A New Prefilter Design for Discrete Multiwavelet Transforms

Xiang-Gen Xia, *Member, IEEE*

***Abstract*— In conventional wavelet transforms, prefiltering is not necessary due to the lowpass property of a scaling function. This is no longer true for multiwavelet transforms. A few research papers on the design of prefilters have appeared recently, but the existing prefilters are usually not orthogonal, which often causes problems in coding. Moreover, the condition on the prefilters was imposed based on the first-step discrete multiwavelet decomposition. In this paper, we propose a new prefilter design that combines the ideas of the conventional wavelet transforms and multiwavelet transforms. The prefilters are orthogonal but nonmaximally decimated. They are derived from a very natural calculation of multiwavelet transform coefficients. In this new prefilter design, multiple step discrete multiwavelet decomposition is taken into account. Our numerical examples (by taking care of the redundant prefiltering) indicate that the energy compaction ratio with the Geronimo–Hardin–Massopust 2 wavelet transform and our new prefiltering is better than the one with Daubechies D_4 wavelet transform.**

I. Introduction

NOW THAT single wavelet transforms are well-understood, multiwavelets recently have attracted much attention in the research community; see, for example, [1]–[20], [26]–[32], where several wavelet functions and scaling functions are used to expand a signal. The multiwavelet functions constructed by Geronimo *et al.* [2]–[4] have more desired properties than any single wavelet function, such as short support, symmetry, and smoothness. Although, in theory, they look more attractive than single wavelets, not much more advantages in practical applications over single wavelets have been found so far. In this author's opinion, the main reason behind this fact might be because of their improper discrete implementations. For single wavelet transforms, the discrete implementation automatically follows from their multiresolution structure, i.e., tree-structured two-channel filterbanks. In the tree-structured filterbank, lowpass and highpass filters are explicitly used, which is tight with the lowpass and the bandpass properties of the scaling and wavelet functions, respectively. Although, for multiwavelet transforms, the discrete implementation also follows from their multiresolution structure, the tree-structured filterbank becomes a tree-structured vector filterbank [1], [8] (or time-variant filterbank [13]). For a tree-structured vector filterbank, the lowpass and the highpass properties for the two vector filters are not as clear as those for the two filters in single wavelet transforms. It has been found in [1], [16]–[17] that in order to have a reasonable decomposition for discrete multiwavelet transforms, prefiltering is necessary. A prefilter design method was introduced in [1], [16]–[17], where the idea is based on the computability of the multiwavelet transform coefficients from uniformly sampled signals. Moreover, an interpretation of the "lowpass" and "highpass" properties for vector filters was introduced in [1] for the prefilter design criterion. The criterion is, however, only good for the first step discrete multiwavelet transform decomposition. The prefilters designed with this method may be nonorthogonal, which might kill the gain of the energy compaction in the transform domain after the decoding is performed. In [31], a different approach was proposed for perserving the orthogonality by using the approximation order criterion. In [32], balanced multiwavelets were studied, where prefiltering for these kinds of multiwavelets is not necessary, but other properties, such as the short supportness and the smoothness, are not as good as the GHM multiwavelets. Notice that in [1] and [8], it was also mentioned that when the "lowpass" filter $\mathbf{H}(\omega)$ satisfies $\mathbf{H}(0) = I$, prefiltering is not necessary.

In this paper, we introduce a new prefilter design by combining ideas in single wavelet transforms and multiwavelet transforms as follows. We first construct a function $\phi(t)$ with the lowpass property, i.e., its Fourier transform $\hat{\phi}(\omega)$ is 1 at $\omega = 0$, or $\hat{\phi}(0) = 1$, from the multiscaling functions and their translations such that $\phi(t-n)$, $n \in \mathbf{Z}$ form an orthonormal set. Notice that the function ϕ does not have to be a scaling function since the nested property is not required, i.e., a dilation equation may not be satisfied. Due to the lowpass property, a signal $f(t)$ can be well approximated by a linear combination of $2^{J/2}\phi(2^J t - n)$, $n \in \mathbf{Z}$ for a large J; meanwhile, $f(t)$ can also be well approximated by a linear combination of the multiscaling functions and their translations due to their multiresolution approximation property. Because of the lowpass property of ϕ and the orthogonality of $\phi(t-n)$, the coefficients in the linear combination of $2^{J/2}\phi(2^J t - n)$, $n \in \mathbf{Z}$ are proportational to $f(n/2^J)$; see, for example, [23]–[25], and [35]. The conversion between these two approximations naturally suggests a prefiltering for computing the multiwavelet transform coefficients at the highest resolution (or called approximation coefficients) from the samples $f(n/2^J)$ of the signal f. Then, the rest of

Manuscript received September 12, 1996; revised January 27, 1998. This work was supported in part by an initiative grant from the Department of Electrical and Computer Engineering, University of Delaware, the Air Force Office of Scientific Research (AFOSR) under Grant F49620-97-1-0253, and the National Science Foundation CAREER Program under Grant MIP-9703377. The associate editor coordinating the review of this paper and approving it for publication was Dr. Truong Q. Nguyen.

The author is with the Department of Electrical and Computer Engineering, University of Delaware, Newark, DE 19716 USA (e-mail: xxia@ee.udel.edu).

Publisher Item Identifier S 1053-587X(98)03918-X.

the multiwavelet transform coefficients (the lowest resolution coefficients and the detailed coefficients) follows from a tree-structured vector filterbank [1], [8]. We will see later that the lowpass condition imposed on the function ϕ is strongly related to the lowpass condition imposed on the combined filters of the prefilters and the multiscaling functions, which also relates to the one imposed on the combined filters of the prefilters and the cascaded vector filterbanks, i.e., multiple steps of the discrete multiwavelet transform decompositions. Notice that the above prefilter structure was first used in [30], but neither the *lowpass condition* on the function ϕ nor any *rationale* for introducing such ϕ was mentioned. Instead, in [30], signal-dependent optimal prefilters, in terms of the energy compaction criterion, were designed. The drawbacks are 1) that the computational load is high and 2) the signal dependency. In this paper, we systematically study the prefilter structure and its rationale. The prefilters are signal independent and orthogonal, and they only depend on multiwavelets.

II. Approximation of Lowpass Functions Using Multiscaling Functions and New Prefilter Structure

In this section, we want to motivate a new prefiltering for multiwavelet transform coefficient computation by approximating a lowpass function using multiscaling functions. To do so, let us first briefly review multiwavelets and matrix dilation equations. For more details about multiwavelets, see, for example, [1]–[20] and [26]–[32].

Consider N compactly supported scaling functions $\phi_l(t)$, $l = 1, 2, \ldots, N$ and their corresponding N mother wavelet functions $\psi_l(t)$, $l = 1, 2, \ldots, N$, where all the translations $\phi_l(t-k)$, $k \in \mathbf{Z}$, $l = 1, 2, \ldots, N$ are mutually orthogonal, and $\psi_{l,j,k} \triangleq 2^{j/2}\psi_l(2^j t - k)$, $j, k \in \mathbf{Z}$, $l = 1, 2, \ldots, N$ form an orthonormal basis for $L^2(\mathbf{R})$. Let $\mathbf{H}(\omega)$ and $\mathbf{G}(\omega)$ be their corresponding $N \times N$ matrix quadrature mirror filters with $N \times N$ impulse response constant matrices H_k and G_k, $k \in \mathbf{Z}$, respectively. Let

$$\Phi(t) \triangleq (\phi_1(t), \ldots, \phi_N(t))^T, \quad \Psi(t) \triangleq (\psi_1(t), \ldots, \psi_N(t))^T.$$

Then, we have the following matrix dilation equations.

$$\Phi(t) = 2\sum_k H_k \Phi(2t-k) \tag{2.1}$$

$$\Psi(t) = 2\sum_k G_k \Phi(2t-k). \tag{2.2}$$

The orthogonality implies

$$\mathbf{H}(\omega)\mathbf{H}^\dagger(\omega) + \mathbf{H}(\omega+\pi)\mathbf{H}^\dagger(\omega+\pi) = I_N \tag{2.3}$$

$$\mathbf{G}(\omega)\mathbf{G}^\dagger(\omega) + \mathbf{G}(\omega+\pi)\mathbf{G}^\dagger(\omega+\pi) = I_N \tag{2.4}$$

$$\mathbf{H}(\omega)\mathbf{G}^\dagger(\omega) + \mathbf{H}(\omega+\pi)\mathbf{G}^\dagger(\omega+\pi) = 0_N \tag{2.5}$$

where † means the complex conjugate transpose, and I_N and 0_N denote the $N \times N$ identity and the all-zero matrix, respectively.

For each fixed $j \in \mathbf{Z}$, let V_j be the closure of the linear span of $\phi_{l,j,k} \triangleq 2^{j/2}\phi_l(2^j t - k)$, $l = 1, 2, \ldots, N$, $k \in \mathbf{Z}$. Then, the spaces V_j, $j \in \mathbf{Z}$ form an orthogonal multiresolution analysis for $L^2(\mathbf{R})$.

Let $f \in V_J$; then

$$f(t) = \sum_{l=1}^{N}\sum_{k\in\mathbf{Z}} c_{l,J,k}\phi_{l,J,k}(t) \tag{2.6}$$

$$\begin{aligned} &= \sum_{l=1}^{N}\sum_{k\in\mathbf{Z}} c_{l,J_0,k}\phi_{l,J_0,k}(t) \\ &\quad + \sum_{l=1}^{N}\sum_{J_0\le j<J}\sum_{k\in\mathbf{Z}} d_{l,j,k}\psi_{l,j,k}(t) \end{aligned} \tag{2.7}$$

where $J_0 < J$, and

$$c_{l,j,k} = \int f(t)\phi_{l,j,k}(t)\,dt$$

and

$$d_{l,j,k} = \int f(t)\psi_{l,j,k}(t)\,dt.$$

Let

$$\mathbf{c}_{j,k} \triangleq (c_{1,j,k}, \ldots, c_{N,j,k})^T$$

and

$$\mathbf{d}_{j,k} \triangleq (d_{1,j,k}, \ldots, d_{N,j,k})^T.$$

Then, by the matrix dilations (2.1)–(2.2)

$$\mathbf{c}_{j-1,k} = \sqrt{2}\sum_n H_n \mathbf{c}_{j,2k+n} \tag{2.8}$$

$$\mathbf{d}_{j-1,k} = \sqrt{2}\sum_n G_n \mathbf{c}_{j,2k+n} \tag{2.9}$$

and

$$\mathbf{c}_{j,n} = \sqrt{2}\sum_k (H_k \mathbf{c}_{j-1,2k+n} + G_k \mathbf{d}_{j-1,2k+n}). \tag{2.10}$$

Thus, to determine the multiwavelet transform coefficients $\mathbf{c}_{J_0,k}$ and $\mathbf{d}_{j,k}$ for $J_0 \le j < J$, $k \in \mathbf{Z}$ from f, it is good enough to determine the coefficients $\mathbf{c}_{J,k}$ for $k \in \mathbf{Z}$ from f.

Unlike single wavelets, where $c_{J,k}$ is proportional to the samples $f(k/2^J)$ when J is large enough due to the lowpass property of a single scaling function, the determination of $\mathbf{c}_{J,k}$ for multiwavelet transforms from the samples of $f(t)$ is not trivial. When the multiscaling functions have the interpolating property, the determination was given in [1] and [16]–[17]. Furthermore, a necessary and sufficient condition for the solvability of $\mathbf{c}_{J,k}$ from the samples of f was also given in [1]. The relationship between the samples of f and the coefficients $\mathbf{c}_{J,k}$ automatically provides a prefiltering for the multiwavelet transform computation from the samples of f. For more details, see [1]. Unfortunately, the prefiltering based on this relationship is usually not orthogonal, which seems to limit the gain in the compression applications.

In order to present our new prefilter design method, i.e., a new relationship between the samples of f and $\mathbf{c}_{J,k}$, let us look at the conventional wavelet transform coefficient computation, which is usually referred as the Mallat algorithm.

Let $\phi(t)$ be a single orthogonal scaling function. Then, for any signal $f(t)$, there exists $J > 0$ such that $f(t)$ can be well approximated by $\phi_{J,k}(t) \triangleq 2^{J/2}\phi(2^J t - k)$, $k \in \mathbf{Z}$, i.e.,

$$f(t) \approx \sum_k c_{J,k}\phi_{J,k}(t) \tag{2.11}$$

where

$$c_{J,k} = \int f(t)\phi_{J,k}(t)\,dt \propto f\left(\frac{k}{2^J}\right). \tag{2.12}$$

The relationship $\propto$ in the above formula is because of the lowpass property of $\phi(t)$, i.e., $\hat{\phi}(0) = 1$, see, for example, [23]–[25] and [35]. The rest of wavelet transform coefficients can be calculated recursively from $c_{J,k}$. The *key point* for the validation of (2.11)–(2.12) is that the scaling function $\phi(t)$ has the lowpass property, and $\phi(t-k)$, $k \in \mathbf{Z}$ are orthogonal.

Motivated from the above observation, we now want to construct a function $\phi(t)$ from the multiscaling functions $\phi_l(t)$, $l = 1, 2, \ldots, N$ such that $\phi(t)$ has the lowpass property, and its translations $\phi(t-k)$, $k \in \mathbf{Z}$ are orthogonal to each other. Notice that such $\phi(t)$ may not be a scaling function because it may not satisfy any dilation equation. As long as $\phi(t)$ has the lowpass property and the orthogonality, the properties (2.11)–(2.12) hold for a signal f.

Let

$$\phi(t) = \sum_{l=1}^{N}\sum_n a_l[n]\phi_l(t-n) \tag{2.13}$$

where $a_l[n]$ are real constants. Then

$$\hat{\phi}(\omega) = \sum_{l=1}^{N} A_l(\omega)\hat{\phi}_l(\omega) \tag{2.14}$$

where

$$A_l(\omega) = \sum_n a_l[n]e^{-jn\omega}. \tag{2.15}$$

The lowpass property implies

$$\hat{\phi}(0) = \sum_{l=1}^{N} A_l(0)\hat{\phi}_l(0) = 1. \tag{2.16}$$

The orthogonality of $\phi(t-n)$, $n \in \mathbf{Z}$ is equivalent to

$$\sum_n |\hat{\phi}(\omega + 2\pi n)|^2 = 1. \tag{2.17}$$

Write out the right-hand side of (2.17) as

$$\begin{aligned}
&\sum_n |\hat{\phi}(\omega + 2\pi n)|^2 \\
&\quad = \sum_n \left(\sum_{l_1=1}^{N} A_{l_1}(\omega)\hat{\phi}_{l_1}(\omega + 2\pi n)\right) \\
&\qquad \times \left(\sum_{l_2=1}^{N} A_{l_2}^*(\omega)\hat{\phi}_{l_2}^*(\omega + 2\pi n)\right) \\
&\quad = \sum_{l_1=1}^{N}\sum_{l_2=1}^{N} A_{l_1}(\omega)A_{l_2}^*(\omega)\sum_n \hat{\phi}_{l_1}(\omega + 2\pi n)\hat{\phi}_{l_2}^*(\omega + 2\pi n).
\end{aligned}$$

By the orthognality of $\phi_l(t-n)$, $l = 1, 2, \ldots, N$, $n \in \mathbf{Z}$, it is not hard to see that

$$\sum_n \hat{\phi}_{l_1}(\omega + 2\pi n)\hat{\phi}_{l_2}^*(\omega + 2\pi n) = \delta(l_1 - l_2).$$

Therefore

$$\sum_n |\hat{\phi}(\omega + 2\pi n)|^2 = \sum_{l=1}^{N} |A_l(\omega)|^2.$$

This implies that the orthogonality of $\phi(t-n)$, $n \in \mathbf{Z}$ is equivalent to

$$\sum_{l=1}^{N} |A_l(\omega)|^2 = 1. \tag{2.18}$$

In conclusion, we have proved the following lemma.

Lemma 1: A linear combination $\phi(t)$ in (2.13) of multiscaling functions $\phi_l(t)$ and their translations has the lowpass property and the orthogonality of its translations if and only if the properties (2.16) and (2.18) hold.

We now assume $\phi(t)$ in (2.13) satisfies the lowpass property (2.16) and the orthogonality (2.18). For a given signal $f(t)$, by the lowpass property of $\phi(t)$, there exists a $J > 0$ such that (see, for example, [35, Prop. 5.3.2, p. 142])

$$f(t) \approx \sum_n b_n 2^{J/2}\phi(2^J t - n) \tag{2.19}$$

where

$$b_n = \int f(t)2^{J/2}\phi(2^J t - n)\,dt.$$

An estimate of the difference

$$\left| f(t) - \sum_n b_n 2^{J/2}\phi(2^J t - n) \right|$$

is given in the Appendix. Notice that the only condition on $\phi(t)$ for the relationship (2.12) to hold is the lowpass property, i.e., $\hat{\phi}(0) = 1$. Therefore, similar to (2.12), we have

$$b_n \propto f\left(\frac{n}{2^J}\right), \quad \text{for large } J.$$

Without loss of the generality, we may assume $J = 0$ for simplicity. Then

$$f(t) \approx \sum_n b_n\phi(t-n) \quad \text{and} \quad b_n \propto f(n).$$

From (2.13)

$$\begin{aligned}
f(t) &\approx \sum_n b_n\phi(t-n) \\
&= \sum_n b_n \sum_{l=1}^{N}\sum_m a_l[m]\phi_l(t-n-m) \\
&= \sum_{l=1}^{N}\sum_k \left(\sum_m b_{k-m}a_l[m]\right)\phi_l(t-k).
\end{aligned}$$

This implies that

$$\sum_m b_{k-m}a_l[m] \approx c_{l,0,k}, \quad l = 1, 2, \ldots, N, k \in \mathbf{Z} \tag{2.20}$$

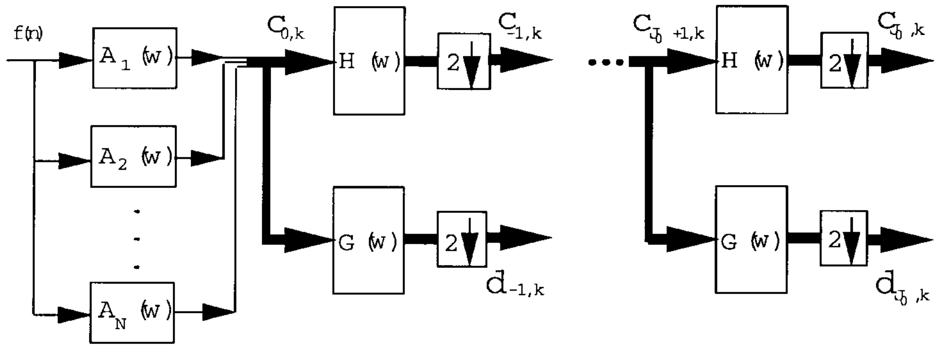


Fig. 1. New prefiltering: Decomposition.

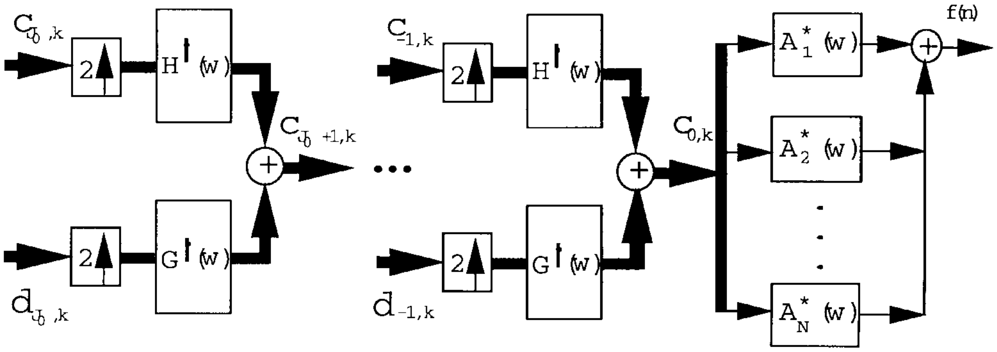


Fig. 2. New prefiltering: Reconstruction.

where $b_n \propto f(n),\ n \in \mathbf{Z}$. The above result (2.20) suggests the following new relationship, i.e., a new prefilter, between the samples $f(n)$ of $f(t)$ and the multiwavelet transform coefficients $c_{l,0,k}$

$$c_{l,0,k} = \sum_m f(k-m)a_l[m] \tag{2.21}$$

which is shown in Fig. 1.

By the orthogonalities of multiwavelets (2.3)–(2.5) and prefilters (2.18), the reconstruction can be shown in Fig. 2.

The difference between the above prefilter bank and the prefilter bank proposed in [1] is the following. The above prefilter bank is not maximally decimated, i.e., redundancies are introduced. Actually, the number of coefficients in the transform domain is increased by N times. The prefilter bank in [1] is, however, maximally decimated, and no redundancy is introduced. We might want to ask, since we are usually interested in reducing the redundancies, why we need to introduce redundancy here. The answer here is two-fold. First, proper overcomplete (or redundant) transforms plus vector quantizations might perform better than nonredundant transforms. This suggests that including redundancy in the transform might not be a bad idea due to its better tolerance of noise than nonredundant transforms. Second, from our numerical examples, the energy compaction with this new prefiltering is better than the one with Daubechies D_4 wavelet transform after the nonmaximality of the decimation in prefiltering has been taken into account.

Notice that the energy of $f(n)$ is preserved after the whole discrete multiwavelet transform in Fig. 1 is performed due to the orthogonalities of the multiwavelet transform and the prefilter bank, although the prefilter bank is nonmaximally decimated.

Motivated from the above prefiltering and the one in [1], we propose the following general prefiltering for discrete multiwavelet transforms, which is shown in Fig. 3, where $1 \le K \le N$ and the pre/post filterbank shown in Fig. 4 have the perfect reconstruction property. Specifically, when the filterbank in Fig. 4 is paraunitary, the prefiltering in Fig. 3 is orthogonal.

III. Prefilter Design and Examples

In this section, we first study the general N wavelet case and then study the case of $N = 2$. Finally, we look at two examples. One is the Geronimo–Hardin–Massopust 2 wavelet prefilter design, and the other is the prefilter design for one of the 2 wavelets obtained by Chui and Lian in [20].

A. General N Wavelet Prefilter Design

Although, for the general prefilter bank in Fig. 3 (i.e., general K), the interpretation in the previous section does not hold, the design of a prefilter bank $A_l(\omega)$ can be done using the same criterion given in [1], where $K = N$. In the following, we focus on the case of $K = 1$ and use the interpretation in Section II to design the prefilter bank $A_l(\omega)$. Moreover, we are

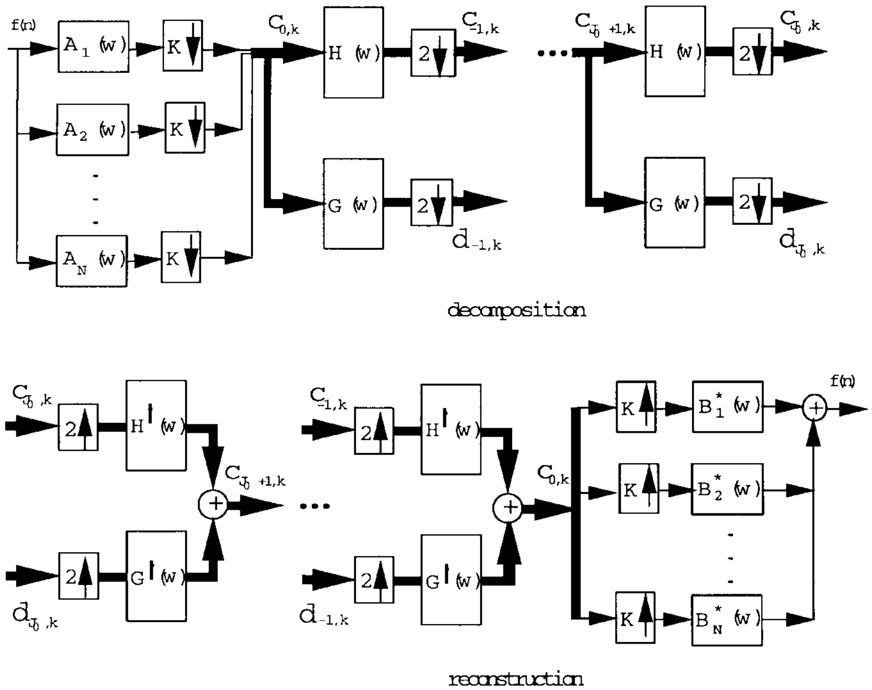


Fig. 3. General prefiltering: Decomposition and reconstruction.

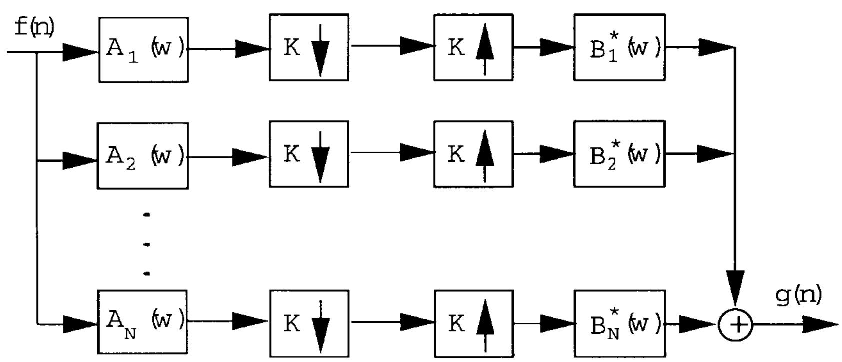


Fig. 4. Pre/post filterbank.

only interested in designing FIR prefilter banks. The lowpass and the orthogonality conditions (2.16) and (2.18) will be used.

Due to its orthogonality, any FIR prefilter bank $A_l(\omega)$ can be factorized as (see, for example, [33], [34])

$$\begin{pmatrix} A_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ A_N(\omega) \end{pmatrix} = U_\rho(\omega) \cdots U_1(\omega) \begin{pmatrix} A_1(0) \\ \cdot \\ \cdot \\ \cdot \\ A_N(0) \end{pmatrix} \tag{3.1}$$

where

$$\sum_{l=1}^{N} |A_l(0)|^2 = 1 \tag{3.2}$$

and

$$U_r(\omega) = I_N + (e^{-j\omega} - 1)\mathbf{u}_r^\dagger \mathbf{u}_r \tag{3.3}$$

where $\mathbf{u}_r = (u_{r1}, \ldots, u_{rN})$ and the norm of the vector $\mathbf{u}_r$ is 1, i.e.,

$$\sum_{l=1}^{N} |u_{rl}|^2 = 1.$$

From the matrix dilation equation, we have

$$\begin{pmatrix} \hat{\phi}_1(0) \\ \cdot \\ \cdot \\ \cdot \\ \hat{\phi}_N(0) \end{pmatrix} = \mathbf{H}(0) \begin{pmatrix} \hat{\phi}_1(0) \\ \cdot \\ \cdot \\ \cdot \\ \hat{\phi}_N(0) \end{pmatrix}. \tag{3.4}$$

When $\mathbf{H}(\omega)$ is known, the vector $(\hat{\phi}(0), \ldots, \hat{\phi}_N(0))$ can be solved. Then, the orthogonality and the lowpass property (2.16) and (2.18) are equivalent to

$$\sum_{l=1}^{N} |A_l(0)|^2 = 1 \quad \text{and} \quad \sum_{l=1}^{N} A_l(0)\hat{\phi}_l(0) = 1. \tag{3.5}$$

The only constraint for the parameters u_{rl} is that they need to be of the unit norm for $r = 1, 2, \ldots, \rho$. The parameter ρ determines the prefilter length and is called the *order* of the prefilter $(A_l(\omega))_{l=1,2,\ldots,N}$. When there is no $U_r(\omega)$ term in (3.1), we set $\rho = 0$, i.e., the order of the prefilter is zero.

Additional conditions may be imposed on the above parameters. An important one is that the combined filters of $A_l(\omega)$ and $\mathbf{H}(\omega)$ need to be lowpass filters, and the combined filters of $A_l(\omega)$ and $\mathbf{G}(\omega)$ need to be highpass filters. The reason for this condition is the same as what was proposed in [1], i.e., we need to keep the "lowpass" part and decompose it again and again but quantize the "highpass" part and therefore keep the "highpass" part as small as possible. This means the "highpass" part needs to be the high-frequency part; otherwise, it will have a lot of energy.

By thinking of the multiscaling vectors as the cascaded version of the "lowpass" vector filter $\mathbf{H}(\omega)$, the new lowpass property (2.16) for the function ϕ means the lowpass property for the combined filters of the prefilters $A_l(\omega)$ and cascaded vector filters $\mathbf{H}(\omega)$. Therefore, the above two lowpass conditions [the new one (2.16) and the old one in [1]] somewhat guarantee the lowpass properties of the all-approximation multiwavelet transform coefficients $\mathbf{c}_{j,k}$ for $J_0 \le j < 0$. The old lowpass condition in [1] is for the lowpass property of the first step decomposition $\mathbf{c}_{-1,k}$ and the new lowpass condition in this paper is for the follow-up decompositions $\mathbf{c}_{j,k}$ for $J_0 \le j < -1$. The old lowpass condition in [1] can be stated as follows.

There are N combined filters of $A_k(\omega)$ and $\mathbf{H}(\omega)$ and N combined filters of $A_l(\omega)$ and $\mathbf{G}(\omega)$. They are

$$H_l(\omega) \triangleq \sum_{k=1}^{N} H_{l,k}(\omega) A_k(\omega), \quad l = 1, 2, \ldots, N \tag{3.6}$$

and

$$G_l(\omega) \triangleq \sum_{k=1}^{N} G_{l,k}(\omega) A_k(\omega), \quad l = 1, 2, \ldots, N \tag{3.7}$$

respectively, where $\mathbf{H}(\omega) = (H_{l,k}(\omega))_{N\times N}$, and $\mathbf{G}(\omega) = (G_{l,k}(\omega))_{N\times N}$. Then, the prefiltering, the first step multiwavelet transform decomposition, and their combined filters can be shown in Fig. 5.

The lowpass property on $H_l(\omega)$ is

$$\sum_{k=1}^{N} H_{l,k}(\pi) A_k(\pi) = 0, \quad l = 1, 2, \ldots, N. \tag{3.8}$$

The highpass property on $G_l(\omega)$ is

$$\sum_{k=1}^{N} G_{l,k}(0) A_k(0) = 0, \quad l = 1, 2, \ldots, N. \tag{3.9}$$

In *conclusion*, the above four conditions [i.e., (3.1), (3.5), (3.8), and (3.9)] need to be imposed on the prefilter design given a multiwavelet.

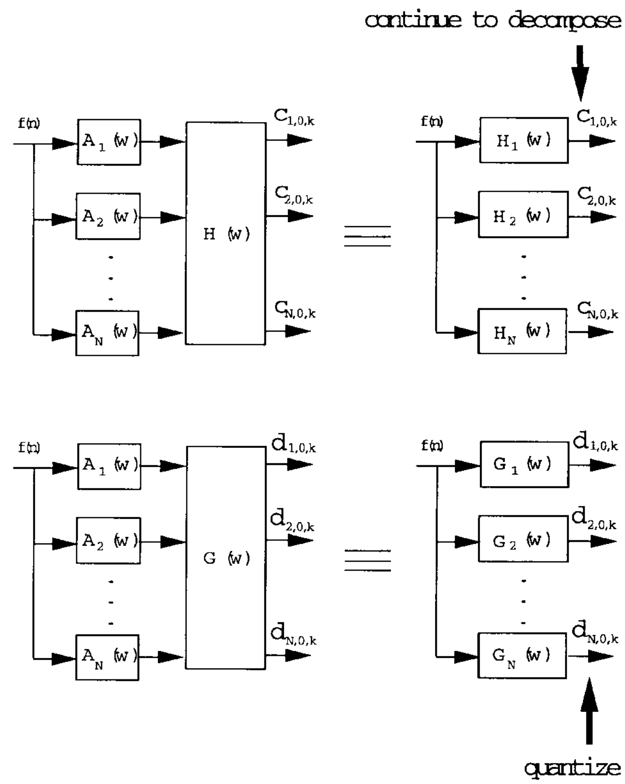


Fig. 5. Combined filters of prefilters and multiwavelet filters.

B. Theory for 2 Wavelets

Since there always exists a solution for (3.4), there exist two real constants a and b such that

$$a\hat{\phi}_1(0) + b\hat{\phi}_2(0) = 0. \tag{3.10}$$

Without loss of generality, we may assume $\hat{\phi}_1(0) = c\hat{\phi}_2(0)$ for a real constant c. Then, by (3.5)

$$cA_1(0) + A_2(0) = 1/\hat{\phi}_2(0) = x, \quad \text{or} \quad A_2(0) = x - cA_1(0) \tag{3.11}$$

and

$$(1 + c^2)A_1^2(0) - 2xcA_1(0) + x^2 - 1 = 0, \quad \text{or}$$
$$A_1(0) = \frac{xc \pm \sqrt{1 + c^2 - x^2}}{1 + c^2} \tag{3.12}$$

where x is an arbitrary constant. This implies that there always exist solutions for (3.5).

When matrix $\mathbf{G}(0)$ has full rank, the only solution for (3.9) is $A_1(0) = A_2(0) = 0$, which does not satisfy (3.5).

When matrix $\mathbf{G}(0)$ does not have full rank, there exist solutions for $A_l(0)$, $l = 1, 2$ in (3.9), i.e., there exist two real constants d and e such that

$$dA_1(0) = eA_2(0). \tag{3.13}$$

Clearly, there exists a solution for $A_1(0)$ and $A_2(0)$ in (3.11)–(3.13).

Now, the only condition left is (3.8). Although the existence of the zeroth-order prefilter $(A_1(\omega), A_2(\omega)) = (A_1(0), A_2(0))$ in (3.8) depends on the form of $(A_1(0), A_2(0))$ and $\mathbf{H}(0)$ (we will see later that there does not exist any zeroth-order prefilter that satisfies (3.8) for the GHM 2 wavelets, but there does exist for one of the 2 wavelets obtained by Chui and Lian in [20]), we may analyze first order prefilters. In this case

$$\begin{pmatrix} A_1(\omega) \\ A_2(\omega) \end{pmatrix} = \left(I_2 + (e^{-j\omega} - 1) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta, \sin\theta) \right) \times \begin{pmatrix} A_1(0) \\ A_2(0) \end{pmatrix} \tag{3.14}$$

and

$$\begin{pmatrix} A_1(\pi) \\ A_2(\pi) \end{pmatrix} = \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} A_1(0) \\ A_2(0) \end{pmatrix}$$

where θ is an arbitrary angle.

For the same reason as before, when matrix $\mathbf{H}(\pi)$ has full rank, there are no solutions for (3.5) and (3.8). Therefore, we assume that matrix $\mathbf{H}(\pi)$ does not have full rank. Then, there exist two real constants u, v such that

$$uA_1(\pi) + vA_2(\pi) = 0.$$

By (3.14)

$$(vA_2(0) - uA_1(0))\cos(2\theta) = (uA_2(0) + vA_1(0))\sin(2\theta).$$

Thus, there exists an angle θ such that the above equation holds. This proves the following theorem.

Theorem 1: There exists a first-order prefilter $(A_1(\omega), A_2(\omega))$ that satisfies all conditions [i.e., (3.1), (3.5), (3.8), and (3.9)] if and only if none of matrices $\mathbf{H}(\pi)$ and $\mathbf{G}(0)$ has full rank.

As pointed out by one of the referees of this manuscript, the condition in the above theorem always holds if a constant can be expressed by a linear combination of the translates $\phi_1(t-k)$ and $\phi_2(t-k)$ of two scaling functions $\phi_1(t)$ and $\phi_2(t)$.

C. Design Examples for the GHM 2 Wavelets

We first want to see the Geronimo–Hardin–Massopust 2 wavelets with the following matrix impulse responses of the vector filters $\mathbf{H}(\omega)$ and $\mathbf{G}(\omega)$, respectively.

$$H_0 = \begin{pmatrix} 3/10 & 2\sqrt{2}/5 \\ -\sqrt{2}/40 & -3/20 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 3/10 & 0 \\ 9\sqrt{2}/40 & 1/2 \end{pmatrix}$$
$$H_2 = \begin{pmatrix} 0 & 0 \\ 9\sqrt{2}/40 & -3/20 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2}/40 & 0 \end{pmatrix}$$

and

$$G_0 = \begin{pmatrix} -\sqrt{2}/40 & -3/20 \\ -1/20 & -3\sqrt{2}/20 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 9\sqrt{2}/40 & -1/2 \\ 9/20 & 0 \end{pmatrix}$$
$$G_2 = \begin{pmatrix} 9\sqrt{2}/40 & -3/20 \\ -9/20 & 3\sqrt{2}/20 \end{pmatrix}, \quad G_3 = \begin{pmatrix} -\sqrt{2}/40 & 0 \\ 1/20 & 0 \end{pmatrix}.$$

In this case

$$\mathbf{H}(0) = \begin{pmatrix} \frac{3}{5} & \frac{2\sqrt{2}}{5} \\ \frac{2\sqrt{2}}{5} & \frac{1}{5} \end{pmatrix}, \quad \mathbf{H}(\pi) = \begin{pmatrix} 0 & \frac{2\sqrt{2}}{5} \\ 0 & -\frac{4}{5} \end{pmatrix}$$
$$\mathbf{G}(0) = \begin{pmatrix} \frac{2\sqrt{2}}{5} & -\frac{4}{5} \\ 0 & 0 \end{pmatrix}.$$

From (3.4)

$$\hat{\phi}_1(0) - \sqrt{2}\hat{\phi}_2(0) = 0. \tag{3.15}$$

Solving (3.4) and (3.5), we have

$$A_1(0) = \frac{x\sqrt{2} \pm \sqrt{3 - x^2}}{3} \quad \text{and} \quad A_1(0) = \frac{x \mp \sqrt{3 - x^2}}{3} \tag{3.16}$$

where x is a real constant with $|x| \le \sqrt{3}$. The condition (3.9) implies

$$\sqrt{2}A_2(0) = A_1(0).$$

Therefore, we solve for $A_l(0)$ as

$$A_1(0) = \frac{\sqrt{6}}{3} \quad \text{and} \quad A_2(0) = \frac{\sqrt{3}}{3}.$$

Then, the prefilters in (3.1) can be written as

$$\begin{pmatrix} A_1(\omega) \\ A_2(\omega) \end{pmatrix} = U_\rho(\omega) \cdots U_1(\omega) \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{3.17}$$

where

$$U_r(\omega) = I_2 + (e^{-j\omega} - 1) \begin{pmatrix} u_{r1} \\ u_{r2} \end{pmatrix} (u_{r1}, u_{r2})$$

with $u_{r1}^2 + u_{r2}^2 = 1$ for two real constants u_{r1} and u_{r2}. It is clear that (3.8) implies that the order ρ in (3.17) must be greater than or equal to 1. Since matrices $\mathbf{H}(\pi)$ and $\mathbf{G}(0)$ do not have full rank, by Theorem 1, there exists a first-order prefilter satisfying the conditions. Let us see what it looks like.

$$\begin{pmatrix} A_1(\omega) \\ A_2(\omega) \end{pmatrix} = \left(I_2 + (e^{-j\omega} - 1) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta, \sin\theta) \right) \times \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \frac{1}{\sqrt{3}} \tag{3.18}$$

where θ is an angle. Thus

$$\begin{pmatrix} A_1(\pi) \\ A_2(\pi) \end{pmatrix} = \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \frac{1}{\sqrt{3}}.$$

Therefore, (3.8) implies

$$-\sqrt{2}\sin 2\theta + \cos 2\theta = 0, \quad \text{or} \quad \theta = \frac{1}{2}\arctan\frac{\sqrt{2}}{2}. \tag{3.19}$$

This proves the following theorem.

Theorem 2: The prefilter in (3.18) with the θ in (3.19) satisfies all conditions we want, including the following.

1) the lowpass property of $\phi(t)$, i.e., $\hat{\phi}(0) = 1$;
2) the orthogonality of $\phi(t-n)$, $n \in \mathbf{Z}$ and the orthogonality of the prefilter bank $A_l(\omega)$ for $l = 1, 2$;
3) the lowpass property of the combined filters $H_l(\omega)$ for $l = 1, 2$ of $A_l(\omega)$, $l = 1, 2$, and $\mathbf{H}(\omega)$;
4) the highpass property of the combined filters $G_l(\omega)$ for $l = 1, 2$ of $A_l(\omega)$, $l = 1, 2$, and $\mathbf{G}(\omega)$.

As mentioned earlier, the zeroth-order prefilter, i.e., without any term $U_r(\omega)$ in (3.18), does not satisfy the above property 3), although it satisfies all the rest, i.e., 1), 2), and 4). Notice that the above zeroth-order prefilter was first used in [16] and [17]. When the order ρ of a prefilter increases, better lowpass and highpass combined filters $H_l(\omega)$ and $G_l(\omega)$, respectively, may be expected, and the length of a prefilter also increases. The final version of the two prefilters in (3.18) can be expressed as

$$\begin{aligned} A_1(\omega) = & \frac{\sqrt{2}}{\sqrt{3}}\sin^2\theta - \frac{1}{2\sqrt{3}}\sin 2\theta \\ & + \left(\frac{\sqrt{2}}{\sqrt{3}}\cos^2\theta + \frac{1}{2\sqrt{3}}\sin 2\theta\right)e^{-j\omega} \end{aligned} \tag{3.20}$$

$$\begin{aligned} A_2(\omega) = & -\frac{1}{\sqrt{6}}\sin 2\theta + \frac{1}{\sqrt{3}}\cos^2\theta \\ & + \left(\frac{1}{\sqrt{6}}\sin 2\theta + \frac{1}{\sqrt{3}}\sin^2\theta\right)e^{-j\omega}. \end{aligned} \tag{3.21}$$

D. Another Design Example

The second example of 2 wavelets is obtained by Chui and Lian in [20]. The matrix impulse responses are

$$H_0 = \frac{1}{2}\begin{pmatrix} 1/2 & 1/2 \\ -\sqrt{7}/4 & -\sqrt{7}/4 \end{pmatrix}, \quad H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$
$$H_2 = \frac{1}{2}\begin{pmatrix} 1/2 & -1/2 \\ \sqrt{7}/4 & -\sqrt{7}/4 \end{pmatrix}$$

and

$$G_0 = \frac{1}{2}\begin{pmatrix} -1/2 & -1/2 \\ 1/4 & 1/4 \end{pmatrix}, \quad G_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{7}/2 \end{pmatrix}$$
$$G_2 = \frac{1}{2}\begin{pmatrix} -1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix}.$$

The multiscaling and multiwavelet functions are supported in $[0,2]$ and have symmetry and certain smoothness. It is clear that

$$\mathbf{H}(0) = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & \frac{1-\sqrt{7}}{2} \end{pmatrix}, \quad \mathbf{H}(\pi) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1+\sqrt{7}}{2} \end{pmatrix}$$
$$\mathbf{G}(0) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1+\sqrt{7}}{2} \end{pmatrix}.$$

Conditions (3.9) and (3.5) imply that $A_1(0) = \pm 1$ and $A_2(0) = 0$. In this case, the zeroth-order $(A_1(\omega), A_2(\omega)) = (A_1(0), A_2(0)) = (1, 0)$ already satisfies (3.8). As pointed out by one of the referees of this manuscript, this result holds not only for the above Chui–Lian multiwavelets but for other multiwavelets as well as long as one of two scaling functions is symmetric and the other of two scaling functions is antisymmetric.

E. Numerical Simulations for the Combined Filters $H_l(\omega)$ and $G_l(\omega)$

In this section, we want to illustrate the combined filters $H_l(\omega)$ and $G_l(\omega)$ for $l = 1, 2$ for the GHM 2 wavelets. Three sets of these combined filters are illustrated: without prefiltering [Fig. 6(a) and (b)]; old zeroth-order orthogonal prefiltering in [1] [Fig. 7(a) and (b)]; new orthogonal prefiltering in Theorem 2 [Fig. 8(a) and (b)].

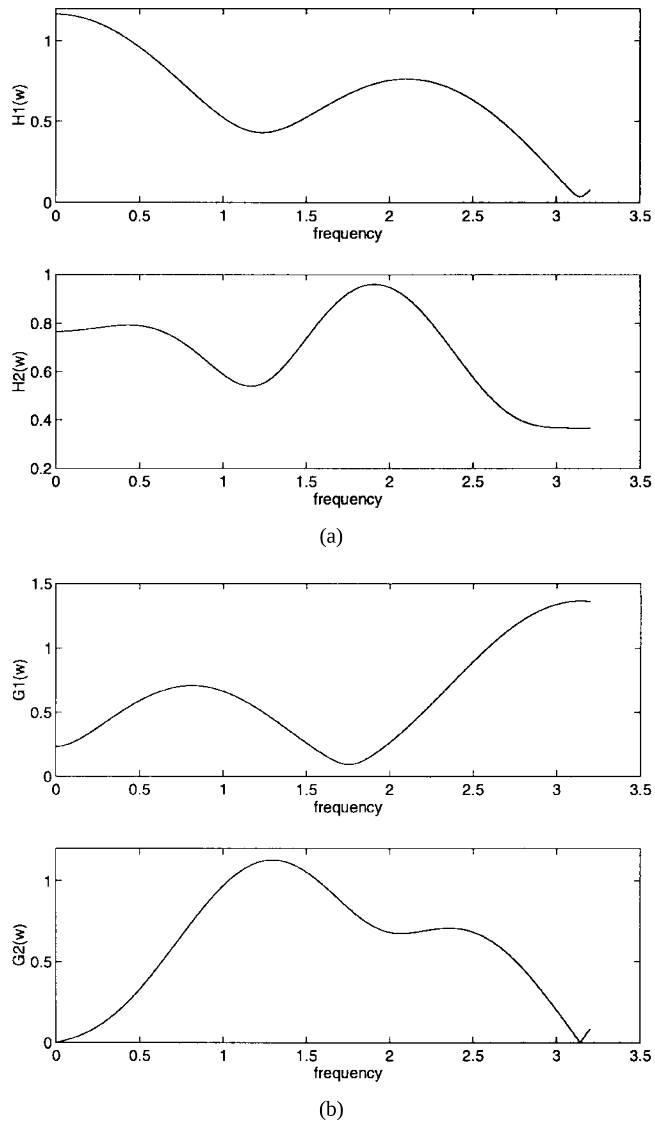
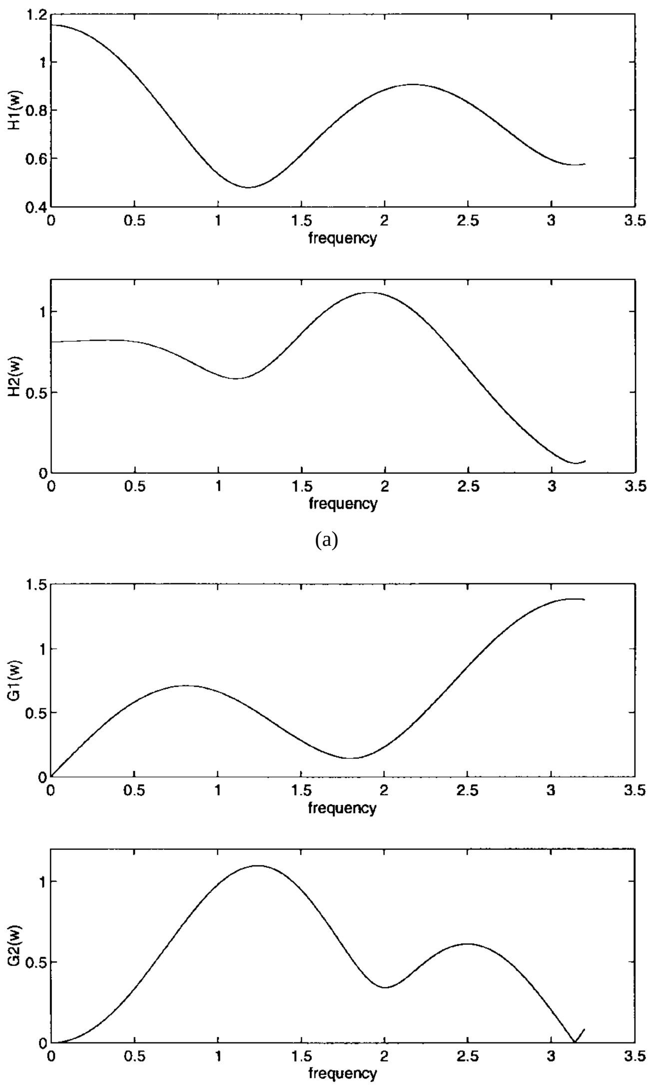


Fig. 6. Combined filters of the GHM 2 wavelets without prefiltering. (a) $|H_l(\omega)|$. (b) $|G_l(\omega)|$.

IV. Numerical Experiments

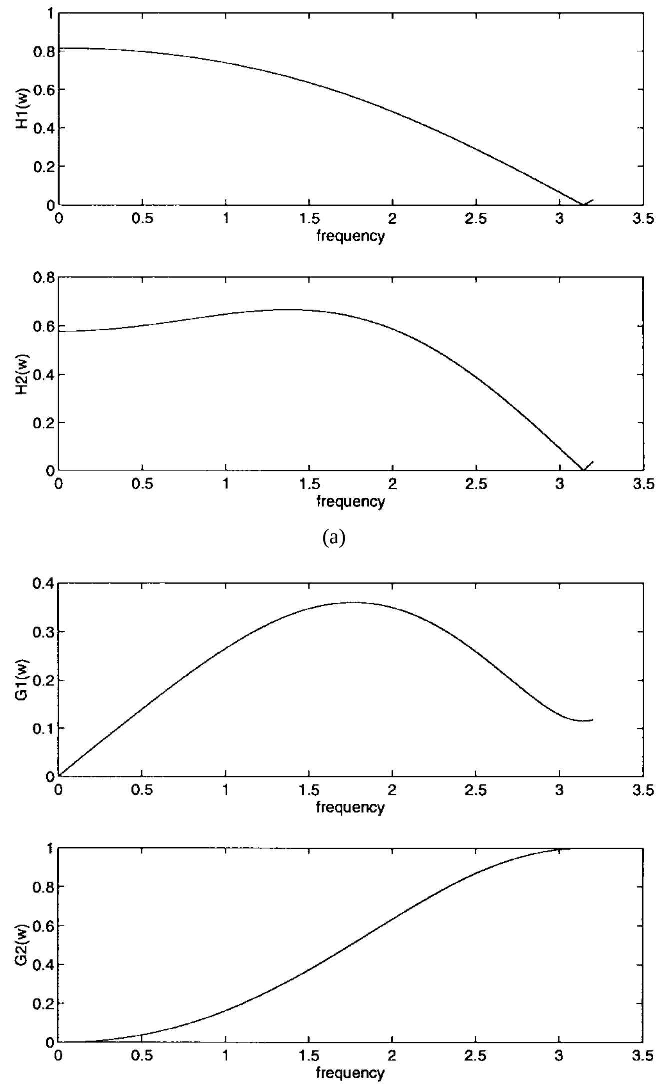
In this section, we want to see the performance of our new prefiltering scheme through some simple numerical examples. The first test signal is the one hundredth horizontal line of the Cameraman image with size 256×256, which is shown in Fig. 9. Six experiments on energy compaction of the following six transforms are done. The first transform T_1 is the GHM 2 wavelets without prefiltering. The second transform T_2 is the GHM 2 wavelets with the old zeroth-order orthogonal prefiltering with $\epsilon_1 = 1/(10\sqrt{3})$ and $\epsilon_2 = 7/(5\sqrt{6})$ in (3.29) in [1]. The third one T_3 is the Daubechies D_4 wavelets. The forth and the fifth are the GHM 2 wavelets with our new orthogonal prefiltering of the zeroth and the first order, respectively. The sixth transform T_6 is the Chui–Lian multiwavelet transform in Section III-D with the zeroth-order prefiltering $(A_1(\omega), A_2(\omega)) = (1, 0)$. Two step decompositions, i.e., $J_0 =$



(a)

(b)

Fig. 7. Combined filters of the GHM 2 wavelets with the old zeroth-order orthogonal prefiltering in [1]. (a) $|H_l(\omega)|$. (b) $|G_l(\omega)|$.



(a)

(b)

Fig. 8. Combined filters of the GHM 2 wavelets with their first-order orthogonal prefiltering. (a) $|H_l(\omega)|$. (b) $|G_l(\omega)|$.

-2 and $J = 0$, in the first three transforms are performed, where the lowpass part of the transformed signal is of length 64, whereas the bandpass part is of length 192. Since our new prefiltering is nonmaximally decimated and the signal size in the discrete multiwavelet transform domain is twice of the input signal (or the output signals of the first three transforms), three step decompositions, i.e., $J_0 = -3$ and $J = 0$, of the discrete multiwavelet transform with our new prefiltering are performed for the last three transforms, where the length of the lowpass part of the transformed signal is also 64, whereas the length of the bandpass part is $512 - 64 = 448$. Therefore, we have the following energy compaction ratio definitions.

The energy compaction ratioes for the first three transforms T_k for $k = 1, 2, 3$ are defined by

$$r = \frac{\sum_{n=65}^{256} |y[n]|^2}{\sum_{n=1}^{256} |y[n]|^2}$$

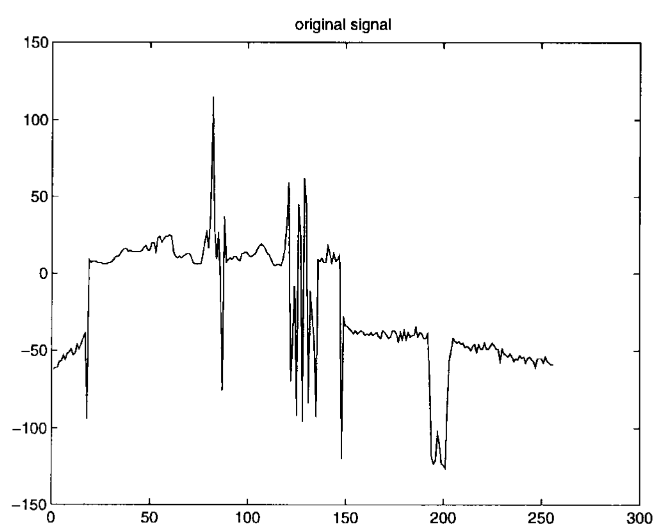


Fig. 9. First test signal.

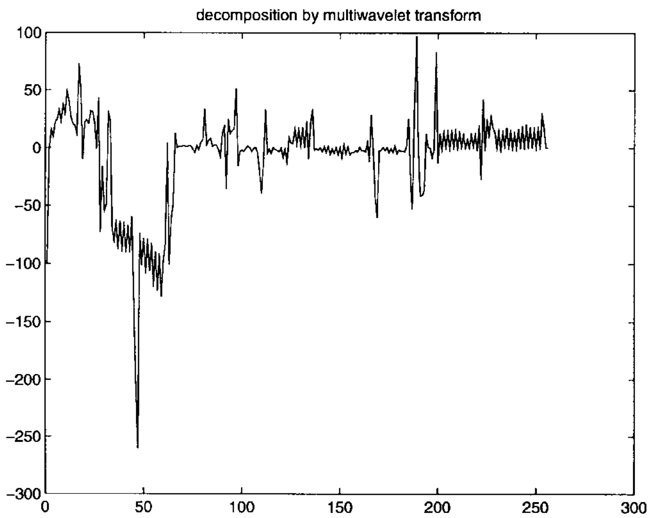


Fig. 10. Decomposition of the first test signal using the GHM 2 wavelets without prefiltering.

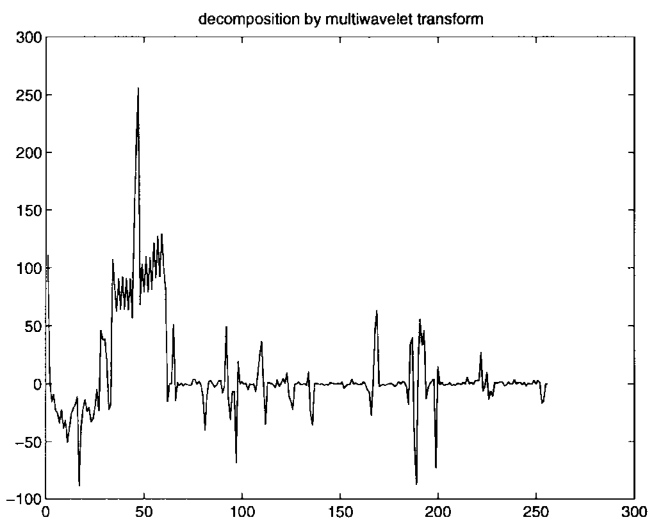


Fig. 11. Decomposition of the first test signal using the GHM 2 wavelets with the old zeroth-order orthogonal prefiltering in [1].

TABLE I
ENERGY COMPACTION RATIO COMPARISON FOR THE FIRST TEST SIGNAL

	r
GHM 2 wavelets without prefiltering	0.1374
GHM 2 wavelets with the old 0th order orthogonal prefiltering in [1]	0.1247
Daubechies D_4 wavelets	0.1123
GHM 2 wavelets with the new 0th order orthogonal prefiltering	0.0896
GHM 2 wavelets with the new 1th order orthogonal prefiltering	0.0722
Chui-Lian 2 wavelets with the 0th order orthogonal prefiltering	0.0944

where $y[n]$ are the signals in the transform domain. The energy compaction ratioes for the rest three transforms, i.e., with the new prefiltering, are defined by

$$r = \frac{\sum_{n=65}^{512} |y[n]|^2}{\sum_{n=1}^{512} |y[n]|^2}.$$

The transformed signals with the first three transforms are shown in Figs. 10–12, respectively. The transformed signals with the new orthogonal prefiltering of the zeroth- order and the first-order for the GHM multiwavelets are shown in Figs. 13 and 14, respectively. The transformed signal with

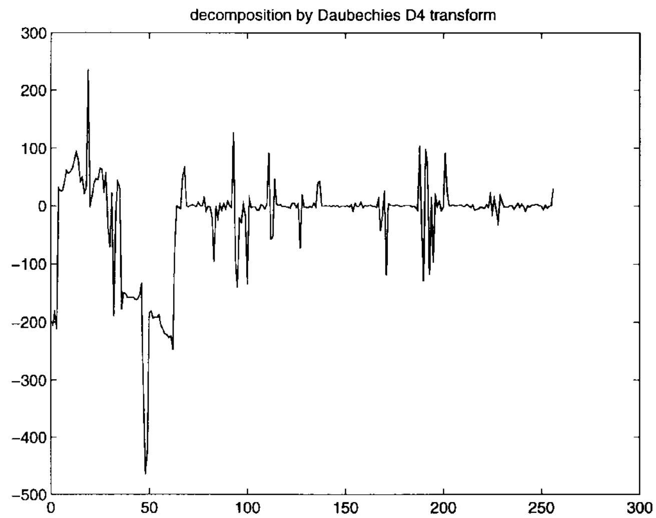


Fig. 12. Decomposition of the first test signal using Daubechies D_4 wavelets.

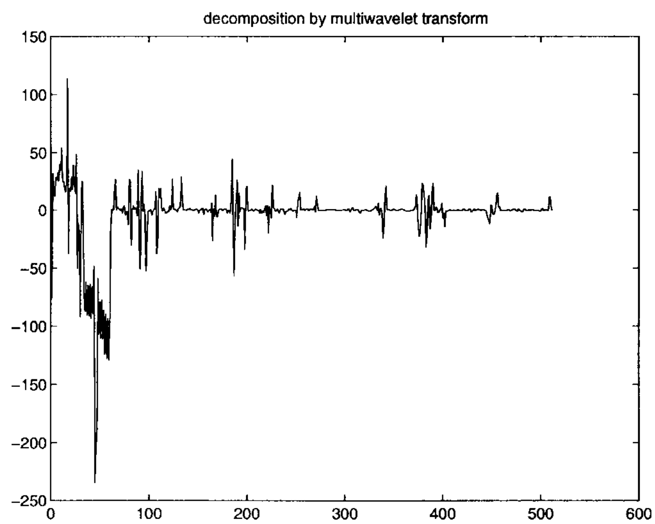


Fig. 13. Decomposition of the first test signal using the GHM 2 wavelets with the new zeroth-order orthogonal prefiltering.

TABLE II
ENERGY COMPACTION RATIO COMPARISON FOR THE SECOND TEST SIGNAL

	r
Daubechies D_4 wavelets	0.0110
GHM 2 wavelets with the new 1th order orthogonal prefiltering	0.0071
Chui-Lian 2 wavelets with the 0th order orthogonal prefiltering	0.0065

the zeroth-order orthogonal prefiltering for the Chui-Lian multiwavelet is shown in Fig. 15. Their energy compaction ratioes are listed in Table I.

The second test signal is the two hundred and fiftieth horizontal line of the Einstein image with size 256×256. The original signal, the transformed signal with transform T_3 (Daubechies D_4 wavelets), the transformed signal with transform T_5 (the GHM 2 wavelets with the first-order orthogonal prefiltering), and transform signal with transform T_6 (the Chui-Lian 2 wavelets with the zeroth-order orthogonal prefiltering) are shown in Figs. 16–19. Their energy compaction ratioes are listed in Table II with the same definitions as above.

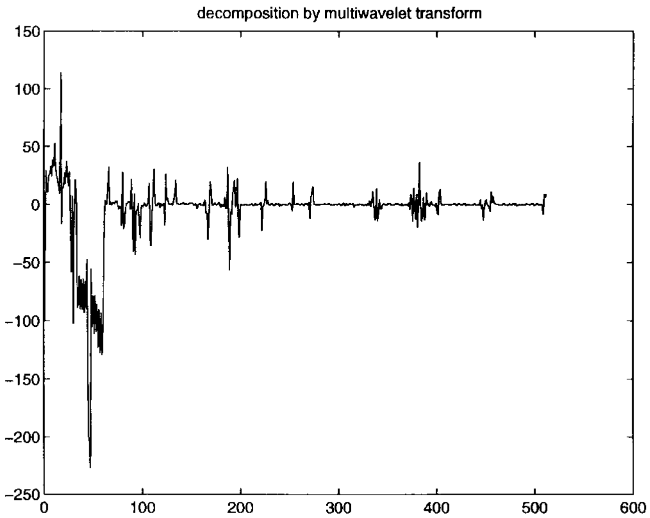


Fig. 14. Decomposition of the first test signal using the GHM 2 wavelets with the new first-order orthogonal prefiltering.

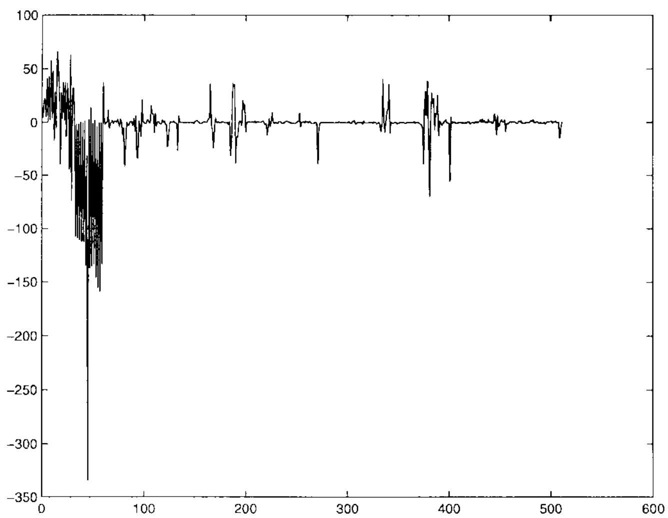


Fig. 15. Decomposition of the first test signal using the Chui-Lian 2 wavelets with the new zeroth-order orthogonal prefiltering.

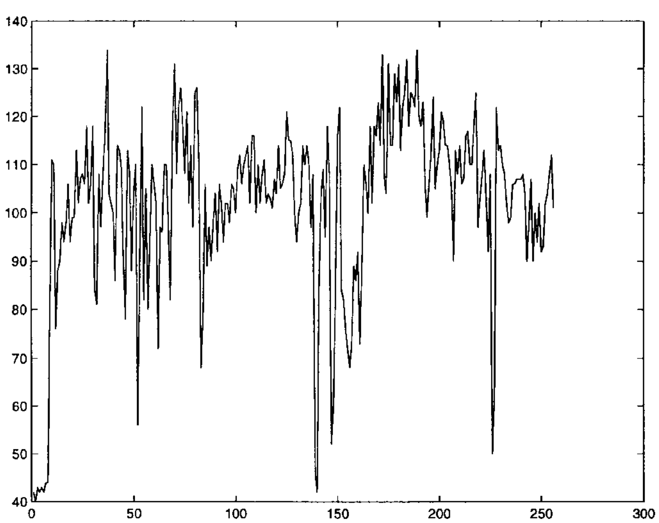


Fig. 16. Second test signal.

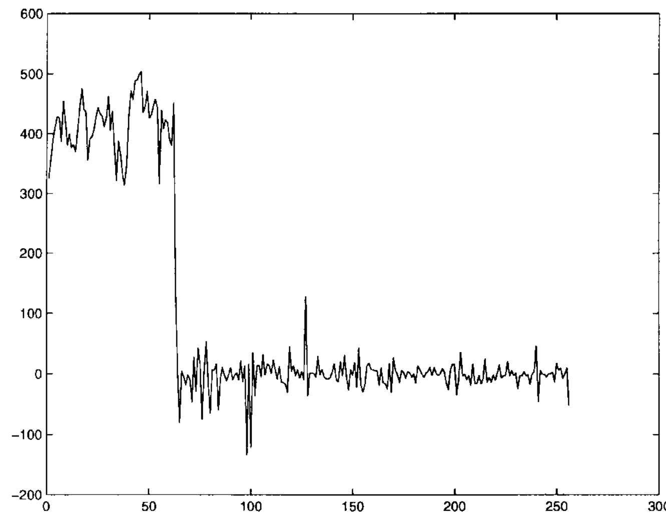


Fig. 17. Decomposition of the second test signal using Daubechies D_4 wavelets.

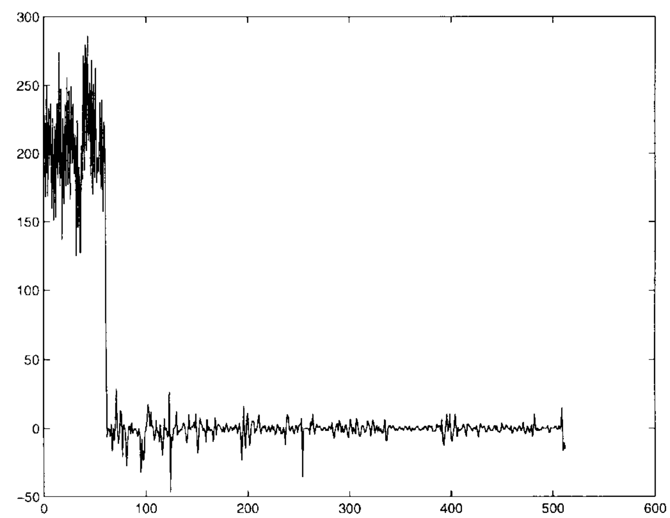


Fig. 18. Decomposition of the second test signal using the GHM 2 wavelets with the new first-order orthogonal prefiltering.

A better energy compaction with the new orthogonal prefilter than with others can be seen from the above tables.

V. Conclusion

In this paper, we have introduced a new prefilter design technique for discrete multiwavelet transforms. The new technique is based on approximating a function with the lowpass property and the orthogonality of their translations by using linear combinations of multiscaling functions and their translations. The new prefiltering is orthogonal but not maximally decimated. It deals with all decomposition steps for discrete multiwavelet transforms, whereas the prefiltering in [1] only focuses on the first step decomposition. The decimation nonmaximality allows one to have more freedom in designing a prefilter so that more desired conditions on the prefilters and the combined filters of the prefilters and multiwavelet vector filters are satisfied. Our numerical examples show that a better energy compaction ratio with the GHM 2 wavelets and the Chui-Lian

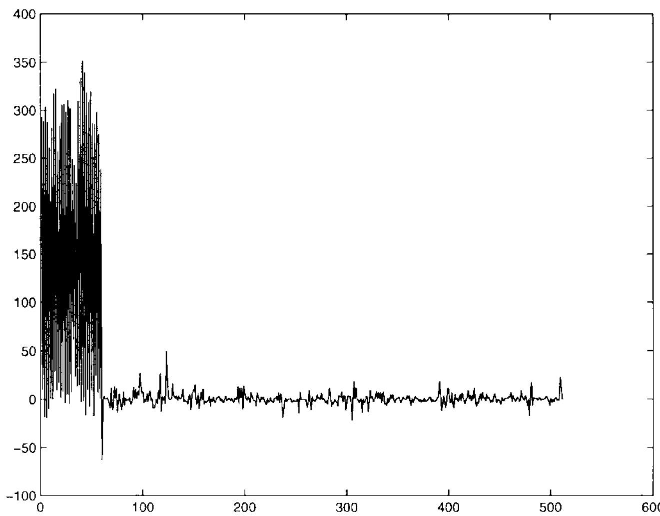


Fig. 19. Decomposition of the second test signal using the Chui-Lian 2 wavelets with the new zeroth-order orthogonal prefiltering.

2 wavelets with the new orthogonal prefiltering than the one with the D_4 wavelet transform is achieved. This suggests the potential applications of discrete multiwavelet transforms in image compression/denoising.

It is known that any nonredundant orthogonal transform keeps the energy. For example, the error energy after the quantization in the transform domain in the compression is equal to the error energy in the reconstruction domain in the decompression. This no longer holds for the redundant prefiltering/postfiltering studied in this paper. In the case when the quantization errors are random, it can be easily shown that the error energy in the reconstruction domain in the decompression is one fourth of the error energy in the transform domain in the compression.

APPENDIX

The error

$$\left| f(t) - \sum_n b_n 2^{J/2} \phi(2^J t - n) \right|$$

where

$$b_n = \int f(t) 2^{J/2} \phi(2^J t - n)\, dt$$

can be estimated as follows. In the Fourier transform domain, the L^2 error can be expressed as

$$\begin{aligned} &\int \left| f(t) - \sum_n b_n 2^{J/2} \phi(2^J t - n) \right|^2 dt \\ &= \sqrt{\frac{2^J}{2\pi}} \int \left| \hat{f}(2^J\omega) - \hat{\phi}(\omega) \sum_n \hat{\phi}(-\omega + 2n\pi) \right. \\ &\qquad \left. \times\ \hat{f}(2^J(\omega - 2n\pi)) \right|^2 d\omega. \end{aligned}$$

When f is bandlimited with bandwidth $2^J\pi$, the error can be simplied as

$$\begin{aligned} &\int \left| f(t) - \sum_n b_n 2^{J/2} \phi(2^J t - n) \right|^2 dt \\ &= \frac{1}{2\pi} \int_{-2^J\pi}^{2^J\pi} |\hat{f}(\omega)|^2 \left(1 - \left| \hat{\phi}\left(\frac{\omega}{2^J} \right) \right|^2 \right) d\omega. \end{aligned}$$

Notice that $\hat{\phi}(0) = 1$. When J is large enough and the bandwidth W of the signal f is much smaller than $2^J\pi$, i.e., $W \ll 2^J\pi$, then

$$\begin{aligned} &\int \left| f(t) - \sum_n b_n 2^{J/2} \phi(2^J t - n) \right|^2 dt \\ &\approx \frac{1}{2\pi} \int_{-W}^{W} |\hat{f}(\omega)|^2 \left(1 - \left| \hat{\phi}\left(\frac{\omega}{2^J} \right) \right|^2 \right) d\omega \approx 0. \end{aligned}$$

This is because $\phi(\omega/2^J) \approx 1$ for large J for $|\omega| \leq W$.

ACKNOWLEDGMENT

The author would like to thank the referees for their useful comments and suggestions that have improved the clarity of this manuscript.

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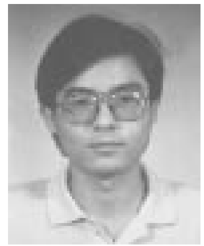

Xiang-Gen Xia (M'97) received the B.S. degree in mathematics from Nanjing Normal University, Nanjing, China, the M.S. degree in mathematics from Nankai University, Tianjin, China, and the Ph.D. degree in electrical engineering from the University of Southern California, Los Angeles, in 1983, 1986, and 1992, respectively.

He was a Lecturer at Nankai University from 1986 to 1988, a Teaching Assistant at the University of Cincinnati, Cincinnati, OH, from 1988 to 1990, a Research Assistant at the University of Southern California from 1990 to 1992, and a Research Scientist at the Air Force Institute of Technology, Wright–Patterson AFB, OH, from 1993 to 1994. He was a Senior/Research Staff Member at Hughes Research Laboratories, Malibu, CA, from 1995 to 1996. In September 1996, he joined the Department of Electrical Engineering, University of Delaware, Newark, where he is currently an Assistant Professor. His current research interests include communication systems including equalization and coding, wavelet transform and multirate filterbank theory and applications, time–frequency analysis and synthesis, and numerical analysis and inverse problems in signal/image processing.

Dr. Xia received the National Science Foundation Faculty Early Career Development (CAREER) Program Award in 1997 and the 1998 Office of Naval research (ONR) Young Investigator Program (YIP) Award. He is currently an Associate Editor of the IEEE TRANSACTIONS ON SIGNAL PROCESSING. He is also a member of the American Mathematical Society.