transmit antennas remains open.

[2] The results in this section remain true if each entry of G is multiplied by an arbitrary phase offset $e^{\mathbf{i}\phi}$.

3.1. GCODs of Square Size

For GCODs of square size, i.e., GCODs in Definition 3.1 with $p = n$, the problem of what is the maximum achievable rate has been solved completely. Tarokh, Jafarkhani and Calderbank [6] first proved that the GCOD of square size with the maximum rate 1 exists only for $n = 2$ transmit antennas. Later, Ganesan and Stoica [9] connected this problem to the amicable design theory [22, 24] which is essentially a generalization of the Hurwitz theorem [21, 23, 25]. Recently, Tirkkonen and Hottinen [10] revisited the Hurwitz theorem and provided a realization of GCODs with the maximum achievable rates directly. In this subsection, we review the Hurwitz theorem at first, then clarify the relationships between the Hurwitz theorem and the problem of the maximum achievable rate of GCODs, and finally give a simple and intuitive interpretation of the realization.

A set of $n \times n$ complex matrices $\{C_1, C_2, \ldots, C_l\}$ is said to be *Hurwitz family*[3] *of order n* [23], *if*

$$C_i^2 = -I_n, \quad 1 \le i \le l; \tag{3.2}$$

$$C_i C_j = -C_j C_i, \quad 1 \le i \ne j \le l. \tag{3.3}$$

Denote $H(n) - 1$ be the maximum number of complex matrices in a Hurwitz family of order n, then the Hurwitz theorem can be stated as follows[4] ([23], [21], [25] p. 86).

THEOREM 3.1. *(Hurwitz) If $n = 2^a \cdot b$, b odd, then*

$$H(n) = 2a + 2.$$

Observing that, when n is odd, the maximum number of complex matrices in a Hurwitz family of order n is 1. Josefiak in [21] presented a general realization of Hurwitz families as follows: if $\{C_1, C_2, \ldots, C_l\}$ is a Hurwitz family of l matrices of order n, then the set

$$\{M \otimes I_n, \mathbf{i}P \otimes I_n\} \cup \{Q \otimes C_i : i = 1, 2, \ldots, l\}, \tag{3.4}$$

[3] Hurwitz family here is different from the Hurwitz–Radon family (see [20, 6, 24]). The matrices in a Hurwitz family are complex, while those in a Hurwitz–Radon family are real.

[4] It is well known that the maximum number of real matrices in a Hurwitz–Radon family of order n is denoted as $\rho(n) - 1$. A similar result is the Hurwitz–Radon theorem [20, 6, 24]: if $n = 2^a \cdot b$, b odd, $a = 4c + d$ with $c \ge 0$ and $0 \le d \le 3$, then

$$\rho(n) = 8c + 2^d.$$

The relationship between $\rho(n)$ and $H(n)$ is:

$$H(n) = \begin{cases} \rho(n) + 1, & \text{if } a \equiv 0 \ (mod\, 4); \\ \rho(n) + 2, & \text{if } a \equiv 1 \text{ or } 2 \ (mod\, 4); \\ \rho(n), & \text{if } a \equiv 3 \ (mod\, 4). \end{cases}$$

is a Hurwitz family of $l+2$ matrices of order $2n$, where the symbol $\cup$ stands for a union of two sets, the symbol $\otimes$ denotes the tensor product,[5] and

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Notice that $C_1 = \mathbf{i}I_1$ when $n = 1$. Then by induction, one can obtain a Hurwitz family of any even order recursively.

Now we go back to the problem of the maximum achievable rate of GCODs. Assume that G be a GCOD of square size $n \times n$ in variables $x_1, x_2, \ldots, x_k$. We rewrite it as

$$G = \sum_{i=1}^{k} \left[Re(x_i) A_i + Im(x_i) B_i \right], \tag{3.5}$$

where A_i and B_i are $n \times n$ complex matrices, $Re(x_i)$ and $Im(x_i)$ are the real and imaginary parts of x_i, respectively. From Definition 3.1 (ii), we have

$$G^{\mathcal{H}} G = \sum_{i=1}^{k} (|Re(x_i)|^2 + |Im(x_i)|^2) I_n \,. \tag{3.6}$$

Expressions (3.5) and (3.6) imply that

$$A_i^{\mathcal{H}} A_i = I_n, \quad B_i^{\mathcal{H}} B_i = I_n, \quad 1 \le i \le k;$$

$$A_i^{\mathcal{H}} A_j = -A_j^{\mathcal{H}} A_i, \quad B_i^{\mathcal{H}} B_j = -B_j^{\mathcal{H}} B_i, \quad 1 \le i \ne j \le k;$$

$$A_i^{\mathcal{H}} B_j = -B_j^{\mathcal{H}} A_i, \quad 1 \le i, j \le k.$$

Let $A_{k+i} = B_i$, $i = 1, 2, \ldots, k$, then we have

$$\begin{aligned} A_i^{\mathcal{H}} A_i &= I_n, \quad 1 \le i \le 2k; \\ A_i^{\mathcal{H}} A_j &= -A_j^{\mathcal{H}} A_i, \quad 1 \le i \ne j \le 2k. \end{aligned}$$

Furthermore, let $C_i = A_1^{\mathcal{H}} A_i$, $i = 1, 2, \ldots, 2k$, then $C_1 = I_n$, and

$$\begin{aligned} C_i^2 &= -I_n, \quad 2 \le i \le 2k; \\ C_i C_j &= -C_j C_i, \quad 2 \le i \ne j \le 2k. \end{aligned}$$

Therefore, $\{C_2, C_3, \ldots, C_{2k}\}$ is a Hurwitz family of $2k-1$ complex matrices of order n. According to the Hurwitz theorem, the number of complex matrices, $2k-1$, cannot be greater than $H(n) - 1$, i.e., $2k \le H(n)$. Thus we have the following corollary.

[5] Let $A = \{\alpha_{i,j}\}$ be a $s \times t$ matrix and B be an arbitrary matrix, the tensor product $A \otimes B$ is given by

$$A \otimes B = \begin{bmatrix} \alpha_{11} B & \cdots & \alpha_{1t} B \\ \vdots & \ldots & \vdots \\ \alpha_{s1} B & \cdots & \alpha_{st} B \end{bmatrix}.$$

The rth tensor power of matrix B is defined as $\otimes^r B = \underbrace{B \otimes B \otimes \cdots \otimes B}_{r \text{ times}}$.

COROLLARY 3.1. *If $n = 2^a \cdot b$, b odd, then the rate of any GCOD of square size*

$$R = \frac{k}{p} \leq \frac{H(n)}{2n} = \frac{a+1}{2^a \cdot b},$$

and the bound can be achieved.

We observe that, when $n = 2$, $R \leq 1$; when $n = 4$, $R \leq 3/4$; when $n = 8$, $R \leq 1/2$; ...; when $n = 2^r$, $R \leq (r+1)/2^r$. The GCODs achieving the bound can be constructed from the Josefiak's realization in (3.4). For example, when $n = 2$,

$$\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} \right\}$$

is a Hurwitz family of order 2. Then, by the notations in (3.5), the realization of a 2×2 GCOD can be expressed as

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix},$$

in which besides the Hurwitz family, the 2×2 identity matrix is also used.

Ganesan and Stoica [9] obtained the result in Corollary 3.1 via the amicable design theory [22, 24] which is essentially a generalization of the Hurwitz theorem [23, 21, 25]. There are also realizations of GCODs from the amicable designs [22, 24, 9] which are similar to the Josefiak's realization in (3.4). Recently, Tirkkonen and Hottinen [10] revisited the Hurwitz theorem and provided a realization of GCODs with the maximum achievable rate directly for $n = 2^r,\ r \geq 1$ as follows:

$$\begin{aligned} G_{2^r}(x_1, x_2, \ldots, x_{r+1}) &= x_1 \left(I_n + \otimes^r \delta\right)/2 + x_1^* \left(I_n - \otimes^r \delta\right)/2 \\ &\quad + \sum_{i=2}^{r+1} \left(\otimes^{r+1-i} I_2\right) \otimes \begin{bmatrix} 0 & x_i \\ -x_i^* & 0 \end{bmatrix} \otimes \left(\otimes^{i-2} \delta\right), \end{aligned}$$

where $\delta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In the following, we present a simple and intuitive interpretation of the realization of GCODs with the maximum achievable rate for $n = 2^r,\ r \geq 0$. Let $G_1(x_1) = x_1 I_1$, and

$$G_{2^r}(x_1, x_2, \ldots, x_{r+1}) = \begin{bmatrix} G_{2^{r-1}}(x_1, x_2, \cdots, x_r) & x_{r+1} I_{2^{r-1}} \\ -x_{r+1}^* I_{2^{r-1}} & G_{2^{r-1}}^{\mathcal{H}}(x_1, x_2, \cdots, x_r) \end{bmatrix},$$
$$r = 1, 2, 3, \ldots .$$

We can check that (in Appendix A)

$$(G_{2^r}(x_1, x_2, \ldots, x_{r+1}))^{\mathcal{H}} G_{2^r}(x_1, x_2, \ldots, x_{r+1}) = (|x_1|^2 + |x_2|^2 + \ldots + |x_{r+1}|^2) I_{2^r},$$

and the rate $R = (r+1)/2^r,\ r \geq 0$.

To be explicit, here are some examples:

$$G_2(x_1, x_2) = \begin{bmatrix} G_1(x_1) & x_2 \\ -x_2^* & G_1^{\mathcal{H}}(x_1) \end{bmatrix} = \left[\begin{array}{c|c} x_1 & x_2 \\ \hline -x_2^* & x_1^* \end{array}\right];$$

$$G_4(x_1,x_2,x_3) = \begin{bmatrix} G_2(x_1,x_2) & x_3 I_2 \\ -x_3^* I_2 & G_2^{\mathcal{H}}(x_1,x_2) \end{bmatrix} = \left[\begin{array}{cc|cc} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ \hline -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_3^* & x_2^* & x_1 \end{array}\right];$$

$$\begin{aligned} G_8(x_1,x_2,x_3,x_4) &= \begin{bmatrix} G_4(x_1,x_2,x_3) & x_4 I_4 \\ -x_4^* I_4 & G_4^{\mathcal{H}}(x_1,x_2,x_3) \end{bmatrix} \\ &= \left[\begin{array}{cccc|cccc} x_1 & x_2 & x_3 & 0 & x_4 & 0 & 0 & 0 \\ -x_2^* & x_1^* & 0 & x_3 & 0 & x_4 & 0 & 0 \\ -x_3^* & 0 & x_1^* & -x_2 & 0 & 0 & x_4 & 0 \\ 0 & -x_3^* & x_2^* & x_1 & 0 & 0 & 0 & x_4 \\ \hline -x_4^* & 0 & 0 & 0 & x_1^* & -x_2 & -x_3 & 0 \\ 0 & -x_4^* & 0 & 0 & x_2^* & x_1 & 0 & -x_3 \\ 0 & 0 & -x_4^* & 0 & x_3^* & 0 & x_1 & x_2 \\ 0 & 0 & 0 & -x_4^* & 0 & x_3^* & -x_2^* & x_1^* \end{array}\right]. \end{aligned}$$

3.2. GCODs of Non-Square Size

Tarokh, Jafarkhani and Calderbank mentioned in [6] that it is not necessary for the GCODs to be of square size in order to construct space-time block codes. Actually, space-time block codes can be constructed from GCODs of non-square size as shown in [6]. In this subsection, we prove that the maximum rate of GCODs of non-square size cannot be greater than $3/4$ for $n \geq 3$.

Let G be a GCOD in variables $x_1, x_2, \ldots, x_k$ of size $p \times n$. We rewrite it as

$$G = [E_1\mathbf{x} + F_1\overline{\mathbf{x}} \quad E_2\mathbf{x} + F_2\overline{\mathbf{x}} \quad \cdots \quad E_n\mathbf{x} + F_n\overline{\mathbf{x}}], \tag{3.7}$$

where E_i and F_i are $p \times k$ complex matrices, $\mathbf{x} = (x_1, x_2, \ldots, x_k)^T$ and $\overline{\mathbf{x}} = (x_1^*, x_2^*, \ldots, x_k^*)^T$. Here, the superscript T stands for the transpose of a matrix or a vector, and $\overline{\mathbf{x}}$ is the complex conjugate of $\mathbf{x}$. Clearly, $(\overline{\mathbf{x}})^T = \mathbf{x}^{\mathcal{H}}$. We use these notations throughout the paper. From Definition 3.1 (ii), we obtain the constraints on E_i and F_i, $i = 1, 2, \ldots, n$, as follows [16].

PROPERTY 3.1. *$G^{\mathcal{H}}G = (|x_1|^2 + |x_2|^2 + \cdots + |x_k|^2)I_n$ is true, where G is represented in (3.7), if and only if*

$$E_i^{\mathcal{H}}E_i + F_i^T\overline{F}_i = I_k, \quad 1 \leq i \leq n; \tag{3.8}$$

$$E_i^{\mathcal{H}}E_j + F_j^T\overline{F}_i = \mathbf{0}_{k\times k}, \quad 1 \leq i \neq j \leq n; \tag{3.9}$$

$$E_i^{\mathcal{H}}F_j + F_j^T\overline{E}_i = \mathbf{0}_{k\times k}, \quad 1 \leq i,\ j \leq n. \tag{3.10}$$

For convenience, we specify some definitions at first. A column (row) of a matrix is said to be *zero*, if all elements of this column (row) are zeros. A matrix is said to be *monomial*, if there is at most one non-zero element per row and column. Clearly, the rank of a monomial matrix is equal to the number of non-zero elements in this matrix. Two matrices A and B of same sizes are said to be *disjoint*, if a column (row) in A is non-zero, then the same column (row) in B must be zero; and conversely if a column (row) in B is non-zero, then the same column (row) in A must be zero. From Definition 3.1 (i), we know that the entries of G cannot be linear combination of $x_1, x_2, \ldots, x_k$ or their conjugates $x_1^*, x_2^*, \ldots, x_k^*$. Together with (3.8) in Property 3.1, we have the following property.

PROPERTY 3.2. *For any* i, $1 \leq i \leq n$,

(i) *both* E_i *and* F_i *are monomial, so is* $E_i + F_i$*;*
(ii) *the pair of* E_i *and* F_i *is disjoint;*
(iii) $E_i^{\mathcal{H}} F_i = \mathbf{0}_{k\times k}$, $E_i F_i^{\mathcal{H}} = \mathbf{0}_{p\times p}$*;*
(iv) $rank(E_i + F_i) = rank(E_i) + rank(F_i)$.

Proof. See Appendix B. □

Notice that G allows row and column permutations. Without loss of generality, we may assume that the first column of G be $[x_1\ x_2\ \cdots\ x_k\ 0\ \cdots\ 0]^T$, i.e., $E_1 = \begin{bmatrix} I_k \\ \mathbf{0}_{(p-k)\times k} \end{bmatrix}$ and $F_1 = \mathbf{0}_{p\times k}$. If there are some x_i^* in the first column of G, we can always obtain the form of $[y_1\ y_2\ \cdots\ y_k\ 0\ \cdots\ 0]^T$ by changing variables. Let

$$E_i = \begin{bmatrix} E_{i1} \\ E_{i2} \end{bmatrix}, \quad F_i = \begin{bmatrix} F_{i1} \\ F_{i2} \end{bmatrix}, \quad i = 2, 3, \ldots, n,$$

where E_{i1} and F_{i1} are $k \times k$ complex matrices, E_{i2} and F_{i2} are $(p-k)\times k$ complex matrices. From (3.9), we have

$$E_1^{\mathcal{H}} E_i + F_i^T \overline{F}_1 = \mathbf{0}_{k\times k}, \quad i = 2, 3, \ldots, n.$$

It implies that $E_{i1} = \mathbf{0}_{k\times k}$ for any $i = 2, 3, \ldots, n$. From (3.10), we have

$$E_1^{\mathcal{H}} F_i + F_i^T \overline{E}_1 = \mathbf{0}_{k\times k}, \quad i = 2, 3, \ldots, n.$$

It implies that $F_{i1} + F_{i1}^T = \mathbf{0}_{k\times k}$ for any $i = 2, 3, \ldots, n$. From the above arguments, we have the following property.

PROPERTY 3.3. *If* $E_1 = \begin{bmatrix} I_k \\ \mathbf{0}_{(p-k)\times k} \end{bmatrix}$ *and* $F_1 = \mathbf{0}_{p\times k}$*, then*

$$E_{i1} = \mathbf{0}_{k\times k}, \quad F_{i1} + F_{i1}^T = \mathbf{0}_{k\times k}, \quad i = 2, 3, \ldots, n.$$

For any i, $i = 2, 3, \ldots, n$, denote $\beta_{i,2}$, $\gamma_{i,1}$ and $\gamma_{i,2}$ as the ranks of E_{i2}, F_{i1} and F_{i2}, respectively. Since $E_i + F_i$, E_i and F_i are monomial, and the pair of E_i and F_i is disjoint, so we have the following property.

PROPERTY 3.4. *For any* $i, i = 2, 3, \ldots, n$,

$$\beta_{i,2} + \gamma_{i,1} + \gamma_{i,2} = k,$$

and

$$p \geq k + \beta_{i,2} + \gamma_{i,2}\,.$$

The following rank equalities and inequalities are useful in the rest of this paper [26]:

- For any complex matrix A,

  $$rank(A^{\mathcal{H}}A) = rank(A^{\mathcal{H}}) = rank(A);$$

- For two complex matrices A and B of same sizes,

  $$rank(A+B) \leq rank(A) + rank(B);$$

- For two complex matrices A of size $s \times t$ and B of size $t \times l$,

  $$rank(A) + rank(B) - t \leq rank(AB) \leq \min\{rank(A),\ rank(B)\}.$$

Now we can prove our main result.

THEOREM 3.2. *For $n \geq 3$, the rate of GCODs cannot be greater than $3/4$. More precisely,*

(i) *When $k = 3l,\ l = 1, 2, 3, \ldots,\ R \leq 3/4$;*
(ii) *When $k = 3l-1,\ l = 1, 2, 3, \ldots,\ R \leq (3l-1)/(4l)$;*
(iii) *When $k = 3l-2,\ l = 1, 2, 3, \ldots,\ R \leq (3l-2)/(4l-1)$;*

and the above upper bounds can be reached for $n = 3$ and 4.

Proof. For any $n \geq 3$, from (3.9) in Property 3.1, we have $E_2^{\mathcal{H}}E_3 + F_3^T\overline{F}_2 = \mathbf{0}_{k\times k}$. Replacing $E_2,\ E_3,\ F_2$ and F_3 by their partitions, we obtain

$$E_{22}^{\mathcal{H}}E_{32} + F_{31}^T\overline{F}_{21} + F_{32}^T\overline{F}_{22} = \mathbf{0}_{k\times k}\,. \tag{3.11}$$

On one hand, we have an upper bound on the rank of $E_{22}^{\mathcal{H}}E_{32} + F_{32}^T\overline{F}_{22}$,

$$\begin{aligned} rank(E_{22}^{\mathcal{H}}E_{32} + F_{32}^T\overline{F}_{22}) &\leq rank(E_{22}^{\mathcal{H}}E_{32}) + rank(F_{32}^T\overline{F}_{22}) \\ &\leq \frac{1}{2}(\beta_{2,2}+\beta_{3,2}) + \frac{1}{2}(\gamma_{3,2}+\gamma_{2,2}). \end{aligned} \tag{3.12}$$

On the other hand, we have a lower bound on the rank of $F_{31}^T\overline{F}_{21}$,

$$rank(F_{31}^T\overline{F}_{21}) \geq \gamma_{3,1} + \gamma_{2,1} - k\,. \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), we have

$$\gamma_{2,1} + \gamma_{3,1} - k \leq \frac{1}{2}(\beta_{2,2} + \beta_{3,2} + \gamma_{2,2} + \gamma_{3,2}).$$

It follows that

$$2\gamma_{2,1} - k \leq \beta_{2,2} + \gamma_{2,2}\,,$$

or

$$2\gamma_{3,1} - k \leq \beta_{3,2} + \gamma_{3,2}\,,$$

i.e., $2\gamma_{i_0,1} - k \leq \beta_{i_0,2} + \gamma_{i_0,2}$ is true for $i_0 = 2$ or 3. From Property 3.4, we know that $\beta_{i_0,2} + \gamma_{i_0,1} + \gamma_{i_0,2} = k$. Thus, we have

$$\gamma_{i_0,1} \leq 2(\beta_{i_0,2} + \gamma_{i_0,2}). \tag{3.14}$$

Therefore, the rate of GCODs

$$R = \frac{k}{p} \leq \frac{k}{k + \beta_{i_0,2} + \gamma_{i_0,2}} = \frac{\gamma_{i_0,1} + (\beta_{i_0,2} + \gamma_{i_0,2})}{\gamma_{i_0,1} + 2(\beta_{i_0,2} + \gamma_{i_0,2})} \leq \frac{3}{4},$$

in which the first inequality follows by Property 3.4 that $p \geq k + \beta_{i_0,2} + \gamma_{i_0,2}$, and the second inequality follows by (3.14).

The proof of (i) is covered in the above arguments. For (ii) and (iii), we need to investigate the property of $\gamma_{i_0,1}$ more precisely. When $k = 3l - 1$, $l = 1, 2, 3, \ldots$, we want to prove $R \leq (3l - 1)/(4l)$. Sufficiently, we will prove

$$\frac{k}{k + \beta_{i_0,2} + \gamma_{i_0,2}} \leq \frac{3l - 1}{4l},$$

which is equivalent to $\gamma_{i_0,1} \leq 2l - 2$. From (3.14), we have $\gamma_{i_0,1} \leq \frac{2}{3}k = 2l - \frac{2}{3}$. According to Property 3.3, $F_{i_0,1} + F_{i_0,1}^T = \mathbf{0}_{k\times k}$. Since $F_{i_0,1}$ is monomial, so $\gamma_{i_0,1}$, the rank of $F_{i_0,1}$, is even. Thus $\gamma_{i_0,1} \leq 2l - 2$, which is what we need.

When $k = 3l - 2$, $l = 1, 2, 3, \ldots$, we want to prove $R \leq (3l - 2)/(4l - 1)$. Sufficiently, we will prove

$$\frac{k}{k + \beta_{i_0,2} + \gamma_{i_0,2}} \leq \frac{3l - 2}{4l - 1},$$

which is equivalent to $\gamma_{i_0,1} \leq 2l - 3$. From (3.14), we have $\gamma_{i_0,1} \leq \frac{2}{3}k = 2l - \frac{4}{3}$. It implies that $\gamma_{i_0,1} \leq 2l - 2$. When $\gamma_{i_0,1} \neq 2l - 2$, then $\gamma_{i_0,1} \leq 2l - 3$, which is what we need. When $\gamma_{i_0,1} = 2l - 2$, we prove $R \leq (3l - 2)/(4l - 1)$ directly in Appendix C.

For $n = 3$ and 4, it is easy to obtain GCODs with the upper bounds. We illustrate only for $n = 4$. When $k = 3l$, $l = 1, 2, 3, \ldots,$

$$G(x_1, x_2, \ldots, x_{3l}) = \begin{bmatrix} G_4(x_1, x_2, x_3) \\ G_4(x_4, x_5, x_6) \\ \vdots \\ G_4(x_{3l-2}, x_{3l-1}, x_{3l}) \end{bmatrix}_{4l\times 4}.$$

When $k = 3l - 1$, $l = 1, 2, 3, \ldots$,

$$G(x_1, x_2, \ldots, x_{3l-1}) = \begin{bmatrix} G_4(x_1, x_2, x_3) \\ G_4(x_4, x_5, x_6) \\ \vdots \\ G_4(x_{3l-5}, x_{3l-4}, x_{3l-3}) \\ G_4(x_{3l-2}, x_{3l-1}, 0) \end{bmatrix}_{4l\times 4}.$$

When $k = 3l - 2$, $l = 1, 2, 3, \ldots$,

$$G(x_1, x_2, \ldots, x_{3l-2}) = \begin{bmatrix} G_4(x_1, x_2, x_3) \\ G_4(x_4, x_5, x_6) \\ \vdots \\ G_4(x_{3l-5}, x_{3l-4}, x_{3l-3}) \\ x_{3l-2} I_3 \end{bmatrix}_{(4l-1)\times 4}.$$

□

The result presented in Theorem 3.2 implies that the two GCOD designs in (3.1) presented in [9–11] have already achieved the maximum achievable rate for three and four transmit antennas.

4. Generalized Complex Orthogonal Designs with Linear Processing (GCODs with Linear Processing)

In this section, we discuss GCODs with linear processing.

DEFINITION 4.1. *A generalized complex orthogonal design with linear processing (GCOD with linear processing for short) in variables $x_1, x_2, \ldots, x_k$ is a $p \times n$ matrix $\mathcal{G}$ such that:*

(i) *The entries of $\mathcal{G}$ are complex linear combinations of $x_1, x_1^*, x_2, x_2^*, \ldots, x_k, x_k^*$;*
(ii) *$\mathcal{G}^{\mathcal{H}}\mathcal{G} = D$, where D is an $n \times n$ diagonal matrix with the (i, i)th diagonal element of the form*

$$l_{i,1}|x_1|^2 + l_{i,2}|x_2|^2 + \cdots + l_{i,k}|x_k|^2,$$

where all the coefficients $l_{i,1}, l_{i,2}, \ldots, l_{i,k}$ are strictly positive numbers.

The rate of $\mathcal{G}$ is defined as $R = k/p$.

It is not difficult to see that, for a GCOD G without linear processing in Definition 3.1, each variable x_i appears and only appears once in each column of G, which is, however, different from that for a GCOD with linear processing $\mathcal{G}$, in which each variable x_i can appear multiple times in a column of $\mathcal{G}$.

It has been proved in [6, 7] that if there exists a $p \times n$ GCOD with linear processing in variables $x_1, x_2, \ldots, x_k$ such that

$$l_{i,1} = l_{i,2} = \cdots = l_{i,k} \tag{4.1}$$

for each i, then there exists a GCOD Θ with linear processing in the same variables and of the same size such that

$$\Theta^{\mathcal{H}}\Theta = (|x_1|^2 + |x_2|^2 + \cdots + |x_k|^2)I_n .$$

Clearly, space-time block codes do not need the constraint of $l_{i,1} = l_{i,2} = \cdots = l_{i,k}$ for each i. Any GCOD with linear processing from Definition 4.1 can also provide the advantages of the fast ML decoding and full diversity. The diagonal form of D guarantees the fast ML decoding, since the orthogonal columns of $\mathcal{G}$ can separate the transmitted symbols $x_1, x_2, \cdots, x_k$ from each other at the decoder. And the strictly positive coefficients $l_{i,1}, l_{i,2}, \cdots, l_{i,k}$ imply the full rank of $\mathcal{G}$ as we explained at the end of Section 2. This guarantees the full diversity advantage of coding. For more details about the coding scheme, we refer the reader to [6].

For a fixed n, it is desired to have the rate of GCODs with linear processing as large as possible. Tarokh, Jafarkhani and Calderbank mentioned in [6] that this rate cannot exceed 1. From Liang and Xia's result in [16], we know that the rate of GCODs with linear processing must be strictly less than 1 for $n \geq 3$.

4.1. GCODs with Linear Processing of Square Size

Assume that $\mathcal{G}$ be a GCOD with linear processing of square size $n \times n$ in variables $x_1, x_2, \ldots, x_k$, and denote $D_j = diag(l_{1,j},\ l_{2,j},\ \ldots,\ l_{n,j})$ for each $j\ (1 \le j \le k)$, then from Definition 4.1 (ii) we have

$$\mathcal{G}^{\mathcal{H}}\mathcal{G} = D_1|x_1|^2 + D_2|x_2|^2 + \cdots + D_k|x_k|^2. \tag{4.2}$$

We rewrite $\mathcal{G}$ as

$$\mathcal{G} = \sum_{i=1}^{k} \left[Re(x_i)A_i + Im(x_i)B_i\right], \tag{4.3}$$

where A_i and B_i are $n \times n$ complex matrices, $Re(x_i)$ and $Im(x_i)$ are the real and imaginary parts of x_i respectively. Substituting (4.3) into (4.2) and comparing two sides of the resulting equation, we have

$$A_i^{\mathcal{H}}A_i = D_i, \quad B_i^{\mathcal{H}}B_i = D_i, \quad 1 \le i \le k;$$

$$A_i^{\mathcal{H}}A_j = -A_j^{\mathcal{H}}A_i, \quad B_i^{\mathcal{H}}B_j = -B_j^{\mathcal{H}}B_i, \quad 1 \le i \ne j \le k;$$

$$A_i^{\mathcal{H}}B_j = -B_j^{\mathcal{H}}A_i, \quad 1 \le i, j \le k.$$

Let $A_{k+i} = B_i$ and $D_{k+i} = D_i$, $i = 1, 2, \ldots, k$, then we have

$$\begin{aligned} A_i^{\mathcal{H}}A_i &= D_i, \quad 1 \le i \le 2k; \\ A_i^{\mathcal{H}}A_j &= -A_j^{\mathcal{H}}A_i, \quad 1 \le i \ne j \le 2k. \end{aligned}$$

Furthermore, let $C_i = D_1^{-1/2}A_1^{\mathcal{H}}A_iD_1^{-1/2}$, $i = 1, 2, \ldots, 2k$, then $C_1 = I_n$, and

$$\begin{aligned} C_i^2 &= -D_1^{-1}D_i, \quad 2 \le i \le 2k; \\ C_iC_j &= -C_jC_i, \quad 2 \le i \ne j \le 2k. \end{aligned}$$

Therefore, if $D_1 = D_2 = \cdots = D_k$, then $\{C_2, C_3, \ldots, C_{2k}\}$ is a Hurwitz family of $2k - 1$ complex matrices of order n. According to the Hurwitz theorem, we know that the number of complex matrices $2k - 1$ cannot be greater than $H(n) - 1$, i.e., $2k \le H(n)$. Similar to Corollary 3.1, we have the following result. Notice that the condition $D_1 = D_2 = \cdots = D_k$ is equivalent to $l_{i,1} = l_{i,2} = \cdots = l_{i,k}$ for each $i\ (1 \le i \le n)$.[6]

COROLLARY 4.1. *If $n = 2^a \cdot b$, b odd, and $\mathcal{G}$ be a GCOD with linear processing of square size $n \times n$ satisfying $l_{i,1} = l_{i,2} = \cdots = l_{i,k}$ for each $i\ (1 \le i \le n)$, then the rate of $\mathcal{G}$*

$$R \le (a + 1)/(2^a \cdot b),$$

and the bound can be achieved, which is the same as the one for GCODs without linear processing of square size in Definition 3.1.

[6] Corollary 4.1 still holds if the condition $l_{i,1} = l_{i,2} = \cdots = l_{i,k}$ for each $i\ (1 \le i \le n)$ is relaxed as $d_1D_1 = d_2D_2 = \cdots = d_kD_k$ for some positive constants $d_1,\ d_2,\ \ldots,\ d_k$.

Clearly, the maximum rate can be achieved by the GCODs of square size in Section 3.1. Thus, for this situation, relaxing the definition of GCODs to the definition of GCODs with linear processing fails to provide a higher rate. However, it is unclear whether this conclusion is true or not if there are no positive constants $d_1, d_2, \ldots, d_k$ such that $d_1 D_1 = d_2 D_2 = \cdots = d_k D_k$.

4.2. GCODs with Linear Processing of Non-Square Size

GCODs with linear processing of non-square size have not been well understood by now. In this subsection, we review some existing designs to illustrate the difficulty of this problem.

Tarokh, Jafarkhani and Calderbank in [6] presented a general design with rate 1/2 for any number of transmit antennas as follows. Assume $L_n(a_1, a_2, \ldots, a_k)$ be a *generalized real orthogonal design* [6] in variables $a_1, a_2, \ldots, a_k$ with rate 1 and of size $k \times n$. Let

$$\mathcal{G}_n(x_1, x_2, \ldots, x_k) = \begin{bmatrix} L_n(x_1, x_2, \ldots, x_k) \\ L_n(x_1^*, x_2^*, \ldots, x_k^*) \end{bmatrix}, \tag{4.4}$$

where $L_n(x_1, x_2, \ldots, x_k)$ and $L_n(x_1^*, x_2^*, \ldots, x_k^*)$ are $k \times n$ matrices constructed by replacing the symbols $a_1, a_2, \ldots, a_k$ everywhere in $L_n(a_1, a_2, \ldots, a_k)$ by $x_1, x_2, \ldots, x_k$ and $x_1^*, x_2^*, \ldots, x_k^*$ respectively. Clearly, the size of $\mathcal{G}_n$ in (4.4) is $2k \times n$, and the rate of $\mathcal{G}_n$ is 1/2. For example, for four transmit antennas, the design is given by the following 8×4 matrix

$$\mathcal{G}_4(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \\ x_1^* & x_2^* & x_3^* & x_4^* \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ -x_3^* & x_4^* & x_1^* & -x_2^* \\ -x_4^* & -x_3^* & x_2^* & x_1^* \end{bmatrix}_{8\times 4}.$$

Later in [15], two designs of GCODs with linear processing of rates higher than 1/2 were presented for five and six transmit antennas. For five transmit antennas, the design is an 11×5 matrix given by

$$\mathcal{G}_5(x_1, x_2, \ldots, x_7) = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & x_4 \\ -x_2^* & x_1^* & 0 & x_3 & x_5 \\ x_3^* & 0 & -x_1^* & x_2 & x_6 \\ 0 & x_3^* & -x_2^* & -x_1 & x_7 \\ x_4^* & 0 & 0 & -x_7^* & -x_1^* \\ 0 & x_4^* & 0 & x_6^* & -x_2^* \\ 0 & 0 & x_4^* & x_5^* & -x_3^* \\ 0 & -x_5^* & x_6^* & 0 & x_1 \\ x_5^* & 0 & x_7^* & 0 & x_2 \\ -x_6^* & -x_7^* & 0 & 0 & x_3 \\ x_7 & -x_6 & -x_5 & x_4 & 0 \end{bmatrix}. \tag{4.5}$$

Actually, $\mathcal{G}_5$ is constructed from $G_4(x_1, x_2, x_3)$ in (3.1) as follows. At first, $G_4(x_1, x_2, x_3)$ is considered as a 4×4 sub-matrix of $\mathcal{G}_5$, and then symbols x_4, x_5, x_6, x_7 are added into the fifth column of $\mathcal{G}_5$. Finally, the entries of $\mathcal{G}_5$ from the fifth row to the end are arranged such that all of the 5 columns are orthogonal to each other and the number of the total rows should be as small as possible. From the resulting matrix in (4.5), we can check that $\mathcal{G}_5^{\mathcal{H}}\mathcal{G}_5 = D$, where D is a 5×5 diagonal matrix with the (i, i)th diagonal element $D(i, i)$ of the form

$$D(1,1) = D(2,2) = D(3,3) = D(4,4) = \sum_{m=1}^{7} |x_m|^2$$

and

$$D(5,5) = 2\sum_{m=1}^{3} |x_m|^2 + \sum_{m=4}^{7} |x_m|^2.$$

The rate of $\mathcal{G}_5$ in (4.5) is $R = 7/11 = 0.6364$. Note that the diagonal matrix D here does not satisfy the condition in (4.1).

For six transmit antennas, the design is a 30×6 matrix given by

$$\mathcal{G}_6(x_1, x_2, \ldots, x_{18}) = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & x_4 & x_8 \\ -x_2^* & x_1^* & 0 & x_3 & x_5 & x_9 \\ x_3^* & 0 & -x_1^* & x_2 & x_6 & x_{10} \\ 0 & x_3^* & -x_2^* & -x_1 & x_7 & x_{11} \\ x_4^* & 0 & 0 & -x_7^* & -x_1^* & x_{12} \\ 0 & x_4^* & 0 & x_6^* & -x_2^* & x_{13} \\ 0 & 0 & x_4^* & x_5^* & -x_3^* & x_{14} \\ 0 & x_5^* & -x_6^* & 0 & -x_1 & x_{15} \\ x_5^* & 0 & x_7^* & 0 & x_2 & x_{16} \\ x_6^* & x_7^* & 0 & 0 & -x_3 & x_{17} \\ x_7 & -x_6 & -x_5 & x_4 & 0 & x_{18} \\ x_8^* & 0 & 0 & -x_{11}^* & -x_{15}^* & -x_1^* \\ 0 & x_8^* & 0 & x_{10}^* & x_{16}^* & -x_2^* \\ 0 & 0 & x_8^* & x_9^* & -x_{17}^* & -x_3^* \\ 0 & 0 & 0 & x_{18}^* & x_8^* & -x_4^* \\ 0 & 0 & -x_{18}^* & 0 & x_9^* & -x_5^* \\ 0 & -x_{18}^* & 0 & 0 & x_{10}^* & -x_6^* \\ x_{18}^* & 0 & 0 & 0 & x_{11}^* & -x_7^* \\ 0 & -x_9^* & x_{10}^* & 0 & x_{12}^* & x_1 \\ x_9^* & 0 & x_{11}^* & 0 & x_{13}^* & x_2 \\ -x_{10}^* & -x_{11}^* & 0 & 0 & x_{14}^* & x_3 \\ -x_{12}^* & -x_{13}^* & -x_{14}^* & 0 & 0 & x_4 \\ -x_{16}^* & -x_{15}^* & 0 & -x_{14}^* & 0 & x_5 \\ -x_{17}^* & 0 & x_{15}^* & -x_{13}^* & 0 & x_6 \\ 0 & -x_{17}^* & -x_{16}^* & x_{12}^* & 0 & x_7 \\ 0 & x_{14} & -x_{13} & -x_{15} & x_{11} & 0 \\ x_{14} & 0 & -x_{12} & -x_{16} & x_{10} & 0 \\ -x_{13} & x_{12} & 0 & x_{17} & x_9 & 0 \\ x_{15} & -x_{16} & x_{17} & 0 & x_8 & 0 \\ -x_{11} & x_{10} & x_9 & -x_8 & x_{18} & 0 \end{bmatrix}. \quad (4.6)$$

Similarly, $\mathcal{G}_6$ is constructed form $\mathcal{G}_5$ in (4.5) as follows. At first, $\mathcal{G}_5$ is considered as an 11×5 sub-matrix of $\mathcal{G}_6$, and then symbols $x_8, x_9, \ldots, x_{18}$ are added into the sixth column of $\mathcal{G}_6$. Finally, the entries of $\mathcal{G}_6$ from the twelfth row to the end are arranged such that all of the 6 columns of $\mathcal{G}_6$ are orthogonal to each other and the number of the total rows should be as small as possible. The resulting matrix in (4.6) is of size 30×6. By a tedious check, we have $\mathcal{G}_6^{\mathcal{H}} \mathcal{G}_6 = D$, where D is a 6×6 diagonal matrix with the (i, i)th diagonal element $D(i, i)$ of the form

$$D(1,1) = D(2,2) = D(3,3) = D(4,4) = \sum_{m=1}^{18} |x_m|^2$$

and

$$\begin{aligned} D(5,5) &= \sum_{m=1}^{18} |x_m|^2 + \sum_{m=1}^{3} |x_m|^2 + \sum_{m=8}^{11} |x_m|^2, \\ D(6,6) &= 2\sum_{m=1}^{7} |x_m|^2 + \sum_{m=8}^{18} |x_m|^2. \end{aligned}$$

Clearly, the rate of $\mathcal{G}_6$ in (4.6) is $R = 18/30 = 0.6$.

The same procedure may be used to construct GCODs for other numbers of transmit antennas. However, it is hard to obtain other designs with rate greater than $1/2$. For example, $\mathcal{G}_6$ in (4.6) may be used to construct $\mathcal{G}_7$ for seven transmit antennas as follow. Similarly, we may keep $\mathcal{G}_6$ as a 30×6 sub-matrix of $\mathcal{G}_7$ and add symbols $x_{19}, x_{20}, \ldots, x_{48}$ into the seventh column of $\mathcal{G}_7$. However, it is hard to arrange the entries of $\mathcal{G}_7$ from the thirty-first row to the end such that all of the seven columns are orthogonal to each other and the number of the total rows should be as small as possible. Notice that in this case, the symbols $x_1, x_2, \ldots, x_{18}$ and their complex conjugates $x_1^*, x_2^*, \ldots, x_{18}^*$ should appear in the seventh column of $\mathcal{G}_7$. Therefore, the number of rows in $\mathcal{G}_7$ will be at least $30 + 18 + 18 = 66$.

To our best knowledge, there are no other known designs with rate higher than $1/2$ for more than four transmit antennas by now except for the two designs in (4.5) and (4.6) of rates $7/11$ and $3/5$, respectively. It is interesting to note that there are no known GCOD designs without linear processing of rates more than $1/2$ for five or more transmit antennas. The rate $3/4$ GCOD designs in (3.1) presented in [9–11] are the only known GCOD designs without linear processing of rates above $1/2$ for more than 2 transmit antennas. As we mentioned earlier, the result presented in Theorem 3.2 in this paper tells us that these two GCOD designs have already achieved the maximum rate in all GCODs without linear processing for more than 2 transmit antennas.

5. Conclusion and Some Comments

Orthogonal designs have a long history in mathematics literature, which have been mainly motivated from the compositions of quadratic forms [24, 25]. Recently, orthogonal designs have attracted considerable attention in space-time coding due to their special structure. Real and complex orthogonal designs are used to construct space-time block codes for PAM and PSK/QAM signals, respectively. As the real orthogonal designs are well understood, the complex orthogonal designs are more difficult to deal with but can provide high transmission rates since they permit complex signal constellations. In this paper, at first we provided a tutorial review of space-time block codes from complex orthogonal designs, in particular, the Hurwitz theorem on complex orthogonal designs [18, 19, 23, 25] and its realizations [21, 10]. We then presented a simple and intuitive interpretation of the realization. For GCODs of square size, the designs of the maximum rates have been known from the Hurwitz theorem [18, 19, 23, 25, 10] or amicable design theory [22, 24, 9]. For GCODs without linear processing of non-square size, we proved that the maximum rate cannot be greater than $3/4$ for more than two transmit antennas. Recently, Wang and Xia in [17] showed that this upper bound, i.e., $3/4$, still holds for some GCODs with linear processing and also provided an upper bound ($4/5$) on the rates of GCODs with linear processing for more than two transmit antennas.

What we have known about GCODs with or without linear processing is only a tip of the iceberg as pointed out in [6]. There are many interesting and important problems unsolved. We list a few of them below.

The first open problem is: what is the maximum rate of a GCOD with or without linear processing for a given number (>2) of transmit antennas, and if it is known, then how to achieve it, i.e., how to construct a GCOD with or without linear processing with the maximum rate?

From Corollaries 3.1 and 4.1, for some cases of designs with square size, relaxing the definition of GCODs to the definition of GCODs with linear processing fails to provide higher rate. Thus, the second open problem is: is there any difference of the maximum rates for GCODs with or without linear processing? In other words, is there any GCOD with linear processing that has a rate higher than the maximum rate of GCODs without linear processing for the same number of transmit antennas?

Another open problem is to construct GCODs with or without linear processing of rates higher than $1/2$ for more than six transmit antennas.

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Appendix A

CLAIM 1. $G_{2^r}^{\mathcal{H}}(x_1, x_2, \ldots, x_{r+1})G_{2^r}(x_1, x_2, \ldots, x_{r+1}) = (|x_1|^2 + |x_2|^2 + \cdots + |x_{r+1}|^2)I_{2^r}$, $r \geq 0$.

Proof. For simplicity, we omit the variables. Since for any $r \geq 1$,

$$G_{2^r}^{\mathcal{H}} G_{2^r} = \begin{bmatrix} G_{2^{r-1}}^{\mathcal{H}} G_{2^{r-1}} + |x_{r+1}|^2 & \mathbf{0} \\ \mathbf{0} & G_{2^{r-1}} G_{2^{r-1}}^{\mathcal{H}} + |x_{r+1}|^2 \end{bmatrix},$$

$$G_{2^r} G_{2^r}^{\mathcal{H}} = \begin{bmatrix} G_{2^{r-1}} G_{2^{r-1}}^{\mathcal{H}} + |x_{r+1}|^2 & \mathbf{0} \\ \mathbf{0} & G_{2^{r-1}}^{\mathcal{H}} G_{2^{r-1}} + |x_{r+1}|^2 \end{bmatrix},$$

and $G_1^{\mathcal{H}} G_1 = |x_1|^2 I_1 = G_1 G_1^{\mathcal{H}}$, so by induction on r, we have

$$G_{2^r}^{\mathcal{H}} G_{2^r} = G_{2^r} G_{2^r}^{\mathcal{H}} = (|x_1|^2 + |x_2|^2 + \cdots + |x_{r+1}|^2)I_{2^r}, \quad r \geq 0,$$

which is Claim 1. □

Appendix B

Proof of Property 3.2. (i) Since the entries of G are $0, \pm x_1, \pm x_2, \ldots, \pm x_k$, or their conjugates $\pm x_1^*, \pm x_2^*, \ldots, \pm x_k^*$, or multiples of them by $\mathbf{i}$, so there is at most one non-zero element per row of E_i, F_i and $E_i + F_i$, and the modulus of non-zero elements is 1. If there are at least two non-zero elements in k_0th column of E_i, then the (k_0, k_0)th element of $E_i^{\mathcal{H}} E_i$ is greater than 1. It follows that the (k_0, k_0)th element of $E_i^{\mathcal{H}} E_i + (F_i^{\mathcal{H}} F_i)^T$ is greater than 1, which is contradictory to (3.8) that $E_i^{\mathcal{H}} E_i + F_i^T \overline{F}_i = I_k$. Thus, there is at most one non-zero element per column of E_i. According to the definition, E_i is monomial. Similarly, we can prove that F_i is monomial.

To prove $E_i + F_i$ is monomial, sufficiently we need only to prove (ii) that the pair of E_i and F_i is disjoint. Suppose that both the k_0th columns of E_i and F_i are non-zero, then both the (k_0, k_0)th elements of $E_i^{\mathcal{H}} E_i$ and $F_i^{\mathcal{H}} F_i$ are greater than or equal to 1. It follows that the (k_0, k_0)th element of $E_i^{\mathcal{H}} E_i + \left(F_i^{\mathcal{H}} F_i\right)^T$ is greater than 1, which is contradictory to (3.8) that $E_i^{\mathcal{H}} E_i + F_i^T \overline{F}_i = I_k$. Thus, the columns in E_i and F_i cannot be non-zero at the same time. On the other hand, we know that there is at most one non-zero element per row of $E_i + F_i$. So the rows in E_i and F_i can not be non-zero at the same time. According to the definition, the pair of E_i and F_i is disjoint.

(iii) Assume that the k_1th column of E_i is $[e_{1k_1}\ e_{2k_1}\ \cdots\ e_{pk_1}]^T$, and the k_2th column of F_i is $[f_{1k_2}\ f_{2k_2}\ \cdots\ f_{pk_2}]^T$, then the (k_1, k_2)th element of $E_i^{\mathcal{H}} F_i$ is $\sum_{j=1}^{p} e_{jk_1}^* f_{jk_2}$. From (ii), the pair of E_i and F_i is disjoint, so if e_{jk_1} is non-zero then f_{jk_2} must be zero. Thus $\sum_{j=1}^{p} e_{jk_1}^* f_{jk_2} = 0$ for arbitrary k_1 and k_2. It follows that $E_i^{\mathcal{H}} F_i = \mathbf{0}_{k \times k}$. Similarly, we can prove that $E_i F_i^{\mathcal{H}} = \mathbf{0}_{p \times p}$.

Since $E_i + F_i$, E_i and F_i are monomial, and the pair of E_i and F_i is disjoint, so we have (iv) immediately. □

Appendix C

CLAIM 2. *If $k = 3l - 2$, $l = 1, 2, 3, \ldots$, and $\gamma_{i_0,1} = 2l - 2$, then $R \leq (3l - 2)/(4l - 1)$.*

Proof. Without loss of generality, we assume $i_0 = 2$, i.e., $\gamma_{2,1} = 2l - 2$. We further assume that $\gamma_{3,1} \geq 2l - 2$. Otherwise, if $\gamma_{3,1} \leq 2l - 3$, then we have

$$R = \frac{k}{p} \leq \frac{k}{k + \beta_{3,2} + \gamma_{3,2}} = \frac{k}{2k - \gamma_{3,1}} \leq \frac{3l - 2}{4l - 1},$$

which is the result in Claim 2.

The proof is divided into six steps. For convenience of description, let's say the jth row (column) is a *common zero row (column)* of F_{21} and F_{31} if the jth row (column) of F_{21} is zero while the jth row (column) of F_{31} is also zero. Similarly, we say the jth row (column) is a *common non-zero row (column)* of F_{21} and F_{31} if the jth row (column) of F_{21} is non-zero while the jth row (column) of F_{31} is also non-zero. The ith and jth rows of a monomial matrix are said to be *a pair of relative rows* if both the (i, j)th and (j, i)th elements of the matrix are non-zero. Since F_2 is monomial, so is F_{21}. From Property 3.3, $F_{21}^T = -F_{21}$. We know that there are totally $l - 1$ pairs of relative rows in F_{21} since $\gamma_{2,1}$, the rank of F_{21}, is $2l - 2$.

Step 1. At first, we prove that there are at most $l-1$ common non-zero rows (columns) in F_{21} and F_{31}.

Suppose that there exist s ($s \geq l$) common non-zero rows in F_{21} and F_{31}. Since all $2l-2$ non-zero rows in F_{21} consist in $l-1$ pairs of relative rows, so there is a pair of relative rows in these s rows in F_{21}. Denote that this pair of relative rows is located at the k_1th and k_2th rows, then both the (k_1, k_2)th and (k_2, k_1)th elements of F_{21} are non-zero. From the assumption, both the k_1th and k_2th rows of F_{31} are non-zero. Denote further that the (k_1, k_3)th element in the k_1th row of F_{31} is non-zero. Since both the (k_1, k_3)th element of F_{31} and the (k_1, k_2)th element of F_{21} are non-zero, and both F_{31} and F_{21} are monomial, so the (k_3, k_2)th element of $F_{31}^T \overline{F}_{21}$ is non-zero.

On the other hand, since $F_{31}^T = -F_{31}$ from Property 3.3, and the k_2th row of F_{31} is non-zero, so the k_2th column of F_{31} is non-zero. Then the k_2th column of E_{32} is zero since $E_3 + F_3$ is monomial. It follows that the (k_3, k_2)th element of $E_{22}^{\mathcal{H}} E_{32}$ is zero. Similarly, since the (k_1, k_2)th element of F_{21} is non-zero, we know that the k_2th column of F_{22} is zero because F_2 is monomial. It implies that the (k_3, k_2)th element of $F_{32}^T \overline{F}_{22}$ is zero.

From the above discussion, we know that both the (k_3, k_2)th elements of $E_{22}^{\mathcal{H}} E_{32}$ and $F_{32}^T \overline{F}_{22}$ are zero while the (k_3, k_2)th element of $F_{31}^T \overline{F}_{21}$ is non-zero. Thus, the (k_3, k_2)th element of $E_{22}^{\mathcal{H}} E_{32} + F_{31}^T \overline{F}_{21} + F_{32}^T \overline{F}_{22}$ is non-zero, which is contradictory to (3.11) that $E_{22}^{\mathcal{H}} E_{32} + F_{31}^T \overline{F}_{21} + F_{32}^T \overline{F}_{22} = \mathbf{0}_{k \times k}$.

Thus we conclude that there are at most $l-1$ common non-zero rows in F_{21} and F_{31}. Similarly, we can prove that there are at most $l-1$ common non-zero columns in F_{21} and F_{31}.

Step 2. Denote N_0 be the number of common zero columns in F_{21} and F_{31}. In this step, we will prove that $N_0 \leq 1$ and $\gamma_{3,1} = 2l-2$.

Let $\tilde{N}_0$ be the number of common non-zero columns in F_{21} and F_{31}. From Step 1, we know that the number of common non-zero columns in F_{21} and F_{31} cannot be greater than $l-1$, i.e., $\tilde{N}_0 \leq l-1$. Since both F_{21} and F_{31} are $k \times k$ monomial matrices, so the number of common zero columns in F_{21} and F_{31} satisfies

$$N_0 = k - (\gamma_{2,1} + \gamma_{3,1} - \tilde{N}_0) \leq 1,$$

in which $k = 3l-2$, $\gamma_{2,1} = 2l-2$ and $\gamma_{3,1} \leq 2l-2$ from the assumption. Moreover,

$$\gamma_{2,1} + \gamma_{3,1} - \tilde{N}_0 \leq k,$$

so we obtain another constraint on $\gamma_{3,1}$, i.e.,

$$\gamma_{3,1} \leq k + \tilde{N}_0 - \gamma_{2,1} \leq 2l-1.$$

Therefore, we have $2l-2 \leq \gamma_{3,1} \leq 2l-1$. Since $\gamma_{3,1}$ must be even, so $\gamma_{3,1} = 2l-2$.

Step 3. We prove $rank(F_{31}^T \overline{F}_{21}) \leq l-1$ in this step.

From Step 1, we know that there exist at most $l-1$ common non-zero rows in F_{21} and F_{31}. Since there are totally $2l-2$ non-zero rows in F_{21}, so there exist at least $l-1$ non-zero rows in F_{21} such that the same rows in F_{31} are zero. Without loss of generality, we can assume that the first t ($t \geq l-1$) rows of F_{21} are non-zero while the first t rows of F_{31} are zero, since there exist elementary row permutation operation U such that $F_{31}^T \overline{F}_{21} = (UF_{31})^T U\overline{F}_{21}$ and the first t rows of UF_{21} are non-zero while the first t rows of $\overline{U}F_{31}$ are zero. We denote F_{21} and F_{31} as

$$F_{21} = \begin{bmatrix} W_{1_{t\times k}} \\ W_{2_{(k-t)\times k}} \end{bmatrix}, \quad F_{31} = \begin{bmatrix} \mathbf{0}_{t\times k} \\ W_{3_{(k-t)\times k}} \end{bmatrix},$$

where each row of W_1 is non-zero. So

$$F_{31}^T \overline{F}_{21} = \begin{bmatrix} \mathbf{0}_{k\times t} & W_3^T \end{bmatrix} \cdot \begin{bmatrix} \overline{W}_1 \\ \overline{W}_2 \end{bmatrix} = W_3^T \overline{W}_2 \,.$$

Thus we have

$$rank(F_{31}^T \overline{F}_{21}) \leq rank(\overline{W}_2) = \gamma_{2,1} - t \leq l-1.$$

Step 4. In this step, we show that $E_{22}^{\mathcal{H}} F_{32} = \mathbf{0}_{k\times k}$ and $E_{32}^{\mathcal{H}} F_{22} = \mathbf{0}_{k\times k}$.

Let $E_{22}^{\mathcal{H}} F_{32} = \{\omega_{i,j}\}_{k\times k}$. According to (3.10) in Property 3.1, we have $E_{22}^{\mathcal{H}} F_{32} = -\left(E_{22}^{\mathcal{H}} F_{32}\right)^T$. So $\omega_{i,i} = 0$ for any i ($1 \leq i \leq k$), and $\omega_{i,j} = -\omega_{j,i}$ for any $1 \leq i \neq j \leq k$. If $E_{22}^{\mathcal{H}} F_{32} \neq \mathbf{0}_{k\times k}$, then there exist i_0 and j_0 ($i_0 \neq j_0$) such that $\omega_{i_0,j_0} \neq 0$ and $\omega_{j_0,i_0} \neq 0$. It follows that the i_0th and j_0th columns of E_{22} are non-zero, while the i_0th and j_0th columns of F_{32} are also non-zero. Since both $E_2 + F_2$ and F_3 are monomial, so the i_0th and j_0th columns of F_{21} are zero, while the i_0th and j_0th columns of F_{31} are also zero. Thus there are at least two common zero columns in F_{21} and F_{31}. This is contradictory to the fact that the number of common zero columns in F_{21} and F_{31} can not be greater than 1, which is proved in Step 2. So we have $E_{22}^{\mathcal{H}} F_{32} = \mathbf{0}_{k\times k}$. For the same reason, we have $E_{32}^{\mathcal{H}} F_{22} = \mathbf{0}_{k\times k}$.

Step 5. From (3.11) we have $E_{22}^{\mathcal{H}} E_{32} + F_{31}^T \overline{F}_{21} + F_{32}^T \overline{F}_{22} = \mathbf{0}_{k\times k}$. Thus, we have

$$rank(F_{31}^T \overline{F}_{21}) = rank(E_{22}^{\mathcal{H}} E_{32} + F_{32}^T \overline{F}_{22}).$$

If the following equation is true, (we will show it in next step.)

$$rank(E_{22}^{\mathcal{H}} E_{32} + F_{32}^T \overline{F}_{22}) = rank(E_{22}^{\mathcal{H}} E_{32}) + rank(F_{32}^T \overline{F}_{22}), \tag{C.1}$$

then we have

$$\begin{aligned} rank(F_{31}^T \overline{F}_{21}) &= rank(E_{22}^{\mathcal{H}} E_{32}) + rank(F_{32}^T \overline{F}_{22}) \\ &= rank(E_{22}^{\mathcal{H}} E_{32}) + rank(F_{22}^{\mathcal{H}} F_{32}) \\ &\geq rank(E_{22}^{\mathcal{H}} E_{32} + F_{22}^{\mathcal{H}} F_{32}) \\ &= rank\left[(E_{22} + F_{22})^{\mathcal{H}} (E_{32} + F_{32})\right], \end{aligned} \tag{C.2}$$

in which the last equality follows by $E_{22}^{\mathcal{H}} F_{32} = \mathbf{0}_{k\times k}$ and $F_{22}^{\mathcal{H}} E_{32} = \mathbf{0}_{k\times k}$ from the result of Step 4. From Step 3, we know that $rank(F_{31}^T \overline{F}_{21}) \leq l-1$. So

$$\begin{aligned} l-1 &\geq rank\left[(E_{22} + F_{22})^{\mathcal{H}} (E_{32} + F_{32})\right] \\ &\geq rank(E_{22} + F_{22}) + rank(E_{32} + F_{32}) - (p-k) \\ &= (\beta_{2,2} + \gamma_{2,2}) + (\beta_{3,2} + \gamma_{3,2}) - (p-k), \end{aligned} \tag{C.3}$$

in which the last equality follows by $rank(E_i + F_i) = rank(E_i) + rank(F_i)$, and $E_i + F_i$, E_i and F_i are monomial from Property 3.2. Since $\gamma_{2,1} = 2l - 2$ and $k = 3l - 2$, so $\beta_{2,2} + \gamma_{2,2} = k - \gamma_{2,1} = l$. From the result in Step 2, we know $\gamma_{3,1} = 2l - 2$. Thus, $\beta_{3,2} + \gamma_{3,2} = k - \gamma_{3,1} = l$. Substituting $\beta_{2,2} + \gamma_{2,2} = l$ and $\beta_{3,2} + \gamma_{3,2} = l$ into (C.3), we have

$$p \geq k - (l - 1) + (\beta_{2,2} + \gamma_{2,2}) + (\beta_{3,2} + \gamma_{3,2}) = 4l - 1.$$

Therefore, the rate $R = k/p \leq (3l - 2)/(4l - 1)$. By now we know that if (C.1) is true, then we get the claim. We will prove (C.1) in next step.

Step 6. Finally, we want to show that

$$rank(E_{22}^{\mathcal{H}} E_{32} + F_{32}^T \overline{F}_{22}) = rank(E_{22}^{\mathcal{H}} E_{32}) + rank(F_{32}^T \overline{F}_{22}). \tag{C.4}$$

We observe that (C.4) is equivalent to

$$rank\left[(\overline{E}_{22}U)^T E_{32}U + (\overline{F}_{32}U)^T \overline{F}_{22}U\right] = rank\left[(\overline{E}_{22}U)^T E_{32}U\right] + rank\left[(\overline{F}_{32}U)^T \overline{F}_{22}U\right],$$

for any $k \times k$ elementary column permutation matrix U which is of full rank. Thus, applying column permutation operations on E_2, E_3, F_2 and F_3 at the same time does not effect the result of (C.4). Without loss of generality, we may assume that the first $2l - 2$ columns of F_{21} are non-zero. Since $E_2 + F_2$ is monomial, so the first $2l - 2$ columns of E_{22} are zeros, and the first $2l - 2$ columns of F_{22} are also zeros.

From Step 2, we know that the number of common zero columns in F_{21} and F_{31} cannot be greater than 1. If there is no common zero column in F_{21} and F_{31}, then the last l columns of F_{31} must be non-zero since the last l columns of F_{21} are zeros from the assumption. Because $E_3 + F_3$ is monomial, we know that both the last l columns of E_{32} and F_{32} are zeros. Thus, we have

$$E_{22}^{\mathcal{H}} E_{32} = \begin{bmatrix} \mathbf{0}_{(2l-2)\times(2l-2)} & \mathbf{0}_{(2l-2)\times l} \\ * & \mathbf{0}_{l\times l} \end{bmatrix}, \quad F_{32}^T \overline{F}_{22} = \begin{bmatrix} \mathbf{0}_{(2l-2)\times(2l-2)} & * \\ \mathbf{0}_{l\times(2l-2)} & \mathbf{0}_{l\times l} \end{bmatrix},$$

which implies (C.4). Therefore, (C.4) is true if there is no common zero column in F_{21} and F_{31}.

If there is one common zero column in F_{21} and F_{31}. We assume that the $(2l-1)$th column is the common zero column in F_{21} and F_{31}. Otherwise, we can obtain it by column permutations. Then the last $l - 1$ columns of F_{31} are non-zero since the last $l - 1$ columns of F_{21} are zeros. It follows that both the last $l - 1$ columns of E_{32} and F_{32} are zero since $E_3 + F_3$ is monomial. We prove (C.4) in the following four cases:

Case I. If both the $(2l - 1)$th columns of E_{22} and E_{32} are non-zero, then both the $(2l - 1)$th columns of F_{22} and F_{32} are zeros since both $E_2 + F_2$ and $E_3 + F_3$ are monomial. Thus, we have

$$E_{22}^{\mathcal{H}} E_{32} = \begin{bmatrix} \mathbf{0}_{(2l-2)\times(2l-1)} & \mathbf{0}_{(2l-2)\times(l-1)} \\ * & \mathbf{0}_{l\times(l-1)} \end{bmatrix}, \quad F_{32}^T \overline{F}_{22} = \begin{bmatrix} \mathbf{0}_{(2l-2)\times(2l-1)} & * \\ \mathbf{0}_{l\times(2l-1)} & \mathbf{0}_{l\times(l-1)} \end{bmatrix},$$

which implies (C.4).

Case II. If both the $(2l - 1)$th columns of E_{22} and E_{32} are zeros, then both the $(2l - 1)$th columns of F_{22} and F_{32} are non-zero. In this case, we have

$$E_{22}^{\mathcal{H}} E_{32} = \begin{bmatrix} \mathbf{0}_{(2l-1)\times(2l-2)} & \mathbf{0}_{(2l-1)\times l} \\ * & \mathbf{0}_{(l-1)\times l} \end{bmatrix}, \quad F_{32}^T \overline{F}_{22} = \begin{bmatrix} \mathbf{0}_{(2l-1)\times(2l-2)} & * \\ \mathbf{0}_{(l-1)\times(2l-2)} & \mathbf{0}_{(l-1)\times l} \end{bmatrix},$$

Thus (C.4) is true.

Case III. If the $(2l-1)$th column of E_{22} is non-zero and the $(2l-1)$th column of E_{32} is zero, then the $(2l-1)$th column of F_{22} is zero and the $(2l-1)$th column of F_{32} is non-zero. Now we have

$$E_{22}^{\mathcal{H}}E_{32} = \begin{bmatrix} \mathbf{0}_{(2l-2)\times(2l-2)} & \mathbf{0}_{(2l-2)\times l} \\ * & \mathbf{0}_{l\times l} \end{bmatrix}, \quad F_{32}^{T}\overline{F}_{22} = \begin{bmatrix} \mathbf{0}_{(2l-1)\times(2l-1)} & * \\ \mathbf{0}_{(l-1)\times(2l-1)} & \mathbf{0}_{(l-1)\times(l-1)} \end{bmatrix}.$$

We observe that if the $(2l-1)$th row of $E_{22}^{\mathcal{H}}E_{32} + F_{32}^{T}\overline{F}_{22}$ is zero, then (C.4) is true. From the assumption that the $(2l-1)$th column of F_{31} is zero, we know that the $(2l-1)$th row of $F_{31}^{T}\overline{F}_{21}$ is zero. Since $E_{22}^{\mathcal{H}}E_{32} + F_{32}^{T}\overline{F}_{22} = -F_{31}^{T}\overline{F}_{21}$ from (3.11), so the $(2l-1)$th row of $E_{22}^{\mathcal{H}}E_{32} + F_{32}^{T}\overline{F}_{22}$ is zero.

Case IV. If the $(2l-1)$th column of E_{22} is zero and the $(2l-1)$th column of E_{32} is non-zero, then the $(2l-1)$th column of F_{22} is non-zero and the $(2l-1)$th column of F_{32} is zero. With this situation, we have

$$E_{22}^{\mathcal{H}}E_{32} = \begin{bmatrix} \mathbf{0}_{(2l-1)\times(2l-1)} & \mathbf{0}_{(2l-1)\times(l-1)} \\ * & \mathbf{0}_{(l-1)\times(l-1)} \end{bmatrix}, \quad F_{32}^{T}\overline{F}_{22} = \begin{bmatrix} \mathbf{0}_{(2l-2)\times(2l-2)} & * \\ \mathbf{0}_{l\times(2l-2)} & \mathbf{0}_{l\times l} \end{bmatrix}.$$

We can see that if the $(2l-1)$th column of $E_{22}^{\mathcal{H}}E_{32} + F_{32}^{T}\overline{F}_{22}$ is zero, then (C.4) is true. From the assumption that the $(2l-1)$th column of F_{21} is zero, we know that the $(2l-1)$th column of $F_{31}^{T}\overline{F}_{21}$ is zero. According to (3.11) again, $E_{22}^{\mathcal{H}}E_{32} + F_{32}^{T}\overline{F}_{22} = -F_{31}^{T}\overline{F}_{21}$, we know that the $(2l-1)$th column of $E_{22}^{\mathcal{H}}E_{32} + F_{32}^{T}\overline{F}_{22}$ is actually zero.

Therefore, (C.4) is also true if there is one common zero column in F_{21} and F_{31}. From the above six steps, we have proved Claim 2 completely. □

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