Unitary Signal Constellations for Differential Space–Time Modulation With Two Transmit Antennas: Parametric Codes, Optimal Designs, and Bounds

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***Abstract*—Differential space–time modulation has been recently proposed in the literature for multiple-antenna systems over Rayleigh-fading channels, where neither the transmitter nor the receiver knows the fading coefficients. For the practical success of differential space–time modulation, it has been shown critical to design unitary space–time signal constellations with large diversity product which is a primary property for the signal constellations to have good performance in high signal-to-noise ratio (SNR) scenarios.**

In this paper, we focus on the design of unitary signal constellations for differential space–time modulation with double transmit antennas. By using the parametric form of a two-by-two unitary matrix, we present a class of unitary space–time codes called *parametric codes* and show that this class of unitary space–time codes leads to a five-signal constellation with the largest possible diversity product and a 16-signal constellation with the largest known diversity product. Although the parametric code of size 16 is not a group by itself, we show that it is a subset of a group of order 32. Furthermore, the unitary signal constellations of sizes 32, 64, 128, and 256 obtained by taking the subsets of the parametric codes of sizes 37, 75, 135, and 273, respectively, have the largest known diversity products.

We also use large diversity sum of unitary space–time signal constellations as another significant property for the signal constellations to have good performance in low-SNR scenarios. The newly introduced unitary space–time codes can lead to signal constellations with sizes of 5 and 9 through 16 that have the largest possible diversity sums. Subsequently, we construct a few sporadic unitary signal constellations with the largest possible diversity product or diversity sum. A four-signal constellation which has both the largest possible diversity product and the largest possible diversity sum and three unitary signal constellations with the largest possible diversity sums for sizes of 6, 7, and 8 are constructed, respectively. Furthermore, by making use of the existing results in sphere packing and spherical codes, we provide several upper and lower bounds on the largest possible diversity product and the largest possible diversity sum that unitary signal constellations of any size can achieve.

***Index Terms*—Differential space–time modulation, diversity products, diversity sums, Rayleigh-fading channels, sphere packing, spherical codes, transmitter diversity, unitary matrices, unitary space–time codes, wireless communications.**

I. Introduction

In the last several years, there has been considerable interest in the wireless communication link using multiple transmit antennas for the Rayleigh-fading channel models. The basic information-theoretic results of transmit diversity suggest that the capacity of a communication link with multiple transmit antennas can remarkably exceed that of a single-antenna link [36], [7], [8], [22], [45]. There have also been several coding and modulation schemes proposed to exploit the potential increase in the capacity through space diversity. For the coherent multiple-antenna channel, several transmit diversity methods have been presented in [35], [34], [25] and references therein (see, e.g., [2], [9], [12], [26], [27], [30], [38]–[41]). Specifically, Tarokh, Seshadri, and Calderbank [35] proposed space–time codes which combine signal processing at the receiver with coding techniques appropriate to multiple transmit antennas. For the noncoherent multiple-antenna channel, Marzetta and Hochwald [22] proposed a general signaling scheme, called unitary space–time modulation, and showed that this scheme can achieve a high ratio of channel capacity in combination with channel coding. The design of unitary space–time constellations was investigated in [15] and [1].

In the recent literature [16], [18], [33], [37], differential modulation techniques for multiple transmit antennas have been proposed, which can be regarded as a natural generalization of the standard differential phase-shift keying (DPSK) used in the single-antenna unknown-channel link. In this paper, we focus on the differential unitary space–time modulation scheme independently proposed by Hochwald and Sweldens in [16] and Hughes in [18]. In differential unitary space–time modulation, the information messages are transmitted through the unitary space–time constellations. It has been shown in [16] and [18] by utilizing the result in [13] that, in high signal-to-noise ratio (SNR) situations, the performance of differential space–time modulation, in terms

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of the block error rate, is dominantly determined by the diversity product of the unitary space–time constellations and hence that the design of unitary space–time constellations with large diversity products is crucial for the good performance of differential unitary modulation schemes. In [16] and [18], a diagonal cyclic code was developed. A fast decoding algorithm for diagonal codes was proposed in [4]. In [31] and [19], the unitary space–time group codes with positive diversity product were extensively investigated. When the constellation size is a power of two, Hughes [19] obtained the characterization of space–time group codes which are either diagonal cyclic codes or dicyclic group codes. In a recent work [31], a thorough classification of unitary space–time group codes of any finite order was presented. The best one among the space–time group codes can therefore be found by an exhaustive computer search in a finite set of unitary signal constellations. It is remarked that the group code has the practical merit that every transmitted signal is still a codeword in the group code and, consequently, can be determined by a simple group table lookup.

The primary purpose of the current paper is to design unitary space–time codes for the differential modulation scheme with double transmit antennas. By using the parameterization of unitary groups, we construct a class of unitary signal constellations, called *parametric codes*, for two-transmit-antenna systems. The parametric codes are demonstrated to have a significant performance improvement over the cyclic group codes. Remarkably, the parametric codes lead to a five-signal constellation with the largest possible diversity product and a 16-signal constellation with the largest known diversity product. Compared with the existing unitary space–time codes for two-transmit-antenna systems, the above generated 16-signal constellation has an improvement in terms of the block error rate up to 1 dB at SNR 22 dB in the case of two receive antennas and at SNR 10 dB in the case of five receive antennas. We also show that the unitary signal constellations of sizes 32, 64, 128, and 256 obtained by taking the subsets of the parametric codes of sizes 37, 75, 135, and 273, respectively, have the largest known diversity products in the literature. Furthermore, for two-transmit-antenna systems, we employ the diversity sum of unitary signal constellations as another efficient metric for good performance of the signal constellations in low-SNR situations. The parametric codes can also lead to unitary signal constellations with the largest possible diversity sums for sizes of 5 and 9–16. A few sporadic unitary constellations with the largest possible diversity product and/or sum for sizes of 4, 6, 7, and 8 are also presented. Finally, by making use of extensive results in sphere packing and spherical codes, we present some upper and lower bounds on the largest possible diversity product and the largest possible diversity sum of unitary signal constellations with any size. The *largest possible* diversity product and sum are also called the *optimal* diversity product and sum, respectively. Notice that the optimal diversity product is always *smaller than or equal to* the optimal diversity sum for any constellation size, as we shall see later. It will further be shown that, while the two quantities are equal for constellation sizes of 2 through 5, the optimal diversity product of a unitary signal constellation is *strictly smaller than* the optimal diversity sum for constellation sizes of 6–9. A main result for large-size signal constellations is that for the 2×2 unitary signal constellations, the optimal diversity product and sum are of an order between $L^{-1/3}$ and $L^{-1/4}$, where L is the constellation size. For general $M \times M$ $(M \geq 2)$ unitary signal constellations, the optimal diversity product and sum are of an order not greater than L^{-1/M^2} for large constellation size L.

This paper is organized as follows. In Section II, some preliminaries in differential unitary space–time modulation and a design criterion for unitary signal constellations with large diversity product and/or diversity sum are presented. In Section III, a novel class of unitary space–time signal constellations for double transmit antennas, i.e., the parametric codes, are developed. Some numerical results in terms of the diversity product and sum and the block error rate are also given for the comparison among the existing known unitary space–time codes. In Section IV, we construct a four-signal constellation with the optimal diversity product and sum and three signal constellations of sizes 6, 7, and 8 with the optimal diversity sums, none of which belongs to the class of parametric codes. In Section V, some upper and lower bounds on the optimal diversity product and the optimal diversity sum for unitary signal constellations with any size are obtained. Some asymptotic upper and lower bounds on the optimal diversity product and sum for large-size signal constellations are also presented. In Section VI, we make some concluding remarks.

II. Differential Unitary Space–Time Modulation and a Criterion for Designing Unitary Signal Constellations

In this section, we present some necessary preliminaries about the differential unitary space–time modulation scheme proposed by Hochwald and Sweldens in [16] and Hughes in [18] for the Rayleigh-fading channel model, where the channel fading coefficients are unknown to both the transmitter and the receiver, an upper bound for the block probability of error and a design criterion for unitary signal constellations.

A. Differential Unitary Space–Time Modulation

In what follows, we adopt the relevant notations used in [16]. An $M \times M$ complex matrix V is called unitary if

$$V^{\mathrm{H}}V = I_M$$

where I_M is the $M \times M$ identity matrix and the superscript H stands for the complex conjugate transpose or Hermitian transpose of a complex matrix. An $M \times M$ unitary signal constellation of size $L \geq 2$ means a subset of

$$\mathcal{V} = \{V_\ell \,|\, V_\ell^{\mathrm{H}} V_\ell = I_M,\, \ell \in \mathbb{Z}_L\}$$

where the index set of signals is

$$\mathbb{Z}_L \stackrel{\mathrm{def}}{=} \{0,\, 1,\, \ldots,\, L-1\}.$$

For an $M \times N$ complex matrix $A = (a_{mn})$, its Frobenius norm or Euclidean norm is defined by [21]

$$\|A\|_F = \sqrt{\mathrm{Tr}\,(A^{\mathrm{H}}A)} = \sqrt{\mathrm{Tr}\,(AA^{\mathrm{H}})} = \left(\sum_{m=1}^{M}\sum_{n=1}^{N}|a_{mn}|^2\right)^{1/2}$$

where $\mathrm{Tr}\,(\cdot)$ denotes the trace of the argument matrix.

Consider a communication link with M transmit antennas and N receive antennas operating in a Rayleigh flat-fading environment, which can be described by the following channel model [16]:

$$X_\tau = \sqrt{\rho}\, S_\tau H_\tau + W_\tau, \qquad \tau = 0, 1, 2, \ldots \tag{1}$$

where τ is the index of the time block within which

$$t = \tau M, \tau M + 1, \ldots, \tau M + M - 1$$

time samples are assembled in order, $S_\tau = (s_{tm})$ the transmitted $M \times M$ matrix-valued signal whose expected total power at any time t is normalized to be one, i.e.,

$$\mathbf{E} \sum_{m=1}^{M} |s_{tm}|^2 = 1$$

where $\mathbf{E}$ denotes expectation, $X_\tau = (x_{tn})$ the received $M \times N$ matrix-valued signal, $H_\tau = (h_{mn}^\tau)$ the $M \times N$ channel fading-coefficient matrix, $W_\tau = (w_{tn})$ the additive $M \times N$ matrix-valued noise, and ρ is the expected SNR at each receive antenna, which does not depend on the number of transmit antennas M. Here, the subscripts t, m, and n satisfy

$$\tau M \le t \le \tau M + M - 1, \quad 1 \le m \le M, \quad \text{and } 1 \le n \le N$$

in the τth time block. We assume that the additive noise w_{tn} at time t and receiving antenna n is independent, with respect to both t and n, identically zero-mean and unit-variance complex Gaussian distributed $\mathcal{CN}(0, 1)$ and that the fading coefficients h_{mn}^τ are constant in the τth time block, independently of the time $t = \tau M, \tau M+1, \ldots, \tau M + M - 1$, and also independent identically complex normal $\mathcal{CN}(0, 1)$ distributed with respect to m and n. The fading-coefficient matrix H_τ indexed by τ is assumed to be nearly equal to its adjacent fading-coefficient matrices, i.e., $H_\tau \approx H_{\tau+1}$ for $\tau = 0, 1, 2, \ldots$.

In a single time block of size M there are M channel uses, and a transmission rate R requires $L = 2^{RM}$ different signals. Each signal is an $M \times M$ unitary matrix V_ℓ from a signal constellation $\mathcal{V}$ consisting of $L \ge 2$ such distinct unitary matrices. We assume that the data to be transmitted is an integer sequence $z_1, z_2, \ldots$ with $z_\tau \in \mathbb{Z}_L$ for $\tau = 1, 2, \ldots$. The transmitted signal sequence is then determined by the following *fundamental differential encoding or transmitter equations* [16]:

$$S_\tau = V_{z_\tau} S_{\tau-1}, \qquad \tau = 1, 2, \ldots \tag{2}$$

where the initial transmitted signal S_0 can be any given $M \times M$ unitary matrix. Therefore, the transmitted signal in time block τ is a product of $\tau + 1$ many $M \times M$ unitary matrices as follows:

$$S_\tau = \left(\prod_{k=1}^{\tau} V_{z_{\tau-k+1}} \right) S_0, \qquad \tau = 1, 2, \ldots$$

which is still an $M \times M$ unitary matrix and therefore satisfies the power normalization. If the initial transmitted signal matrix $S_0 = I_M$, then the set of all possibly transmitted signals S_τ for $\tau = 0, 1, 2, \ldots$ is a semigroup [20] which is finitely generated by the signal constellation $\mathcal{V}$. This conclusion is actually true if only the initial transmitted signal matrix S_0 belongs to the above semigroup. Here, the binary operation on the semigroup is the usual matrix multiplication. In the following, for simplicity we assume that the initial transmitted signal matrix $S_0 = I_M$, although it is not necessary for the differential modulation scheme.

Substituting (2) into the channel model (1) and using the assumption of $H_\tau \approx H_{\tau-1}$ for $\tau = 1, 2, \ldots$, we can obtain

$$\begin{aligned} X_\tau &= \sqrt{\rho}\, V_{z_\tau} S_{\tau-1} H_\tau + W_\tau \\ &= V_{z_\tau} X_{\tau-1} + W_\tau - V_{z_\tau} W_{\tau-1}, \qquad \tau = 1, 2, \ldots. \end{aligned} \tag{3}$$

We define

$$W'_\tau = \frac{1}{\sqrt{2}} (W_\tau - V_{z_\tau} W_{\tau-1}) \tag{4}$$

then we can rewrite (3) as the following *fundamental differential receiver equations* [16]:

$$X_\tau = V_{z_\tau} X_{\tau-1} + \sqrt{2}\, W'_\tau, \qquad \tau = 1, 2, \ldots$$

where W'_τ defined by (4) is an $M \times N$ matrix with additive independent $\mathcal{CN}(0, 1)$ distributed noise entries. The maximum-likelihood (ML) demodulator for differential space–time modulation is given by [13], [16]

$$\begin{aligned} \hat{z}_\tau^{\mathrm{ML}} &= \arg\min_{\ell \in \mathbb{Z}_L} \|X_\tau - V_\ell X_{\tau-1}\|_F \\ &= \arg\max_{\ell \in \mathbb{Z}_L} \operatorname{Tr}(X_{\tau-1}^{\mathrm{H}} V_\ell^{\mathrm{H}} X_\tau + X_\tau^{\mathrm{H}} V_\ell X_{\tau-1}), \\ &\qquad \tau = 1, 2, \ldots. \end{aligned} \tag{5}$$

B. An Upper Bound on the Block Error Rate

The pairwise probability $P_{\ell,\ell'}$ of mistaking V_ℓ for $V_{\ell'}$ ($\forall\, \ell, \ell' \in \mathbb{Z}_L,\ \ell \ne \ell'$) or *vice versa* for the ML demodulator (5) has a closed-form expression of [13], [16]

$$\begin{aligned} P_{\ell,\ell'} &= \operatorname{Prob}\{\text{choose } V_{\ell'} \mid V_\ell \text{ transmitted}\} \\ &= \operatorname{Prob}\{\text{choose } V_\ell \mid V_{\ell'} \text{ transmitted}\} \\ &= \frac{1}{2\pi} \int_0^\infty \frac{1}{\omega^2 + \frac{1}{4}} \prod_{m=1}^{M} \left[1 + \frac{\rho^2 \sigma_m^2}{1 + 2\rho} \left(\omega^2 + \frac{1}{4} \right) \right]^{-N} d\omega \\ &= \frac{1}{\pi} \int_0^{\pi/2} \prod_{m=1}^{M} \left[\frac{\cos^2\theta + 1 - \operatorname{sgn}(\sigma_m^2)}{\cos^2\theta + \frac{\rho^2}{4(1+2\rho)}\, \sigma_m^2 + 1 - \operatorname{sgn}(\sigma_m^2)} \right]^N d\theta \end{aligned} \tag{6}$$

where $\sigma_m \stackrel{\text{def}}{=} \sigma_m(V_\ell - V_{\ell'})$ represents the mth singular value of the $M \times M$ difference matrix $V_\ell - V_{\ell'}$ for $m = 1, 2, \ldots, M$, and $\operatorname{Prob}\{\cdot\}$ the probability of a random event, and the function $\operatorname{sgn}(\cdot)$ is the signum function defined as 0 if the argument variable is zero and 1 if the variable is larger than zero, and the last equality in (6) utilizes the following coordinate transformation [14]:

$$\omega = \frac{1}{2} \tan\theta.$$

The pairwise probability of error $P_{\ell,\ell'}$ has the Chernoff upper bound [13], [16]

$$P_{\ell,\ell'} \le \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{\rho^2 \sigma_m^2}{4(1+2\rho)} \right]^{-N}. \tag{7}$$

Furthermore, it is clear that the pairwise probability of error $P_{\ell,\ell'}$ can generally be bounded from above by the following summation of a finite integral and a positive number:

$$P_{\ell,\ell'} \le \frac{1}{2\pi}\int_0^a \frac{1}{\omega^2+\frac{1}{4}} \prod_{m=1}^{M}\left[1+\frac{\rho^2\sigma_m^2}{1+2\rho}\left(\omega^2+\frac{1}{4}\right)\right]^{-N} d\omega + \left(\frac{1}{2}-\frac{\arctan(2a)}{\pi}\right)\prod_{m=1}^{M}\left[1+\frac{\rho^2\sigma_m^2}{1+2\rho}\left(a^2+\frac{1}{4}\right)\right]^{-N} \tag{8}$$

for all $a \ge 0$.

Let the right-hand side of (8) be denoted by $F(a)$, which is a function in terms of $a \ge 0$. Clearly, $F(0)$ is the Chernoff bound (7). The function $F(a)$ has the following properties.

Proposition 1: The function $F(a)$ in terms of $a \ge 0$ given by the right-hand side of (8) satisfies the following conditions.

1) $F(a)$ is monotonically decreasing for $a \ge 0$ and tends to $P_{\ell,\ell'}$ as $a \to +\infty$.
2) When $F(a)$ is used to numerically evaluate the pairwise error probability $P_{\ell,\ell'}$, the nonnegative relative error

$$\frac{F(a)-P_{\ell,\ell'}}{P_{\ell,\ell'}} \le \frac{\pi}{2\arctan(2a)} - 1, \qquad \text{for } a > 0$$

which is less than 0.0032 when $a \ge 100$.

3) When the SNR ρ is large, the pairwise error probability $P_{\ell,\ell'}$ and its Chernoff bound in the right-hand side of (7), i.e., $F(0)$, decay at a rate of the same order. To be precise, if $\tilde{M} \ge 1$ is the number of nonzero singular values of the $M \times M$ difference matrix $V_\ell - V_{\ell'}$, then, for large SNR ρ

$$\frac{P_{\ell,\ell'}}{F(0)} = \prod_{k=1}^{\tilde{M}N}\left(1-\frac{1}{2k}\right) + o_\rho(1)$$

where $o_\rho(1)$ represents a variable in terms of ρ which approaches zero as ρ tends to infinity.

Proof: See Appendix A. □

We assume that the L transmitted unitary signals are equally probable *a priori*. Then, the performance of a general constellation consisting of unitary space–time signals can be measured by the following Chernoff union bound on the block probability of error P_e [15], [16]

$$\begin{aligned} P_e &= \frac{1}{L}\sum_{\ell=0}^{L-1} \mathrm{Prob}\{\text{error} \mid V_\ell \text{ transmitted}\} \\ &\le \frac{2}{L}\sum_{\ell=0}^{L-2}\sum_{\ell'=\ell+1}^{L-1} P_{\ell,\ell'} \\ &\le \frac{1}{L}\sum_{\ell=0}^{L-2}\sum_{\ell'=\ell+1}^{L-1}\prod_{m=1}^{M}\left[1+\frac{\rho^2\sigma_m^2}{4(1+2\rho)}\right]^{-N}. \end{aligned} \tag{9}$$

We shall use the above first inequality and Property 2) in Proposition 1 to evaluate numerically the block probability of error P_e in the subsequent section.

C. A Design Criterion for Unitary Signal Constellations

The Chernoff bound on the pairwise error probability $P_{\ell,\ell'}$, given by the right-hand side of (7), can be rewritten as

$$P_{\ell,\ell'} \le \frac{1}{2}\left[1+\sum_{m=1}^{M}\tilde{\rho}^m E_m\right]^{-N} \tag{10}$$

where

$$\tilde{\rho} \stackrel{\text{def}}{=} \frac{\rho^2}{4(1+2\rho)}$$

and

$$E_m = E_m(V_\ell - V_{\ell'}) \stackrel{\text{def}}{=} \sum_{1\le i_1<\cdots<i_m\le M}\prod_{k=1}^{m}\sigma_{i_k}^2(V_\ell - V_{\ell'})$$

for $m = 1, 2, \ldots, M$. Moreover, the Chernoff union bound on the block probability of error P_e, given by the right-hand side of (9), can also be represented by

$$P_e \le \frac{1}{L}\sum_{\ell=0}^{L-2}\sum_{\ell'=\ell+1}^{L-1}\left[1+\sum_{m=1}^{M}\tilde{\rho}^m E_m\right]^{-N}.$$

We may give a geometrical interpretation of E_m for $m = 1, 2, \ldots, M$ as follows. Let $\Omega = \Omega(\sigma_1^2, \sigma_2^2, \ldots, \sigma_M^2)$ be a hypercube in the M-dimensional Euclidean real space $\mathbb{R}^M$ defined by the set

$$\begin{aligned} &(\lambda_1\sigma_1^2, \lambda_2\sigma_2^2, \ldots, \lambda_M\sigma_M^2)^\top \\ &\quad = \lambda_1(\sigma_1^2, 0, \ldots, 0)^\top + \lambda_2(0, \sigma_2^2, \ldots, 0)^\top \\ &\qquad + \cdots + \lambda_M(0, \ldots, 0, \sigma_M^2)^\top, \\ &\qquad\qquad 0 \le \lambda_m \le 1, \ m = 1, 2, \ldots, M \end{aligned}$$

where the superscript $\top$ denotes the transpose of a vector. Then, E_m is a sum of the m-dimensional volumes of those m-dimensional faces of Ω each of which has the origin in $\mathbb{R}^M$ as one of its vertices. It is clear that there are

$$\binom{M}{m} \stackrel{\text{def}}{=} \frac{M!}{m!(M-m)!} = \prod_{k=1}^{m}\left(1+\frac{M-m}{k}\right)$$

many such m-dimensional faces having a vertex in the origin. For example, in the case of $M = 3$, we have

$$E_1 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$$
$$E_2 = \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2$$

and

$$E_3 = \sigma_1^2\sigma_2^2\sigma_3^2.$$

Moreover, for the general case of $M \ge 1$

$$E_1 = \sum_{m=1}^{M}\sigma_m^2$$

and

$$E_M = \prod_{m=1}^{M}\sigma_m^2$$

which are the sum and product of the squared singular values σ_m^2 for all $m = 1, 2, \ldots, M$, respectively.

It is seen from (10) that the Chernoff bound on the pairwise probability of error $P_{\ell,\ell'}$ is small when the terms E_m for all $m = 1, 2, \ldots, M$ are large. The Chernoff bound (9) on the block probability of error P_e is small when the terms $E_m(V_\ell - V_{\ell'})$

are large for all $m = 1, 2, \ldots, M$ and for all $0 \le \ell < \ell' \le L-1$. Now, we want to introduce some quantities that are closely related to the evaluation of the pairwise probability of error and the block probability of error.

For any two $M \times M$ unitary matrices V_1 and V_2, we define M quantities that reflect the dissimilarity between the two matrices as follows:

$$D_m(V_1, V_2) = \frac{1}{2}\left(\frac{E_m(V_1 - V_2)}{\binom{M}{m}}\right)^{1/(2m)}, \quad m = 1, 2, \ldots, M. \tag{11}$$

In the extreme cases of $m = 1$ and $m = M$, the quantities $D_m(V_1, V_2)$ are related to the Frobenius norm and determinant of the difference matrix $V_1 - V_2$, respectively. We rewrite them as

$$D_{\text{euc}}(V_1, V_2) \stackrel{\text{def}}{=} D_1(V_1, V_2) = \frac{1}{2\sqrt{M}}\|V_1 - V_2\|_F = \frac{1}{2\sqrt{M}}\left(\sum_{m=1}^{M}\sigma_m^2(V_1 - V_2)\right)^{1/2}$$

and

$$D_{\det}(V_1, V_2) \stackrel{\text{def}}{=} D_M(V_1, V_2) = \frac{1}{2}\sqrt[M]{|\det(V_1 - V_2)|} = \frac{1}{2}\left(\prod_{m=1}^{M}\sigma_m(V_1 - V_2)\right)^{1/M}$$

where $\det(\cdot)$ denotes the determinant of the argument matrix. In the following, the quantities $D_{\text{euc}}(V_1, V_2)$ and $D_{\det}(V_1, V_2)$ are called the normalized Euclidean distance and normalized determinant dissimilarity between the two matrices V_1 and V_2, respectively. In addition, the quantity $\|V_1 - V_2\|_F$ is called the Euclidean distance between V_1 and V_2.

For any given $M \times M$ unitary signal constellation $\mathcal{V}$ of size L, namely, $\mathcal{V} = \{V_\ell \mid V_\ell^{\mathrm{H}} V_\ell = I_M, \ \ell \in \mathbb{Z}_L\}$, we may define the following M quantities that reflect the minimum dissimilarity between any two different unitary signals in $\mathcal{V}$ as follows:

$$\xi_m(L, \mathcal{V}) = \min_{0 \le \ell < \ell' \le L-1} D_m(V_\ell, V_{\ell'}), \quad m = 1, 2, \ldots, M. \tag{12}$$

In the extreme cases of $m = 1$ and $m = M$, the quantities $\xi_m(L, \mathcal{V})$ are, respectively,

$$\delta(L, \mathcal{V}) \stackrel{\text{def}}{=} \xi_1(L, \mathcal{V}) = \frac{1}{2\sqrt{M}}\min_{0 \le \ell < \ell' \le L-1}\|V_\ell - V_{\ell'}\|_F = \frac{1}{2\sqrt{M}}\min_{0 \le \ell < \ell' \le L-1}\left(\sum_{m=1}^{M}\sigma_m^2(V_\ell - V_{\ell'})\right)^{1/2} \tag{13}$$

and

$$\zeta(L, \mathcal{V}) \stackrel{\text{def}}{=} \xi_M(L, \mathcal{V}) = \frac{1}{2}\min_{0 \le \ell < \ell' \le L-1}\sqrt[M]{|\det(V_\ell - V_{\ell'})|} = \frac{1}{2}\min_{0 \le \ell < \ell' \le L-1}\left(\prod_{m=1}^{M}\sigma_m^2(V_\ell - V_{\ell'})\right)^{\frac{1}{2M}}. \tag{14}$$

In [16], the quantity $\zeta(L, \mathcal{V})$ is called the *diversity product* of the constellation $\mathcal{V}$, which is represented in terms of the minimum among the *products* of the squared singular values for all difference signal matrices. Analogously, we may call $\delta(L, \mathcal{V})$ the *diversity sum* of the constellation $\mathcal{V}$, since it is represented in terms of the minimum among the *sums* of the squared singular values for all difference signal matrices. The M quantities defined by (12) possess the following properties.

Proposition 2: For any given $M \times M$ unitary signal constellation $\mathcal{V}$ of size L, the nonnegative quantities $\xi_m(L, \mathcal{V})$ given by (12) for $m = 1, 2, \ldots, M$ and $L \ge 2$ satisfy the following conditions.

1) For each $m = 1, 2, \ldots, M-1$
$$\xi_m(L, \mathcal{V}) \ge \xi_{m+1}(L, \mathcal{V})$$
and for each $m = 2, 3, \ldots, M-1$
$$\xi_m^{2m}(L, \mathcal{V}) \ge \xi_{m+1}^{m+1}(L, \mathcal{V})\xi_{m-1}^{m-1}(L, \mathcal{V}).$$

2) If $2 \le L \le 2M^2 + 1$, then
$$\xi_m(L, \mathcal{V}) \le \sqrt{\frac{L}{2(L-1)}}, \qquad \text{for all } m = 1, 2, \ldots, M. \tag{15}$$
In the case $m = 1$, the above inequality holds with equality if and only if any two distinct matrices in $\mathcal{V}$ have the same normalized Euclidean distance and that the sum of all the L signal matrices in $\mathcal{V}$ is an $M \times M$ all-zero matrix.

3) If $2M^2 + 1 < L \le 4M^2$, then
$$\xi_m(L, \mathcal{V}) \le \frac{1}{\sqrt{2}}, \qquad \text{for all } m = 1, 2, \ldots, M.$$

4) If $L > 4M^2$, then
$$\xi_m(L, \mathcal{V}) < \frac{1}{\sqrt{2}}, \qquad \text{for all } m = 1, 2, \ldots, M.$$

Proof: See Appendix B. □

According to (10) and (11), the Chernoff bound on the pairwise probability of error $P_{\ell,\ell'}$ is small when the dissimilarity quantities $D_m(V_\ell, V_{\ell'})$ for all $m = 1, 2, \ldots, M$ are large. Therefore, when the minimum-dissimilarity quantities $\xi_m(L, \mathcal{V})$ of the signal constellation $\mathcal{V}$, defined by (12), are large for all $m = 1, 2, \ldots, M$, the Chernoff bound (9) on the block probability of error P_e becomes small correspondingly, at any SNR ρ. Moreover, it is easy to see that the diversity product, i.e., $\xi_M(L, \mathcal{V})$, is crucial for the performance of the unitary space–time constellations at high-SNR ρ, while the diversity sum, i.e., $\xi_1(L, \mathcal{V})$, is at low-SNR ρ (see also [16], [18]). For the sake of simplicity, we shall only consider to design the unitary signal constellation $\mathcal{V}$ with diversity sum (13) and diversity product (14) as large as possible. If the unitary signal constellation $\mathcal{V}$ has the *largest possible* diversity product (14) (respectively, diversity sum (13)), then we say that the constellation has an *optimal* diversity product (respectively, optimal diversity sum).

In the sequel, we shall focus on the design of unitary signal constellations for differential space–time modulation with $M = 2$ transmit antennas while allowing any number of receiving antennas $N \geq 1$. Then, a signal constellation we shall consider consists of 2×2 unitary matrices. Our design objective is to find a unitary signal constellation with large minimum normalized Euclidean distance and/or normalized determinant dissimilarity, or equivalently, with large diversity sum and/or diversity product. The diversity sum and diversity product of a 2×2 unitary signal constellation

$$\mathcal{V} = \{V_\ell \,|\, V_\ell^{\mathrm{H}} V_\ell = I_2, \, \ell \in \mathbb{Z}_L\}$$

of size $L \geq 2$ are, respectively, given by

$$\delta(L, \mathcal{V}) = \min_{0 \leq \ell < \ell' \leq L-1} D_{\mathrm{euc}}(V_\ell, V_{\ell'}) = \frac{1}{2\sqrt{2}} \min_{0 \leq \ell < \ell' \leq L-1} \|V_\ell - V_{\ell'}\|_F \tag{16}$$

and

$$\zeta(L, \mathcal{V}) = \min_{0 \leq \ell < \ell' \leq L-1} D_{\det}(V_\ell, V_{\ell'}) = \frac{1}{2} \min_{0 \leq \ell < \ell' \leq L-1} \sqrt{|\det(V_\ell - V_{\ell'})|}. \tag{17}$$

III. A Class of Unitary Space–Time Signal Constellations

In this section, we use the parametric form of 2×2 unitary matrices to construct a class of unitary signal constellations. We shall see that this construction method can lead to a five-signal constellation with both the optimal diversity product and the optimal diversity sum and a 16-signal constellation with the largest known diversity product in addition to the optimal diversity sum. Moreover, the unitary signal constellations for sizes of 9–15 in the class have the optimal diversity sums.

A. A Class of Unitary Signal Constellations for Double Transmit Antennas

Let the positive integer $L \geq 2$ denote the size of a unitary signal constellation, $\theta_L \stackrel{\mathrm{def}}{=} 2\pi/L$, and $j \stackrel{\mathrm{def}}{=} \sqrt{-1}$ being the imaginary unit in the complex plane $\mathbb{C}$. For any given three integers $k_1, k_2, k_3 \in \mathbb{Z}_L$, we define the 2×2 unitary matrix $A(k_1, k_2, k_3)$ as a product of three 2×2 unitary matrices as follows:

$$A(k_1, k_2, k_3) = \begin{pmatrix} e^{j\theta_L} & 0 \\ 0 & e^{jk_1\theta_L} \end{pmatrix} \begin{pmatrix} \cos(k_2\theta_L) & \sin(k_2\theta_L) \\ -\sin(k_2\theta_L) & \cos(k_2\theta_L) \end{pmatrix} \times \begin{pmatrix} e^{jk_3\theta_L} & 0 \\ 0 & e^{-jk_3\theta_L} \end{pmatrix}$$

and then construct the following unitary signal constellation of size L:

$$\mathcal{V}(k_1, k_2, k_3) = \{A_\ell(k_1, k_2, k_3) \,|\, \ell \in \mathbb{Z}_L\} \tag{18}$$

where the 2×2 unitary matrix $A_\ell(k_1, k_2, k_3)$ is defined by the following product of three powers of 2×2 unitary matrices:

$$A_\ell(k_1, k_2, k_3) = \begin{pmatrix} e^{j\theta_L} & 0 \\ 0 & e^{jk_1\theta_L} \end{pmatrix}^\ell \begin{pmatrix} \cos(k_2\theta_L) & \sin(k_2\theta_L) \\ -\sin(k_2\theta_L) & \cos(k_2\theta_L) \end{pmatrix}^\ell \times \begin{pmatrix} e^{jk_3\theta_L} & 0 \\ 0 & e^{-jk_3\theta_L} \end{pmatrix}^\ell. \tag{19}$$

For any given constellation size L, we select a unitary signal constellation from the following constellation class:

$$\mathcal{C}_L \stackrel{\mathrm{def}}{=} \{\mathcal{V}(k_1, k_2, k_3) \,|\, k_1, k_2, k_3 \in \mathbb{Z}_L\} \tag{20}$$

such that the unitary signal constellation has the largest diversity product and/or the largest diversity sum in the constellation class (20). We call the found signal constellation $\mathcal{V}(k_1, k_2, k_3)$ for some $k_1, k_2, k_3 \in \mathbb{Z}_L$ as *parametric code*, since every signal matrix in the constellation possesses the *parametric form* of 2×2 unitary matrices, as shown in (19). It is seen that when the condition $k_2 = k_3 = 0$ is imposed in the constellation class (20), the parametric code (18) is exactly the *diagonal cyclic code* in the case $M = 2$ [16], [18], [19].

For the signal constellation class $\mathcal{C}_L$ given by (20), we have the following result.

Theorem 1: Let $L = 2^p$, where $p \in \mathbb{N}$ and $\mathbb{N}$ is the set of all positive integers. The signal constellation $\mathcal{V}(k_1, k_2, k_3)$ in the constellation class $\mathcal{C}_L$ given by (20) has a positive diversity sum (16) if and only if none of the following three cases occurs.

1) $p \geq 3$, $k_1 \equiv 3 \pmod 4$, and $(k_2, k_3) = (\text{odd, odd})$.
2) $p \geq 1$ and $(k_1, k_2, k_3) = (\text{odd, even, odd})$.
3) $p \geq 1$ and $(k_1, k_2, k_3) = (\text{odd, odd, even})$, where $k_i = \text{odd}$ and $k_i = \text{even}$ for $i = 1, 2, 3$ represent that $k_i \in \mathbb{Z}_L$ is an odd and even number, respectively.

Proof: See Appendix C. □

In the case of the constellation size L being a power of 2, Theorem 1 can be utilized to reduce the search range of $k_1, k_2, k_3 \in \mathbb{Z}_L$ such that the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ has the largest diversity sum and/or product in the constellation class $\mathcal{C}_L$ given by (20), especially for large constellation size L. Moreover, Theorem 1 implies that the diagonal signal constellation, which is in the form of (18) with $k_1 \in \mathbb{Z}_L$ and $k_2 = k_3 = 0$ always has a positive diversity sum.

B. Comparison With Previous Unitary Space–Time Codes and Numerical Simulation Results

There have been several classes of 2×2 unitary space–time constellations proposed in the previous works. In [16] and [18], a diagonal code or cyclic group code was introduced. The general $M \times M$ diagonal code is the first that appeared in the literature as a unitary space–time code for the M-antenna differential modulation scheme. The 2×2 diagonal code can be thought of as the above parametric code imposed by the constraints $k_1 \in \mathbb{Z}_L$ and $k_2 = k_3 = 0$. In other words, the parametric code $\mathcal{V}(k_1, k_2, k_3)$ is an extension to the three-parameter case of the 2×2 diagonal code $\mathcal{V}(k_1, 0, 0)$ with a single parameter $k_1 \in \mathbb{Z}_L$. A main difference between the diagonal cyclic code and the parametric code is that the diagonal code has an algebraic group structure while the parametric code is in general a nongroup signal constellation.

Another class of codes, called generalized quaternion codes or dicyclic group codes, was developed in [18] and [19]. The signal constellation is of size $L = 2^Q$, where $Q \in \mathbb{N}$, and, in the case $M = 2$, can be described by

$$\mathcal{V}(k_1) = \left\{ \begin{pmatrix} e^{j2k_1\theta_L} & 0 \\ 0 & e^{-j2k_1\theta_L} \end{pmatrix}^\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^m \,\middle|\, \ell \in \mathbb{Z}_{\frac{L}{2}}, \, m = 0, 1 \right\}$$

for $k_1 \in \mathbb{Z}_L$. The quaternion code is the signal constellation with the largest diversity product in the following constellation class:

$$\left\{ \mathcal{V}(k_1) \,\middle|\, k_1 = 1, 2, \ldots, \frac{L}{2} - 1 \right\}.$$

By a simple calculation, we can see that the diversity sum of a quaternion code is identical to its diversity product.

In [33], a two-antenna differential detection scheme has been proposed, which is based on the well-tailored orthogonal design of 2×2 unitary matrices [2]. The signal constellation in the orthogonal design is of size $L = Q^2$, where $Q \in \mathbb{N}$ and $Q \geq 2$, and can be described by

$$\mathcal{V} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \middle| x = e^{jk_1\theta_Q},\ y = e^{jk_2\theta_Q}, \text{ and } k_1, k_2 \in \mathbb{Z}_Q \right\}$$

where $\theta_Q = 2\pi/Q$, x and y are both the Qth roots of unit 1, and x^* and y^* denote the complex conjugate of x and y, respectively. The diversity sum and diversity product of this signal constellation are the following:

$$\delta(L, \mathcal{V}) = \zeta(L, \mathcal{V}) = \frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{\sqrt{L}}\right).$$

Like the parametric codes, the signal constellations in the orthogonal design are generally nongroup matrix-valued signal sets. Therefore, when these signal constellations are used for differential space–time modulation, the transmitted symbols generated from the fundamental differential encoding equation (2) are possibly arbitrary unitary signals.

In [31], the authors presented a thorough classification of fixed-point free (FPF) unitary group codes of any finite order. An FPF unitary group code itself is a group consisting of the unitary signal matrices in which the difference matrix of any two different unitary signals has a nonzero determinant. The FPF group code is also called a *full-diversity* group code [31]. The group code has the practical merit that every transmitted signal in the differential modulation scheme is still a codeword in the group code and can thus be determined by a simple group table lookup. Moreover, in [31], the authors also investigated the construction methodology of general unitary space–time codes, which may or may not be group codes, inspired by the FPF group codes.

The diversity products and diversity sums of the above unitary space–time codes for some constellation sizes are presented in Tables I and II, respectively. We observe that the FPF group codes can provide better diversity products than the other codes in Table I for relatively large constellation sizes. Parametric codes possess comparable diversity products with those of FPF group codes, as shown in Table I. It is seen from Tables I and II that the parametric codes have equal or better diversity products and sums than those of the cyclic codes, quaternion codes, and the signal constellations in orthogonal design.

According to Property 2) in Proposition 2 and Tables I and II, we can see that the parametric codes can lead to a five-signal constellation $\mathcal{V}(4, 2, 0)$ with the largest possible diversity product and sum of $\sqrt{5/8}$ and a nine-signal constellation achieving the largest possible diversity sum of $3/4$. By using Property 3) in Proposition 2, the 16-signal constellation $\mathcal{V}(3, 4, 2)$ in the class of parametric codes also has the largest possible diversity sum of $\sqrt{2}/2$ in addition to the largest known diversity product of $\sqrt[4]{2}/2$. This also implies that the unitary signal constellations with sizes 10–15, as any of the *subsets* of the above 16-signal constellation $\mathcal{V}(3, 4, 2)$, can also attain the largest possible diversity sum of $\sqrt{2}/2$. It is worth noting that, although the above parametric code of size 16, $\mathcal{V}(3, 4, 2)$, itself is not a group, its finitely generated semigroup is actually a finite group of order 32, i.e., it is a subset of a group of order 32. Thus, like other group codes, when the parametric code $\mathcal{V}(3, 4, 2)$ is used for differential modulation, every transmitted signal can also be determined by a simple table lookup in the above finite group. More precisely, the 16-signal constellation of parametric code

$$\mathcal{V}(3, 4, 2) = \{A_\ell \,|\, \ell \in \mathbb{Z}_{16}\}$$

as shown in Table III, has the following property.

Proposition 3: When the 16-signal constellation

$$\mathcal{V} = \{A_\ell \,|\, \ell \in \mathbb{Z}_{16}\}$$

given in Table III is used for differential modulation, the set of all possibly transmitted signals S_τ for $\tau = 0, 1, 2, \ldots$ encoded by (2) in which the initial transmitted signal $S_0 = I_M$, is a finite group given by

$$\{j^m A_\ell \,|\, \ell \in \mathbb{Z}_{16} \text{ and } m = 0, 1\}.$$

The above conclusion remains true if only the initial transmitted signal S_0 in (2) belongs to the above finite group.

Proof: See Appendix D. □

From Proposition 3, we know that, when the parametric code of size 16, $\mathcal{V}(3, 4, 2)$, given in Table III is used for differential modulation with double transmit antennas, each transmitted 2×2 unitary signal is a diagonal matrix or a matrix with zero diagonal entries. Hence, only one of the two transmit antennas is activated at each time when the signals are transmitted. The computer-simulated performance in terms of the block error rate of the parametric code $\mathcal{V}(3, 4, 2)$, quaternion code, and the signal constellations in orthogonal design, each of size 16, is shown in Figs. 1 and 2. The union bound on the block error probability in Figs. 1 and 2 is obtained by summing all the pairwise error probability divided by L and the pairwise error probability is numerically evaluated by using the right-hand side of (8), where we take $a = 100$. We can see from Figs. 1 and 2 that the union bound and the simulation result of the block error probability fit each other quite well. As shown in Fig. 1, in the case of two receive antennas, the parametric code has an improvement in block error rate of about 1 dB over the existing codes at SNR 22 dB. In the case of five receive antennas, the improvement is over 1 dB at SNR 10 dB as seen from Fig. 2.

Furthermore, from Table I, one can see that, the unitary space–time codes of sizes $32, 64, 128,$ and 256 as the subsets taken from the parametric codes of sizes 37, 75, 135, and 273, respectively, have the largest known diversity products.

TABLE I
DIVERSITY PRODUCTS $\zeta(L, \mathcal{V})$ OF UNITARY SPACE–TIME CODES FOR TWO TRANSMIT ANTENNAS

L	$\zeta(L,\mathcal{V})$	codes and comments
2	1	parametric code of $(k_1,k_2,k_3)=(1,0,0)$
3	$\sqrt{3}/2$	parametric code of $(k_1,k_2,k_3)=(0,0,1)$
4	0.7071	cyclic code, quaternion code, orthogonal design, parametric code
4	$\sqrt{2/3}$	optimal code
5	$\sqrt{5/8}$	parametric code of $(k_1,k_2,k_3)=(4,2,0)$ with the optimal diversity product
8	0.5946	cyclic code
8	0.7071	quaternion code, parametric code of $(k_1,k_2,k_3)=(7,2,0)$
16	0.3827	cyclic code, quaternion code
16	0.5000	orthogonal design
16	$\sqrt[4]{2}/2$	parametric code of $(k_1,k_2,k_3)=(3,4,2)$
24	0.5000	FPF group code, parametric code of $(k_1,k_2,k_3)=(5,6,0)$
32	0.1951	quaternion code
32	0.2494	cyclic code
32	0.3827	parametric code of $(k_1,k_2,k_3)=(7,8,2)$
37	0.4461	parametric code of $(k_1,k_2,k_3)=(30,6,0)$
48	0.3868	FPF group code
55	0.3874	parametric code of $(k_1,k_2,k_3)=(34,15,0)$
64	0.0980	quaternion code
64	0.1985	cyclic code
64	0.2706	orthogonal design
64	0.3070	parametric code of $(k_1,k_2,k_3)=(7,10,0)$
75	0.3535	parametric code of $(k_1,k_2,k_3)=(49,18,0)$
91	0.3451	parametric code of $(k_1,k_2,k_3)=(64,21,0)$
105	0.3116	parametric code of $(k_1,k_2,k_3)=(34,42,0)$
120	0.3090	FPF group code
121	0.1992	orthogonal design
128	0.0491	quaternion code
128	0.1498	cyclic code
128	0.2606	parametric code of $(k_1,k_2,k_3)=(1,8,20)$
135	0.2869	parametric code of $(k_1,k_2,k_3)=(14,34,0)$
145	0.2841	parametric code of $(k_1,k_2,k_3)=(62,38,0)$
203	0.2603	parametric code of $(k_1,k_2,k_3)=(146,35,0)$
217	0.2511	parametric code of $(k_1,k_2,k_3)=(125,84,0)$
240	0.2257	FPF group code
273	0.2152	parametric code of $(k_1,k_2,k_3)=(104,71,0)$

IV. UNITARY SIGNAL CONSTELLATIONS WITH OPTIMAL DIVERSITY SUM AND PRODUCT FOR $L = 4$ AND WITH OPTIMAL DIVERSITY SUMS FOR $L = 6, 7, 8$

In the preceding section, we have shown that the unitary signal constellations in the class of parametric codes for sizes of 2–16, except for $4, 6, 7,$ and 8, can attain the *optimal* diversity sums. Thus, a natural question that arises is what are the unitary signal constellations for sizes of $4, 6, 7,$ and 8 which achieve the *optimal* diversity sums. The question can well be answered by providing several examples of unitary signal constellations, as shown later.

A. *Unitary Signal Constellation With Optimal Diversity Sum and Product for $L = 4$*

The following result indicates that there is a unitary signal constellation of size 4 which has both the optimal diversity sum and the optimal diversity product.

Proposition 4: Let $a_i \in \mathbb{R}$ for $i = 1, 2, 3$ satisfy $a_1^2 + a_2^2 + a_3^2 = 1$. If $a_1^2 + a_3^2 = 2/3$, then the four-signal constellation $\mathcal{V}$ of 2×2 unitary matrices given by

$$V_0 = \begin{pmatrix} ja_1 & a_3 - ja_2 \\ -a_3 - ja_2 & -ja_1 \end{pmatrix}$$
$$V_1 = \begin{pmatrix} -ja_1 & -a_3 - ja_2 \\ a_3 - ja_2 & ja_1 \end{pmatrix}$$
$$V_2 = \begin{pmatrix} -ja_3 & a_1 + ja_2 \\ -a_1 + ja_2 & ja_3 \end{pmatrix}$$
$$V_3 = \begin{pmatrix} ja_3 & -a_1 + ja_2 \\ a_1 + ja_2 & -ja_3 \end{pmatrix}$$

has the optimal diversity sum and product of the same value $\sqrt{2/3}$.

TABLE II
DIVERSITY SUMS $\delta(L, \mathcal{V})$ OF UNITARY SPACE–TIME CODES FOR TWO TRANSMIT ANTENNAS

L	$\delta(L,\mathcal{V})$	codes and comments
2	1	parametric code of $(k_1,k_2,k_3)=(1,0,0)$
3	$\sqrt{3}/2$	parametric code of $(k_1,k_2,k_3)=(0,0,1)$
4	0.7071	cyclic code, quaternion code, orthogonal design, parametric code
4	$\sqrt{2/3}$	optimal code
5	$\sqrt{5/8}$	parametric code of $(k_1,k_2,k_3)=(4,2,0)$ with the optimal diversity sum
6	$\sqrt{3/5}$	optimal code
7	$\sqrt{7/12}$	optimal code
8	0.5946	cyclic code
8	0.7071	quaternion code, parametric code of $(k_1,k_2,k_3)=(7,2,0)$
8	$\sqrt{4/7}$	optimal code
9	3/4	parametric code of $(k_1,k_2,k_3)=(1,2,4)$ with the optimal diversity sum
16	0.3827	quaternion code
16	0.5000	cyclic code, orthogonal design
16	$\sqrt{2}/2$	parametric code of $(k_1,k_2,k_3)=(3,4,2)$ with the optimal diversity sum
32	0.1951	quaternion code
32	0.3827	cyclic code
32	0.5621	parametric code of $(k_1,k_2,k_3)=(8,3,16)$
64	0.0980	quaternion code
64	0.2706	orthogonal design
64	0.2753	cyclic code
64	0.4852	parametric code of $(k_1,k_2,k_3)=(18,5,64)$
121	0.1992	orthogonal design
128	0.0491	quaternion code
128	0.2009	cyclic code
128	0.3936	parametric code of $(k_1,k_2,k_3)=(48,30,64)$

Proof: Let $d_{ik}=\det(V_i-V_k)$ for $0\le i<k\le 3$. By a simple calculation, we have

$$d_{01}=d_{23}=4(a_1^2+a_3^2)=8/3$$

and

$$d_{02}=d_{03}=d_{12}=d_{13}=2(a_1^2+a_3^2)+4a_2^2=8/3.$$

According to (17), the above four-signal constellation $\mathcal{V}$ has a diversity product of

$$\zeta(4,\mathcal{V})=\frac{\sqrt{8/3}}{2}=\sqrt{2/3}.$$

Furthermore, it follows from Properties 1) and 2) in Proposition 2 that

$$\zeta(4,\mathcal{V})\le\delta(4,\mathcal{V})\le\sqrt{2/3}.$$

Therefore, the diversity sum of the above four-signal constellation $\mathcal{V}$ is also $\delta(4,\mathcal{V})=\sqrt{2/3}$. The optimality of the diversity product and sum of value $\sqrt{2/3}$ for $L=4$ is seen from inequality (15) in Proposition 2.

The proof of Proposition 4 is thus completed. □

In Figs. 3 and 4, we present computer simulation results of the block error rate of the above four-signal constellation, where we take $a_1=a_2=a_3=1/\sqrt{3}$, compared with the cyclic code of size 4. It is seen from Fig. 3 that, in the case of two receive antennas, the code given in Proposition 4 has an improvement of 1dB over cyclic code at SNR 14 dB. In the case of five receive antennas, the improvement is 1 dB at SNR 8 dB as shown in Fig. 4.

B. Unitary Signal Constellations With Optimal Diversity Sums for $L=6, 7, 8$

The signal constellations of sizes 6, 7, and 8 with the largest possible diversity sums given as follows are constructed mainly through a computer search. According to inequality (15) in Proposition 2, we can verify the following two results through direct numerical evaluation.

Proposition 5: Let $a=\sqrt{3/5}\,e^{j(\pi/4)}$, $b=\sqrt{2/5}\,e^{j(\pi/6)}$, $c=\sqrt{3/5}\,e^{j(3\pi/4)}$, $d=\sqrt{2/5}\,e^{j(5\pi/6)}$, $h=\sqrt{3/5}-j\sqrt{2/5}$, $\Delta=e^{j\arccos(-1/5)}$. Then, the six-signal constellation $\mathcal{V}$ composed of the following 2×2 unitary matrices

$$V_0=\begin{pmatrix} a & b\\ -b^* & a^*\end{pmatrix},\qquad V_1=\begin{pmatrix} a^* & -b^*\\ b & a\end{pmatrix}$$

$$V_2=\begin{pmatrix} c & d\\ -d^* & c^*\end{pmatrix},\qquad V_3=\begin{pmatrix} c^* & -d^*\\ d & c\end{pmatrix}$$

$$V_4=\begin{pmatrix} 0 & h\\ -h^*\Delta & 0\end{pmatrix},\qquad V_5=\begin{pmatrix} 0 & -h^*\\ h\Delta^* & 0\end{pmatrix}$$

has the optimal diversity sum of

$$\sqrt{3/5}=0.7745966692414834\ldots.$$

TABLE III
THE SIGNAL CONSTELLATION OF PARAMETRIC CODE $\mathcal{V}(3, 4, 2)$ OF SIZE 16 FOR TWO TRANSMIT ANTENNAS

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & e^{j\frac{15}{8}\pi} \\ e^{j\frac{13}{8}\pi} & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} e^{j\frac{7}{4}\pi} & 0 \\ 0 & e^{j\frac{5}{4}\pi} \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & e^{j\frac{5}{8}\pi} \\ e^{j\frac{15}{8}\pi} & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} e^{j\frac{3}{2}\pi} & 0 \\ 0 & e^{j\frac{1}{2}\pi} \end{pmatrix} \qquad A_5 = \begin{pmatrix} 0 & e^{j\frac{11}{8}\pi} \\ e^{j\frac{1}{8}\pi} & 0 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} e^{j\frac{5}{4}\pi} & 0 \\ 0 & e^{j\frac{7}{4}\pi} \end{pmatrix} \qquad A_7 = \begin{pmatrix} 0 & e^{j\frac{1}{8}\pi} \\ e^{j\frac{3}{8}\pi} & 0 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A_9 = \begin{pmatrix} 0 & e^{j\frac{7}{8}\pi} \\ e^{j\frac{5}{8}\pi} & 0 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} e^{j\frac{3}{4}\pi} & 0 \\ 0 & e^{j\frac{1}{4}\pi} \end{pmatrix} \qquad A_{11} = \begin{pmatrix} 0 & e^{j\frac{13}{8}\pi} \\ e^{j\frac{7}{8}\pi} & 0 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} e^{j\frac{1}{2}\pi} & 0 \\ 0 & e^{j\frac{3}{2}\pi} \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 0 & e^{j\frac{3}{8}\pi} \\ e^{j\frac{9}{8}\pi} & 0 \end{pmatrix}$$

$$A_{14} = \begin{pmatrix} e^{j\frac{1}{4}\pi} & 0 \\ 0 & e^{j\frac{3}{4}\pi} \end{pmatrix} \qquad A_{15} = \begin{pmatrix} 0 & e^{j\frac{9}{8}\pi} \\ e^{j\frac{11}{8}\pi} & 0 \end{pmatrix}$$

Proposition 6: We construct a seven-signal constellation $\mathcal{V}$ as follows.

1) The first unitary matrix is

$$V_0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2) Let $\theta_a = 6\pi/7$ and $\theta_{\Delta_{12}} = 12\pi/7$. We define three real numbers of

$$r_a = -\frac{1}{3[\cos(\theta_a) + \cos(\theta_a - \theta_{\Delta_{12}})]} = -\frac{1}{6\cos(6\pi/7)}$$

$r_b = \sqrt{1 - r_a^2}$, and

$$\theta_b = \frac{1}{2}\left[\theta_{\Delta_{12}} - \arccos\left(\frac{7/6 - 2r_a^2\sin^2(\theta_a) - r_b^2}{r_b^2\cos(\theta_{\Delta_{12}})}\right)\right] \bmod 2\pi$$

and three complex numbers of $a = r_a e^{j\theta_a}$, $b = r_b e^{j\theta_b}$, and $\Delta_{12} = e^{j\theta_{\Delta_{12}}}$. The second and third unitary matrices are given by

$$V_1 = \begin{pmatrix} a & b \\ -b^*\Delta_{12} & a^*\Delta_{12} \end{pmatrix}$$

$$V_2 = \begin{pmatrix} a^* & -b^* \\ b\Delta_{12}^* & a\Delta_{12}^* \end{pmatrix}.$$

3) Let $\theta_c = 9\pi/7$ and $\theta_{\Delta_{34}} = 4\pi/7$. We define the three real numbers of

$$r_c = -\frac{1}{3[\cos(\theta_c) + \cos(\theta_c - \theta_{\Delta_{34}})]} = -\frac{1}{6\cos(5\pi/7)}$$

$r_d = \sqrt{1 - r_c^2}$, and

$$\theta_d = \frac{1}{2}\left[\theta_{\Delta_{34}} - \arccos\left(\frac{7/6 - 2r_c^2\sin^2(\theta_c) - r_d^2}{r_d^2\cos(\theta_{\Delta_{34}})}\right)\right] \bmod 2\pi$$

and the three complex numbers of $c = r_c e^{j\theta_c}$, $d = r_d e^{j\theta_d}$, and $\Delta_{34} = e^{j\theta_{\Delta_{34}}}$. The fourth and fifth unitary matrices are given by

$$V_3 = \begin{pmatrix} c & d \\ -d^*\Delta_{34} & c^*\Delta_{34} \end{pmatrix}$$

$$V_4 = \begin{pmatrix} c^* & -d^* \\ d\Delta_{34}^* & c\Delta_{34}^* \end{pmatrix}.$$

4) Let $\theta_{\Delta_{56}} = 6\pi/7$, $\Delta_{56} = e^{j\theta_{\Delta_{56}}}$, and the 2×2 complex matrix

$$\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \stackrel{\text{def}}{=} -(V_0 + V_1 + V_2 + V_3 + V_4).$$

We define the four real numbers of $g_1 = \mathrm{Re}\{\beta_1\}/2$, $h_2 = \mathrm{Im}\{\beta_2\}/2$

$$g_2 = \frac{\mathrm{Re}\{\beta_4\}/2 - g_1\cos(\theta_{\Delta_{56}})}{\sin(\theta_{\Delta_{56}})}$$

and

$$h_1 = \frac{h_2\cos(\theta_{\Delta_{56}}) - \mathrm{Im}\{\beta_3\}/2}{\sin(\theta_{\Delta_{56}})}$$

and the two complex numbers of $g = g_1 + jg_2$ and $h = h_1 + jh_2$, where $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ stand for the real and imaginary parts of a complex number, respectively. The sixth and seventh matrices are given by

$$V_5 = \begin{pmatrix} g & h \\ -h^*\Delta_{56} & g^*\Delta_{56} \end{pmatrix}$$

$$V_6 = \begin{pmatrix} g^* & -h^* \\ h\Delta_{56}^* & g\Delta_{56}^* \end{pmatrix}.$$

Then, the seven-signal constellation $\mathcal{V} = \{V_\ell \mid \ell \in \mathbb{Z}_7\}$ defined in the above consists of 2×2 unitary matrices and has the optimal diversity sum of

$$\sqrt{7/12} = 0.7637626158259734\ldots.$$

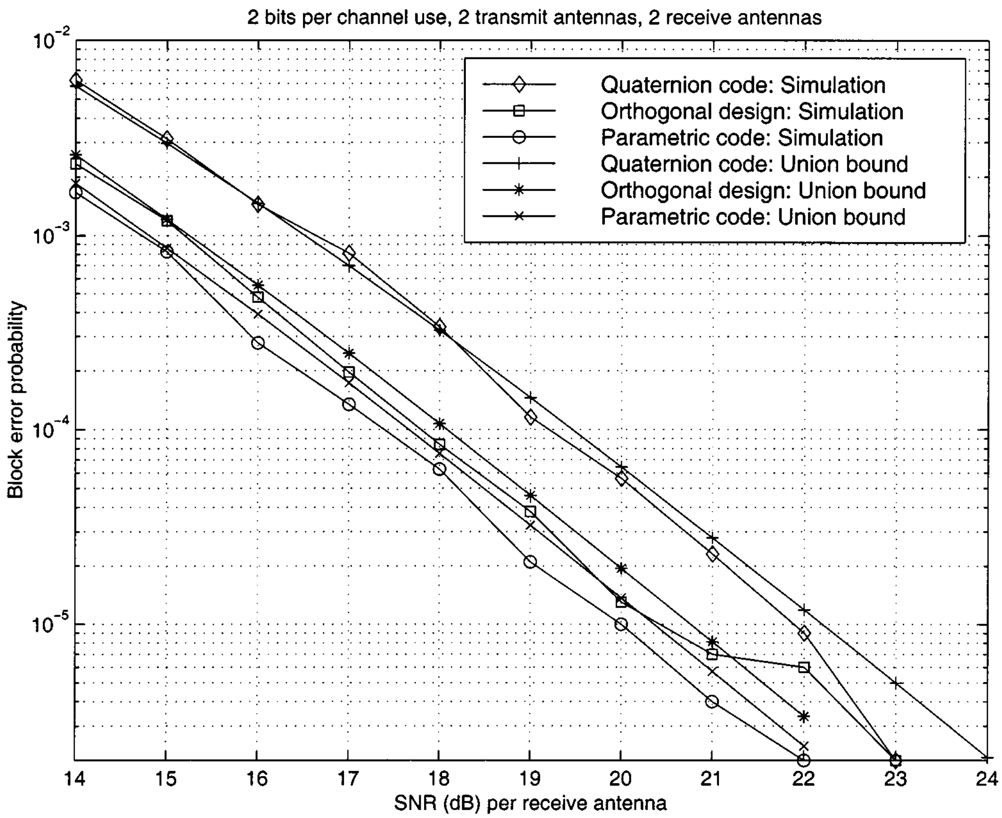

Fig. 1. Simulation results and union bounds on the block error rates for signal constellations of size $L = 16$ with $M = 2$ and $N = 2$.

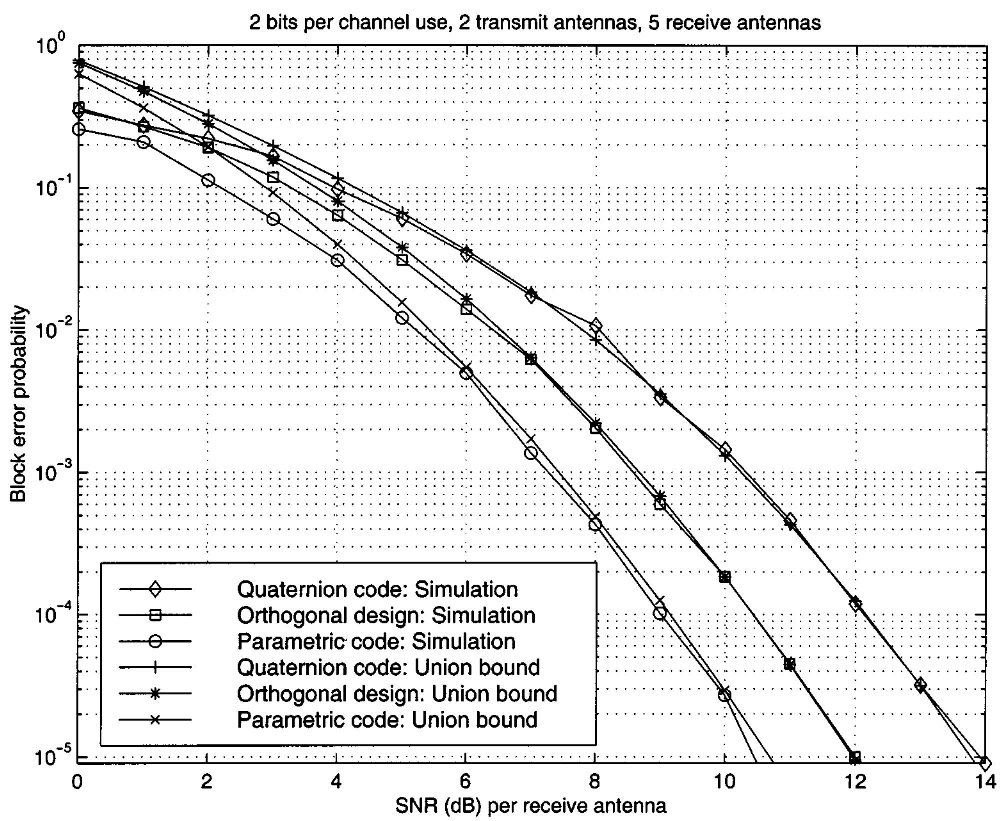

Fig. 2. Simulation results and union bounds on the block error rates for signal constellations of size $L = 16$ with $M = 2$ and $N = 5$.

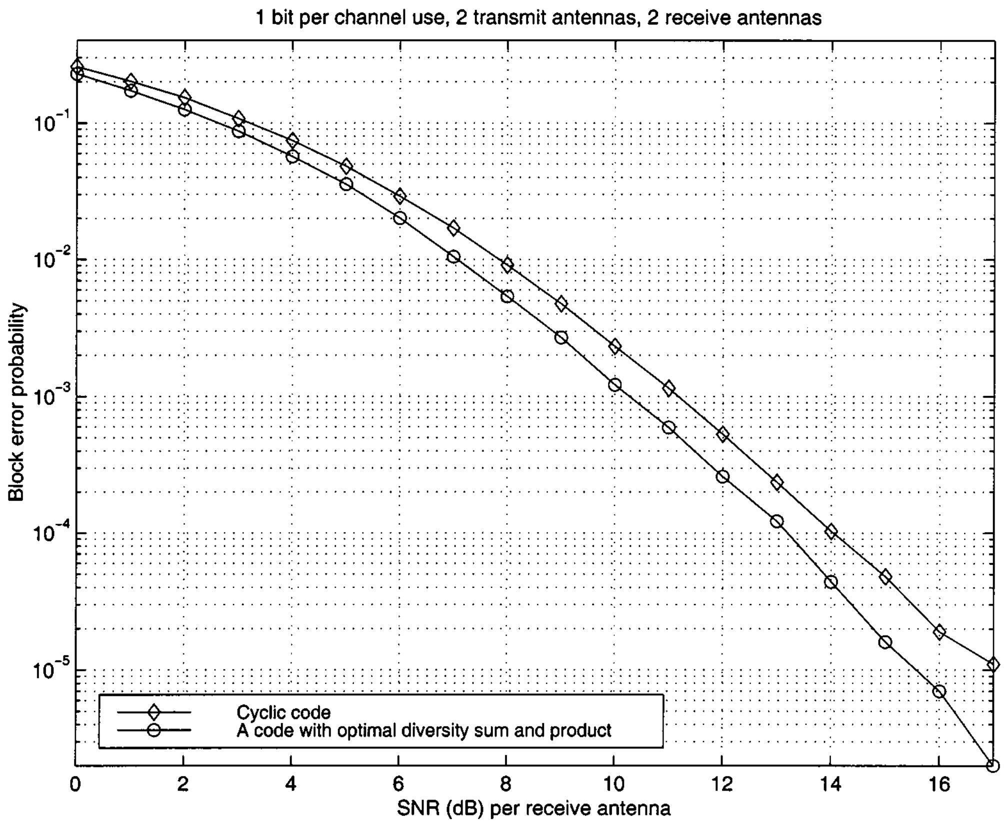

Fig. 3. Simulation results of the block error rates for signal constellations of size $L = 4$ with $M = 2$ and $N = 2$.

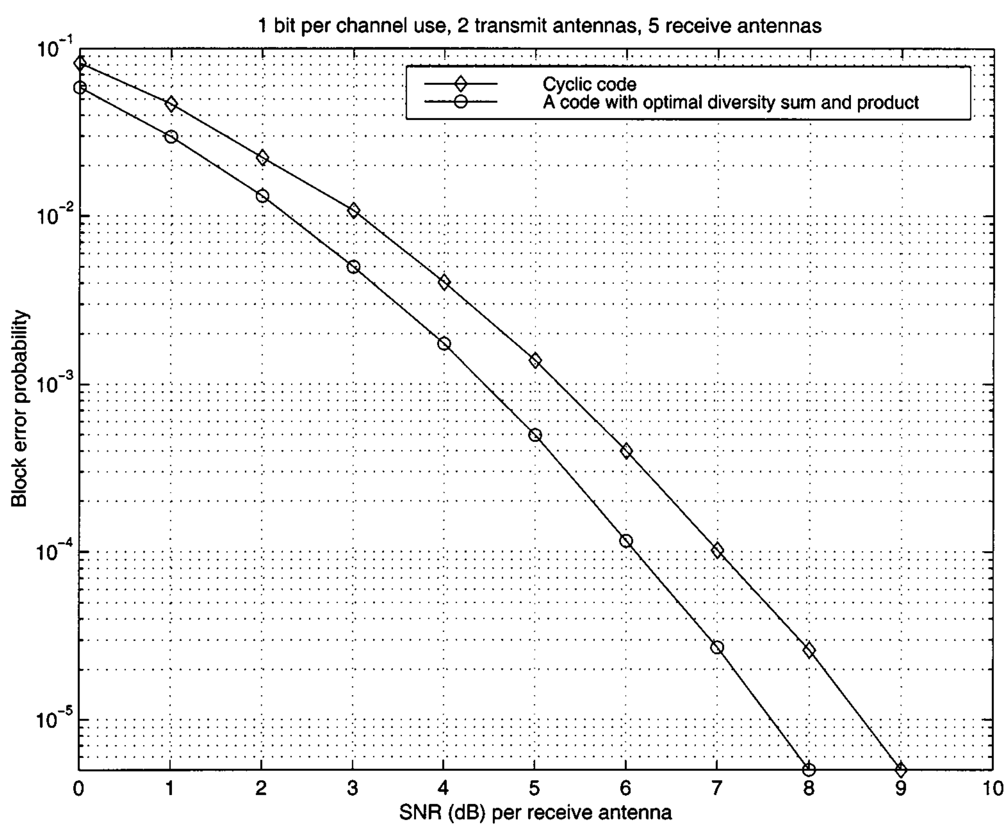

Fig. 4. Simulation results of the block error rates for signal constellations of size $L = 4$ with $M = 2$ and $N = 5$.

In Propositions 5 and 6, the unitary signal constellations are presented directly. Contrary to this, the eight-signal constellation described in what follows is somewhat complicated and determined by three parameters which satisfy a system of *nonlinear* equations in order that the eight-signal constellation has an optimal diversity sum.

Proposition 7: We define $\phi = \arccos(-1/7)$ and

$$\lambda = 2j(e^{j\phi}-1)/(7\sin\phi) = -2/7 - j\left(4\sqrt{3}/21\right).$$

Let $\theta_i \in [0, 2\pi)$ $(i = 1, 2, 3)$ be three free parameters and $f(\theta)$ an auxiliary function in terms of $\theta \in [0, 2\pi)$ given by

$$f(\theta) = -\frac{3+5\cos\theta}{(19+21\cos\theta)\cos\theta}.$$

Let

$$\beta_1 = \beta_1(\theta_1) = \frac{1}{2}\left[\theta_1 + \arccos(f(\theta_1))\right] \bmod 2\pi$$

$$\beta_2 = \beta_2(\theta_2) = \frac{1}{2}\left[\theta_2 - \arccos(f(\theta_2))\right] \bmod 2\pi$$

and

$$\beta_3 = \beta_3(\theta_3) = \frac{1}{2}\left[\theta_3 - \arccos(f(\theta_3))\right] \bmod 2\pi.$$

For $i = 1, 2, 3$, we define the following four numbers of $\Delta_i = e^{j\theta_i}$:

$$a_i = \frac{1}{1+\Delta_i^*} = \frac{1}{2}\left[1 + j\tan(\theta_i/2)\right]$$

$$r_i = \sqrt{1-|\lambda a_i|^2} = \left(1 - \frac{1}{21\cos^2(\theta_i/2)}\right)^{1/2}$$

and $b_i = r_i e^{j\beta_i}$. We introduce the following eight matrices:

$$A_i = A_i(\theta_i) = \begin{pmatrix} \lambda a_i & b_i \\ -b_i^*\Delta_i & \lambda^* a_i^*\Delta_i \end{pmatrix}, \qquad i = 1, 2, 3$$

$$B_i = B_i(\theta_i) = \begin{pmatrix} \lambda a_i^* & b_i^* \\ -b_i\Delta_i^* & \lambda^* a_i\Delta_i^* \end{pmatrix}, \qquad i = 1, 2, 3$$

$$C = \begin{pmatrix} e^{j\phi} & 0 \\ 0 & e^{-j\phi} \end{pmatrix}$$

$$D = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for any $\theta_i \in [0, 2\pi)$ $(i = 1, 2, 3)$ satisfying $|f(\theta_i)| \leq 1$ and $|\lambda a_i| \leq 1$, i.e., that

$$\cos\theta_i \in \left[-1/3 - 4\sqrt{7}/21, -1/7\right] \cup \left[-1/3 + 4\sqrt{7}/21, 1\right]$$

the eight-signal constellation defined by

$$\mathcal{V}(\theta_1, \theta_2, \theta_3) = \{A_1(\theta_1), A_2(\theta_2), A_3(\theta_3), B_1(\theta_1), B_2(\theta_2), B_3(\theta_3), C, D\}$$

consists of 2×2 unitary matrices and satisfies

$$\begin{aligned} \|A_i - B_i\|_F &= \|A_i - C\|_F = \|A_i - D\|_F = \|B_i - C\|_F \\ &= \|B_i - D\|_F = \|C - D\|_F = \sqrt{32/7}, \\ &\qquad \text{for } i = 1, 2, 3 \end{aligned}$$

and $\|A_i - A_k\|_F = \|B_i - B_k\|_F$ and $\|A_i - B_k\|_F = \|A_k - B_i\|_F$ for $1 \leq i < k \leq 3$. Therefore, if the following system of nonlinear equations in terms of $\theta_i \in [0, 2\pi)$ for $i = 1, 2, 3$

$$\|A_i(\theta_i) - A_k(\theta_k)\|_F = \|A_i(\theta_i) - B_k(\theta_k)\|_F = \sqrt{32/7}, \quad 1 \leq i < k \leq 3 \qquad (21)$$

admits a solution $\tilde{\theta}_i \in [0, 2\pi)$ satisfying

$$\cos\tilde{\theta}_i \in \left[-1/3 - 4\sqrt{7}/21, -1/7\right] \cup \left[-1/3 + 4\sqrt{7}/21, 1\right]$$

for $i = 1, 2, 3$, then the above eight-signal constellation $\mathcal{V}(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ possesses the optimal diversity sum of

$$\sqrt{4/7} = 0.7559289460184544\ldots.$$

Proof: See Appendix E. □

We are not able to find an analytical proof of the existence of solutions to the nonlinear equation (21) in terms of $\theta_i \in [0, 2\pi)$ for $i = 1, 2, 3$. However, we find two numerical solutions of (21), namely

$$(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3) = (2.537158998077295, 0.9578023163753372, 4.530318445275019)$$

and

$$\left(2\pi - \tilde{\theta}_1, 2\pi - \tilde{\theta}_2, 2\pi - \tilde{\theta}_3\right)$$

satisfying

$$\cos\tilde{\theta}_i \in \left[-1/3 - 4\sqrt{7}/21, -1/7\right] \cup \left[-1/3 + 4\sqrt{7}/21, 1\right]$$

for $i = 1, 2, 3$. The resultant unitary signal constellation $\mathcal{V}(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ of size 8 has a diversity sum of

$$0.7559289460184543$$

which is approximate to the analytical value $\sqrt{4/7}$ of optimal diversity sum within the precision of $1.110223025 \times 10^{-16}$. It is remarked that the above three unitary signal constellations of sizes 6, 7, and 8 with the optimal diversity sums possess the diversity products of

$$0.6887246539984297$$

$$0.6673838402524986$$

and

$$0.6235414450084460$$

respectively.

V. Upper and Lower Bounds on the Optimal Diversity Sum and Optimal Diversity Product of Unitary Signal Constellations

It is known that a unitary signal constellation can achieve good performance in terms of block error rate if it has large minimum normalized Euclidean distance and/or large minimum normalized determinant dissimilarity. Thus, it is meaningful to examine the largest possible minimum normalized Euclidean distance and the largest possible minimum normalized determinant dissimilarity that a unitary signal constellation of any size $L \geq 2$ can attain. The *optimal* diversity sum and product of order L are, respectively, the above largest possible minimum normalized Euclidean distance and normalized determinant dissimilarity. Clearly, the optimal diversity sum and product for unitary signal constellations of size L also provide the fundamental limits on how well we can separate L elements in the space of $M \times M$ unitary matrices. In what follows, we shall investigate the exact values of the optimal diversity sum and product that can be achieved by unitary signal constellations and

TABLE IV
EXACT VALUES OF OPTIMAL DIVERSITY SUMS $\Delta(L)$ AND PRODUCTS $\Xi(L)$ FOR TWO TRANSMIT ANTENNAS

L	2	3	4	5	6	7	8	9	10 through 16
$\Delta(L)$	1	$\sqrt{3}/2$	$\sqrt{2/3}$	$\sqrt{5/8}$	$\sqrt{3/5}$	$\sqrt{7/12}$	$\sqrt{4/7}$	$3/4$	$\sqrt{2}/2$
$\Xi(L)$	1	$\sqrt{3}/2$	$\sqrt{2/3}$	$\sqrt{5/8}$	unknown				

provide some upper and lower bounds on the optimal diversity sum and product in the case that the exact values are not available.

Let $\mathcal{V} = \{ V_\ell \,|\, V_\ell^{\mathrm{H}} V_\ell = I_M,\, \ell \in \mathbb{Z}_L \}$ be an $M \times M$ unitary signal constellation of size L. The optimal diversity sum and the optimal diversity product of order L for the unitary signal constellation $\mathcal{V}$ are, respectively, given by

$$\begin{aligned}\Delta_M(L) &\stackrel{\text{def}}{=} \max_{\mathcal{V}} \delta(L, \mathcal{V}) \\ &= \frac{1}{2\sqrt{M}} \max_{\mathcal{V}} \min_{0 \le \ell < \ell' \le L-1} \|V_\ell - V_{\ell'}\|_F \end{aligned} \tag{22}$$

and

$$\begin{aligned}\Xi_M(L) &\stackrel{\text{def}}{=} \max_{\mathcal{V}} \zeta(L, \mathcal{V}) \\ &= \frac{1}{2} \max_{\mathcal{V}} \min_{0 \le \ell < \ell' \le L-1} \sqrt[M]{|\det(V_\ell - V_{\ell'})|}. \end{aligned} \tag{23}$$

It is emphasized that each of the optimal values defined in (22) and (23) must be attained at some unitary signal constellation, since the optimization in (22) and (23) is essentially performed over a compact subset of parameters of the unitary matrices in $\mathcal{V}$ [24].

By Property 1) in Proposition 2 and definitions of (13) and (14), the diversity product of a unitary signal constellation cannot exceed its diversity sum. Therefore, we have

$$0 \le \Xi_M(L) \le \Delta_M(L) \le 1$$

for all $L \ge 2$ and for all $M \ge 1$. Specifically, it is clear that

$$\Xi_1(L) = \Delta_1(L) = \sin(\pi/L)$$

for all $L \ge 2$.

For the general case $M \ge 2$, let the $M \times M$ unitary signal constellation $\mathcal{V}$ be composed of $V_\ell = e^{j2\pi\ell/L} I_M$ for all $\ell \in \mathbb{Z}_L$, we can know that

$$\Delta_M(L) \ge \Xi_M(L) \ge \sin(\pi/L), \qquad L \ge 2,\ M \ge 2. \tag{24}$$

Moreover, by using Properties 2)–4) in Proposition 2, we have, for any $M \ge 2$

$$\Xi_M(L) \le \Delta_M(L) \begin{cases} \le \sqrt{\frac{L}{2(L-1)}}, & \text{if } 2 \le L \le 2M^2+1 \\ \le \frac{1}{\sqrt{2}}, & \text{if } 2M^2+1 < L \le 4M^2 \\ < \frac{1}{\sqrt{2}}, & \text{if } L > 4M^2. \end{cases} \tag{25}$$

Therefore, it follows from (24) and (25) that

$$\Xi_M(L) = \Delta_M(L) = \sin(\pi/L)$$

for $L = 2, 3$, and any $M \ge 2$.

In the following, we want to find some exact values of optimal diversity sum $\Delta_2(L)$ and optimal diversity product $\Xi_2(L)$ in the case $M = 2$ and present some upper and lower bounds on $\Delta_2(L)$ and $\Xi_2(L)$ in case their exact values are not available. For simplicity, in the following we denote $\Delta(L) = \Delta_2(L)$ and $\Xi(L) = \Xi_2(L)$ for $L \ge 2$ in the case $M = 2$.

A. The Known Exact Values of Optimal Diversity Sum and Product

The known exact values of optimal diversity sum $\Delta(L)$ for $2 \le L \le 16$ and optimal diversity product $\Xi(L)$ for $2 \le L \le 5$ are summarized in Table IV. For $L = 2$ and $L = 3$, the optimal diversity product and sum are derived from (24) and (25). Proposition 4 presented a unitary code of size 4 with the optimal diversity product and sum. The parametric code of size 5 attains the optimal diversity product and sum. The unitary codes of sizes 6, 7, and 8 with the optimal diversity sums are shown, respectively, in Propositions 5–7. The unitary codes of sizes 9 through 16 with the optimal diversity sums are the parametric code $\mathcal{V}(1, 2, 4)$ of size 9 and the parametric code $\mathcal{V}(3, 4, 2)$ of size 16 and its subsets with the corresponding sizes of 10–15, respectively.

We do not know the exact values of optimal diversity sum $\Delta(L)$ for $L \ge 17$ and optimal diversity product $\Xi(L)$ for $L \ge 6$. In what follows, we shall present some upper and lower bounds on $\Delta(L)$ for $L \ge 17$ and $\Xi(L)$ for $L \ge 6$ and some asymptotic upper and lower bounds on $\Delta(L)$ and $\Xi(L)$ when L is large. It is obvious that the largest known diversity sum and product are also lower bounds on the optimal diversity sum and product, respectively. We shall use the notation $o_L(1)$ to represent a variable in terms of L which approaches 0 when $L \to +\infty$. It is noted that for any fixed $L \ge 2$, the function $L^{-\gamma}$ in terms of $\gamma > 0$ is strictly monotonically decreasing.

B. Upper and Lower Bounds on the Optimal Diversity Product for $L = 6$ Through 16

We see from (25) that the optimal diversity product

$$\Xi(L) \le \sqrt{\frac{L}{2(L-1)}}, \qquad \text{for } L = 6, 7, 8, 9$$

$$\Xi(L) \le \frac{1}{\sqrt{2}}, \qquad \text{for } 10 \le L \le 16$$

and

$$\Xi(L) < \frac{1}{\sqrt{2}}, \qquad \text{for } L \ge 17.$$

For the cases of L from 6 to 9, the following result indicates that the optimal diversity product $\Xi(L)$ is actually *smaller* than $\sqrt{\frac{L}{2(L-1)}}$. That is, there do not exist unitary signal constellations with sizes L of 6 through 9 whose *diversity products* can reach $\sqrt{\frac{L}{2(L-1)}}$, while there are indeed unitary signal constellations whose *diversity sums* can attain the upper bound as seen from Table IV.

TABLE V
BOUNDS ON THE OPTIMAL DIVERSITY PRODUCTS $\Xi(L)$ FOR $6 \le L \le 16$ FOR TWO TRANSMIT ANTENNAS

L	6	7	8	9	10	11	12	13	14	15	16
Upper Bounds	$\sqrt{3/5}$	$\sqrt{7/12}$	$\sqrt{4/7}$	$3/4$	$\sqrt{2}/2$	$\sqrt{2}/2$					
Lower Bounds	$\sqrt{2}/2$			$\sqrt{3/8}$		$\sqrt[4]{2}/2$					

Theorem 2: The optimal diversity product $\Xi(L)$ in the case $M = 2$ has the following property:

$$\Xi(L) < \Delta(L) = \sqrt{\frac{L}{2(L-1)}}, \qquad \text{for } L = 6,\, 7,\, 8,\, 9.$$

Proof: From Table IV, it is known that the optimal diversity sum

$$\Delta(L) = \sqrt{\frac{L}{2(L-1)}}, \qquad \text{for } L = 6,\, 7,\, 8,\, 9.$$

Therefore, to prove Theorem 2, we need only to prove that

$$\Xi(L) < \sqrt{\frac{L}{2(L-1)}}, \qquad \text{for } L = 6,\, 7,\, 8,\, 9$$

which is shown in Appendix F. □

Theorem 2 says that, the optimal diversity product is strictly *smaller* than the optimal diversity sum for the unitary space–time codes of sizes of 6 through 9 in the case $M = 2$. Contrary to this, the optimal diversity product and the optimal diversity sum in the case $M = 2$ are *equal* for orders of 2 through 5, as shown in Table IV. The upper and lower bounds on the optimal diversity products $\Xi(L)$ for $6 \le L \le 16$ are given in Table V, where the upper bounds for $6 \le L \le 9$ are unattainable from Theorem 2 and the lower bounds are the largest known diversity products the parametric codes can attain.

C. Bounds on the Optimal Diversity Product and Sum for $L \ge 17$

The subsequent results about the upper and lower bounds on the optimal diversity product and sum of unitary signal constellations will resort to the arguments in the areas of sphere packing and spherical codes (see, [5] and [10]). Actually, we can regard the design problem of unitary signal constellations as the construction of "spherical codes" in the following complex Stiefel manifold:

$$S(M,\, M) = \left\{V \in \mathbb{C}^{M\times M} \,|\, V^{\mathrm{H}}V = I_M\right\}$$

which is simply the unit-radius circle in $\mathbb{C}$ or $\mathbb{R}^2$ in the case $M = 1$. In the following, we give some preliminaries in sphere packing and spherical codes, which are particularly relevant to our need.

A k-dimensional *spherical code* $\mathcal{C}$ is a finite set of distinct points in $\mathbb{R}^k$ that lie on the surface of the unit radius k-dimensional Euclidean sphere defined by

$$\Omega_k = \left\{(x_1,\, x_2,\, \ldots,\, x_k)^\top \in \mathbb{R}^k \,\middle|\, \sum_{i=1}^{k} x_i^2 = 1\right\}. \qquad (26)$$

Let $|\mathcal{C}| \ge 2$ denote the number of code points in the spherical code $\mathcal{C}$, i.e., the code size of the spherical code $\mathcal{C}$. The *minimum distance* of a k-dimensional spherical code $\mathcal{C} \subset \Omega_k$ is defined as

$$d_{\min}(\mathcal{C}) = \min_{x,y\in\mathcal{C},\, x\ne y} \|x-y\|$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^k$ defined by $\|x\| = (x^\top x)^{1/2}$ for $x \in \mathbb{R}^k$. The *minimum angular separation* of spherical code $\mathcal{C}$ is defined as

$$\phi_{\min}(\mathcal{C}) = 2\arcsin(d_{\min}(\mathcal{C})/2).$$

Equivalently, we have

$$d_{\min}(\mathcal{C}) = 2\sin(\phi_{\min}(\mathcal{C})/2).$$

It is obvious that for any spherical code $\mathcal{C} \subset \Omega_k$ with $|\mathcal{C}| \ge 2$, there must be $0 < d_{\min}(\mathcal{C}) \le 2$, or equivalently, $0 < \phi_{\min}(\mathcal{C}) \le \pi$. For a given space-dimension k, a minimum distance $0 < d \le 2$, and code size L, we define

$$A(k,\, d) = \max\{L \,|\, \text{there is a spherical code } \mathcal{C} \subset \Omega_k \text{ with } L \text{ code points and } d_{\min}(\mathcal{C}) \ge d\} \qquad (27)$$

which denotes the largest number of code points of a k-dimensional spherical code with minimum distance not less than d, and

$$D(k,\, L) = \max\{d \,|\, \text{there is a spherical code } \mathcal{C} \subset \Omega_k \text{ with } L \text{ code points and } d_{\min}(\mathcal{C}) \ge d\} \qquad (28)$$

which is the largest possible minimum distance d that a k-dimensional spherical code $\mathcal{C} \subset \Omega_k$ of given size L can achieve. It is clear that, for any given space dimension $k \ge 2$, $A(k,\, d)$ defined in (27) is a monotonically decreasing staircase-like function in terms of $0 < d \le 2$, and $D(k,\, L)$ defined in (28) is a monotonically decreasing sequence in terms of $L \ge 2$. Furthermore, for each $k \ge 2$, we have

$$\lim_{d\to 0^+} A(k,\, d) = +\infty$$

and

$$\lim_{L\to+\infty} D(k,\, L) = 0.$$

On the other hand, for any fixed $0 < d \le 2$ and $L \ge 2$, $A(k,\, d)$ and $D(k,\, L)$ are both monotonically increasing sequences in terms of $k \ge 2$, since any k-dimensional spherical code $\mathcal{C} \subset \Omega_k$ is also a $(k+1)$-dimensional spherical code in Ω_{k+1}.

The two metrics defined above, namely, $A(k,\, d)$ and $D(k,\, L)$, for evaluating the quality of a spherical code are essentially equivalent. In the current application situation, we are more interested in the quantity $D(k,\, L)$ than $A(k,\, d)$, since we can employ $D(k,\, L)$ to get the upper and lower bounds on the optimal diversity product and sum of unitary signal constellations with any code size through the following approach.

Theorem 3: For any constellation size $L \geq 2$, the optimal diversity product and sum have the following bounds.

1) In the case $M = 2$, we have
$$\Delta(L) \geq \Xi(L) \geq \frac{1}{2} D(4, L).$$
2) In the case $M = 2$, we have
$$\Xi(L) \leq \frac{\sqrt{1+\sqrt{2}}}{2} D(6, L).$$
3) In the case $M = 2$, we have
$$\Delta(L) \leq \frac{\sqrt{3+\sqrt{5}}}{2\sqrt{2}} D(6, L).$$
4) For all $M \geq 1$, we have
$$\Xi_M(L) \leq \Delta_M(L) \leq \frac{1}{2} D(2M^2, L)$$
where the equality holds in the case $M = 1$, i.e.,
$$\Xi_1(L) = \Delta_1(L) = \frac{1}{2} D(2, L) = \sin(\pi/L).$$

Proof: See Appendix G. □

It is noted that the lower bound in Property 1) of Theorem 3 for the optimal diversity product of 2×2 unitary constellations, i.e., $D(4, L)/2$, is actually the maximal possible diversity product of 2×2 Hamiltonian constellations [31], which form the group of Hamiltonian quaternions of norm 1, with size $L \geq 2$.

We emphasize that most studies in the earlier works on spherical codes have used $A(k, d)$ as the figure of merit for a spherical code. As a result, there have been extensive upper and lower bounds on $A(k, d)$, while few ones on $D(k, L)$, in the literature, as shown in the seminal book of Conway and Sloane [5]. Moreover, the upper and lower bounds on $A(k, d)$ were generally given in terms of $A(k, \phi)$ [5], where $\phi = 2\arcsin(d/2)$ satisfying $0 < \phi \leq \pi$, or equivalently, $d = 2\sin(\phi/2)$ satisfying $0 < d \leq 2$. The notation $A(k, \phi)$ means the maximal size of a spherical code $\mathcal{C}$ with minimum angular separation larger than or equal to ϕ, i.e., with the property that $x^\top y \leq \cos\phi$ for any $x, y \in \mathcal{C}$ satisfying $x \neq y$.

In order to make Theorem 3 useful in practice, we have to establish the upper and lower bounds on $D(k, L)$ from the existing ones on $A(k, d)$ in the literature. It is clear that
$$\begin{aligned} A(k, d) &= \max\{L \mid D(k, L) \geq d\} \\ &= \min\{L \mid D(k, L) < d\} - 1 \\ &= \min\{L \mid D(k, L+1) < d\} \\ &\geq \min\{L \mid D(k, L) \leq d\} \end{aligned} \tag{29}$$
and
$$\begin{aligned} D(k, L) &= \max\{d \mid A(k, d) \geq L\} \\ &= \inf\{d \mid A(k, d) < L\}. \end{aligned} \tag{30}$$
From the above relationship between $A(k, d)$ and $D(k, L)$, namely, (29) and (30), we have immediately the following.

Proposition 8: Let $A^+(k, d)$ and $A_-(k, d)$ be, respectively, the upper and lower bounds on $A(k, d)$ for all $k \geq 2$ and $0 < d \leq 2$, i.e.,
$$A_-(k, d) \leq A(k, d) \leq A^+(k, d).$$
Then, for all $k \geq 2$ and for all $L \geq 2$, we have
$$\begin{aligned} \inf\{d \mid A_-(k, d) < L\} &\leq D(k, L) \\ &\leq \sup\{d \mid A^+(k, d) \geq L\} \end{aligned}$$
and
$$\begin{aligned} \sup\{d \mid A_-(k, d) \geq L\} &\leq D(k, L) \\ &\leq \inf\{d \mid A^+(k, d) < L\}. \end{aligned}$$

By applying the preceding method, we can give the following result based on which the upper bound on the optimal diversity product and sum given in Proposition 2 can be derived.

Proposition 9: The quantity $D(k, L)$ satisfies

1) $D(2, L) = 2\sin(\pi/L)$ for all $L \geq 2$;
2) for all $k \geq 3$ and for all $L \geq 2$, we have
$$D(k, L) \begin{cases} = \sqrt{\frac{2L}{L-1}}, & \text{if } 1 < L \leq k+1 \\ = \sqrt{2}, & \text{if } k+1 < L \leq 2k \\ < \sqrt{2}, & \text{if } L > 2k. \end{cases} \tag{31}$$

The above proposition is essentially Rankin's result on the precise values of $A(k, \phi)$ when $\pi/2 \leq \phi \leq \pi$, or equivalently, $A(k, d)$ when $\sqrt{2} \leq d \leq 2$, which can be stated as follows (see, [28, p. 139] and [5, p. 27]).

Lemma 1: For each $k \geq 2$, we have

1) $A(k, \phi) = \lfloor 1 - \sec\phi \rfloor$, for $\operatorname{arcsec}(-k) < \phi \leq \pi$, where $\lfloor\cdot\rfloor$ denotes the integer part of a nonnegative real number;
2) $A(k, \phi) = k + 1$, for $\pi/2 < \phi \leq \operatorname{arcsec}(-k)$;
3) $A(k, \pi/2) = 2k$.

From Lemma 1, it is known that $A(k, \phi)$ cannot take a value between $k + 1$ and $2k$.

Proof of Proposition 9:

1) The quantity $D(2, L)$ means the largest possible minimum distance d of a spherical code consisting of L distinct code points in the unit circle in the two-dimensional plane $\mathbb{R}^2$. From this, it follows that $D(2, L) = 2\sin(\pi/L)$.
2) From Properties 1) and 2) in Lemma 1, it follows that $A(k, \phi) = L$ for $\operatorname{arcsec}(-L) < \phi \leq \operatorname{arcsec}(-L+1)$ and $1 < L \leq k$ and that $A(k, \phi) = k+1$ for $\pi/2 < \phi \leq \operatorname{arcsec}(-k)$. By using
$$D(k, L) = \max\{\, d \mid A(k, d) \geq L \,\}$$
where $d = 2\sin(\phi/2)$, we can have that for $1 < L \leq k+1$
$$D(k, L) = 2\sin\left(\frac{1}{2}\operatorname{arcsec}(-L+1)\right).$$

From the definition of

$$\sec\theta = \frac{1}{\cos\theta} \in (-\infty,\, -1] \cup [1,\, +\infty)$$

for $\theta \in \mathbb{R}$ satisfying $\cos\theta \neq 0$, we know that

$$\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right) \in [0,\, \pi/2) \cup (\pi/2,\, \pi], \qquad \text{for } x \in \mathbb{R}$$

satisfying $|x| \geq 1$. By virtue of these facts and making use of

$$\sin\theta = \sqrt{\frac{1-\cos(2\theta)}{2}}, \qquad \text{for } \theta \in [0,\, \pi/2]$$

we obtain that for $1 < L \leq k+1$

$$D(k,\, L) = 2\sin\left(\frac{1}{2}\arccos\left(\frac{1}{1-L}\right)\right) = \sqrt{\frac{2L}{L-1}}.$$

When $k+1 < L \leq 2k$, by Lemma 1, it holds that

$$\begin{aligned} D(k,\, L) &= \max\{d \,|\, A(k,\, d) \geq L\} \\ &= \max\{d \,|\, A(k,\, d) \geq 2k\} \\ &= 2\sin(\pi/4) = \sqrt{2}. \end{aligned}$$

If $L > 2k$, then

$$\begin{aligned} D(k,\, L) &= \max\{d \,|\, A(k,\, d) \geq L\} \\ &\leq \max\{d \,|\, A(k,\, d) \geq 2k+1\} \\ &< 2\sin(\pi/4) = \sqrt{2} \end{aligned}$$

where the last strict inequality is implied by Property 3) in Lemma 1.

The proof of Proposition 9 is thus completed. □

By using the result (31), we can see that the strict inequality in (29) can hold in some case such as

$$A\left(8,\, \sqrt{2}\right) = 16 > 10 = \min\left\{L \,\middle|\, D(8,\, L) \leq \sqrt{2}\right\}.$$

It is interesting to note that we also have $D(k,\, A(k,\, d)) \geq d$ and $A(k,\, D(k,\, L)) \geq L$ for any $k \geq 2$, $0 < d \leq 2$, and $L \geq 2$, where the strict inequality may hold in some cases such as

$$D(8,\, A(8,\, d)) = D(8,\, 9) = 3/2 > d, \qquad \text{for } \sqrt{2} < d < 3/2$$

and

$$A(8,\, D(8,\, 10)) = A\left(8,\, \sqrt{2}\right) = 16 > 10.$$

Now, we want to present Yaglom's lower bound on the optimal diversity product and sum of 2×2 unitary signal constellations. We need some knowledge in sphere packing (see [5] and [10]). A *sphere packing* (or simply *packing*) is a set of mutually disjoint, equal radius, open spheres. The *packing radius* is the radius of the spheres in a packing. The packing radius is normally as large as possible such that there are tangent spheres in the packing but no overlapping spheres. As defined in [29], a packing is said to have *density* Δ if the ratio of the volume of the part of a hypercube covered by the spheres of the packing to the volume of the whole hypercube tends to the limit Δ, as the side length of the hypercube tends to infinity. That is, the density is the fraction of space occupied by the spheres of the packing. The highest density of any sphere packing in $\mathbb{R}^k$, denoted by Δ_k^{pack}, is the density of the densest k-dimensional sphere packing. It is known that $\Delta_1^{\text{pack}} = 1$ and $\Delta_2^{\text{pack}} = \frac{\pi}{\sqrt{12}}$. They are the only two known and provably precise values among Δ_k^{pack} for $k \in \mathbb{N}$ [5]. A long-standing conjecture, the so-called "Kepler conjecture," stated that no packing of three-dimensional spheres can have a greater density than that of the face-centered cubic lattice. The density of the three-dimensional face-centered cubic lattice is $\pi/\sqrt{18} = 0.74048\ldots$. The current status of Kepler conjecture has been examined in [5] and [32]. Moreover, Muder [23] obtained the upper bound of $0.773055\ldots$ on the density Δ_3^{pack}.

Yaglom's lower bound on $A(k,\, \phi)$ for $0 < \phi \leq \pi$ can be stated as follows (see [44], [5, p. 265]).

Lemma 2: For all $k \geq 2$ and $0 < \phi \leq \pi$, we have

$$A(k,\, \phi) \geq (\sin(\phi/2))^{-(k-1)}\Delta_{k-1}^{\text{pack}} = \left(\frac{2}{d}\right)^{k-1}\Delta_{k-1}^{\text{pack}}$$

where $d = 2\sin(\phi/2)$.

Therefore, according to (30) and Proposition 8, we have, for all $k \geq 2$ and for all $L \geq 2$

$$D(k,\, L) \geq 2\left(\Delta_{k-1}^{\text{pack}}\right)^{\frac{1}{k-1}} L^{-\frac{1}{k-1}}. \tag{32}$$

From (32) and Property 1) in Theorem 3, we get the following corollary.

Corollary 1: In the case $M = 2$, we have

$$\begin{aligned} \Delta(L) \geq \Xi(L) &\geq \left(\frac{\Delta_3^{\text{pack}}}{L}\right)^{\frac{1}{3}} \\ &\geq \left(\frac{\pi}{3\sqrt{2}}\right)^{\frac{1}{3}} L^{-\frac{1}{3}} \\ &\gtrsim 0.904699\, L^{-\frac{1}{3}}, \qquad \text{for } L \geq 2 \end{aligned} \tag{33}$$

where the notation $a \gtrsim b$ means that the number a is greater than but similar to b.

It is noted that the lower bound on the optimal diversity product and sum given in Corollary 1 is represented *explicitly* for all $L \geq 2$. That is, the lower bound given in Corollary 1 is a function of the constellation size L in an *explicit* form. More available bounds on the optimal diversity product and sum can only be obtained generally through an *implicit* method in the sense that the bounds are written as the *implicit* functions, in terms of the constellation size L, which can be determined by an optimization procedure.

We can derive Coxeter's upper bound on the optimal diversity product and sum by using Coxeter's upper bound on $A(k,\, \phi)$ for $0 < \phi \leq \pi$, which employs the Schläfli's function defined recursively by

$$F_{k+1}(\alpha) = \frac{2}{\pi}\int_{(\operatorname{arcsec} k)/2}^{\alpha} F_{k-1}(\beta)\, d\theta$$

where $\sec(2\beta) = \sec(2\theta) - 2$ and the initial conditions are $F_0(\alpha) = F_1(\alpha) = 1$. Coxeter [6] conjectured and Böröczky [3] proved the following result.

Lemma 3: For all $k \geq 2$ and $0 < \phi \leq \pi$, we have

$$A(k, \phi) \leq A_{\text{coxeter}}(k, \phi) \stackrel{\text{def}}{=} \frac{2F_{k-1}(\alpha)}{F_k(\alpha)}$$

where α is determined by

$$\sec(2\alpha) = \sec\phi + k - 2 = \frac{2}{2 - d^2} + k - 2.$$

Then, according to Properties 2)–4) in Theorem 3 and Proposition 8, we can obtain

$$\Xi(L) \leq \frac{\sqrt{1+\sqrt{2}}}{2} \sup\{d \,|\, A_{\text{coxeter}}(6, \phi) \geq L\}, \qquad \text{for } L \geq 2 \tag{34}$$

$$\Delta(L) \leq \frac{\sqrt{3+\sqrt{5}}}{2\sqrt{2}} \sup\{d \,|\, A_{\text{coxeter}}(6, \phi) \geq L\}, \qquad \text{for } L \geq 2 \tag{35}$$

and

$$\Delta(L) \leq \frac{1}{2} \sup\{d \,|\, A_{\text{coxeter}}(8, \phi) \geq L\}, \qquad \text{for } L \geq 2 \tag{36}$$

where the functions $A_{\text{coxeter}}(k, \phi)$ for $k = 6$ and $k = 8$ are defined in Lemma 3 and $\phi = 2\arcsin(d/2)$.

We emphasize that, by applying Lemma 3 and Proposition 8, we can provide an upper bound on the maximal possible diversity product of 2×2 Hamiltonian constellations [31] with any size $L \geq 2$, namely, $D(4, L)/2$, given by

$$\frac{1}{2}D(4, L) \leq \frac{1}{2} \sup\{d \,|\, A_{\text{coxeter}}(4, \phi) \geq L, \phi = 2\arcsin(d/2)\}, \qquad \text{for } L \geq 2 \tag{37}$$

where the function $A_{\text{coxeter}}(4, \phi)$ is defined in Lemma 3. With the help of inequality (37), we can obtain, by using a computer program, an upper bound on the maximal possible diversity product of 2×2 Hamiltonian constellations with *16* unitary signals being $0.58196571\ldots$, which is *smaller than* the diversity product of the parametric code $\mathcal{V}(3, 4, 2)$ of size 16, given in Table III, being $0.59460356\ldots$. This indicates that the parametric code of size 16 has a diversity product better than that of any Hamiltonian constellation of the same size 16 generated from a spherical code lying on Ω_4 defined in (26).

In a similar way as for obtaining Coxeter's upper bound in the inequalities (34)–(36), we can also derive the Rankin's upper bound on the optimal diversity product and sum by using Rankin's upper bound on $A(k, \phi)$ for $0 < \phi < \pi/2$ [28] as stated in the following lemma.

Lemma 4: For all $k \geq 2$ and $0 < \phi < \pi/2$, we have

$$A(k, \phi) \leq A_{\text{rankin}}(k, \phi) \stackrel{\text{def}}{=} \frac{\sqrt{\pi}\,\Gamma\left(\frac{k-1}{2}\right) \sin\beta \tan\beta}{2\,\Gamma\left(\frac{k}{2}\right) \int_0^\beta (\sin\theta)^{k-2}(\cos\theta - \cos\beta)\, d\theta}$$

where $\beta = \arcsin(\sqrt{2}\sin(\phi/2))$ and $\Gamma(x)$ is the usual gamma function defined by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt, \qquad \text{for } x \in \mathbb{R} \text{ and } x > 0.$$

It is remarked that when ϕ lying in $(0, \pi/2)$ is very close to $\pi/2$, the above function $A_{\text{rankin}}(k, \phi)$ tends to infinity. In fact

$$\lim_{\phi \to (\pi/2)^-} A_{\text{rankin}}(k, \phi) = \lim_{\beta \to (\pi/2)^-} \frac{C_k}{\cos\beta} = +\infty$$

where the positive constant C_k does not depend on ϕ and β. Therefore, the preceding Rankin's upper bound on $A(k, \phi)$ is not effective in estimating $A(k, \phi)$ in practice when $\phi \in (0, \pi/2)$ and is very close to $\pi/2$. Rankin's upper bound can apply to the unitary signal constellations with size $L \geq 17$ and be given by

$$\Xi(L) \leq \frac{\sqrt{1+\sqrt{2}}}{2} \sup\{d \,|\, A_{\text{rankin}}(6, \phi) \geq L\}, \qquad \text{for } L \geq 17 \tag{38}$$

$$\Delta(L) \leq \frac{\sqrt{3+\sqrt{5}}}{2\sqrt{2}} \sup\{d \,|\, A_{\text{rankin}}(6, \phi) \geq L\}, \qquad \text{for } L \geq 17 \tag{39}$$

and

$$\Delta(L) \leq \frac{1}{2} \sup\{d \,|\, A_{\text{rankin}}(8, \phi) \geq L\}, \qquad \text{for } L \geq 17 \tag{40}$$

where the functions $A_{\text{rankin}}(k, \phi)$ for $k = 6$ and $k = 8$ are defined in Lemma 1 and $\phi = 2\arcsin(d/2)$.

It can be shown by asymptotic analysis that, for large constellation size L, the above Coxeter's upper bounds of (34)–(36) on the optimal diversity product and sum are better than Rankin's upper bounds of (38)–(40), respectively, the details of which are omitted here due to the limitation of space. For $17 \leq L \leq 128$, the comparison between them can be made through numerical evaluation, as shown in Fig. 5. It is seen from Fig. 5 that Coxeter and Rankin's upper bounds on $D(k, L)$ satisfy

$$\sup\{d \,|\, A_{\text{coxeter}}(k, \phi) \geq L\} < \sup\{d \,|\, A_{\text{rankin}}(k, \phi) \geq L\}$$

for $k = 6$ and $k = 8$ and for $17 \leq L \leq 128$. Hence, for $17 \leq L \leq 128$, Coxeter's upper bounds of (34)–(36) on the optimal diversity product and sum are, respectively, better than Rankin's upper bounds of (38)–(40).

The numerical values of Coxeter's upper bounds of (34)–(36) for $17 \leq L \leq 128$ are plotted in Fig. 6, which shows that the upper bound (36) is the best among the above three upper bounds. However, for large L, the upper bounds of (34) and (35) should be better than (36), since the former two bounds can be shown to be of the order of $L^{-1/5}$ while the upper bound (36) is of the order of $L^{-1/7}$ for large constellation size L.

Besides Yaglom's lower bound given in Corollary 1, another lower bound on the optimal diversity product and sum can be deduced by using the following Wyner's lower bound on $A(k, \phi)$ for $0 < \phi < \pi/2$ [42].

Lemma 5: For all $k \geq 2$ and $0 < \phi < \pi/2$, we have

$$A(k, \phi) \geq A_{\text{wyner}}(k, \phi) \stackrel{\text{def}}{=} \frac{k}{k-1} \sqrt{\pi} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} \left[\int_0^\phi (\sin\theta)^{k-2}\, d\theta\right]^{-1}.$$

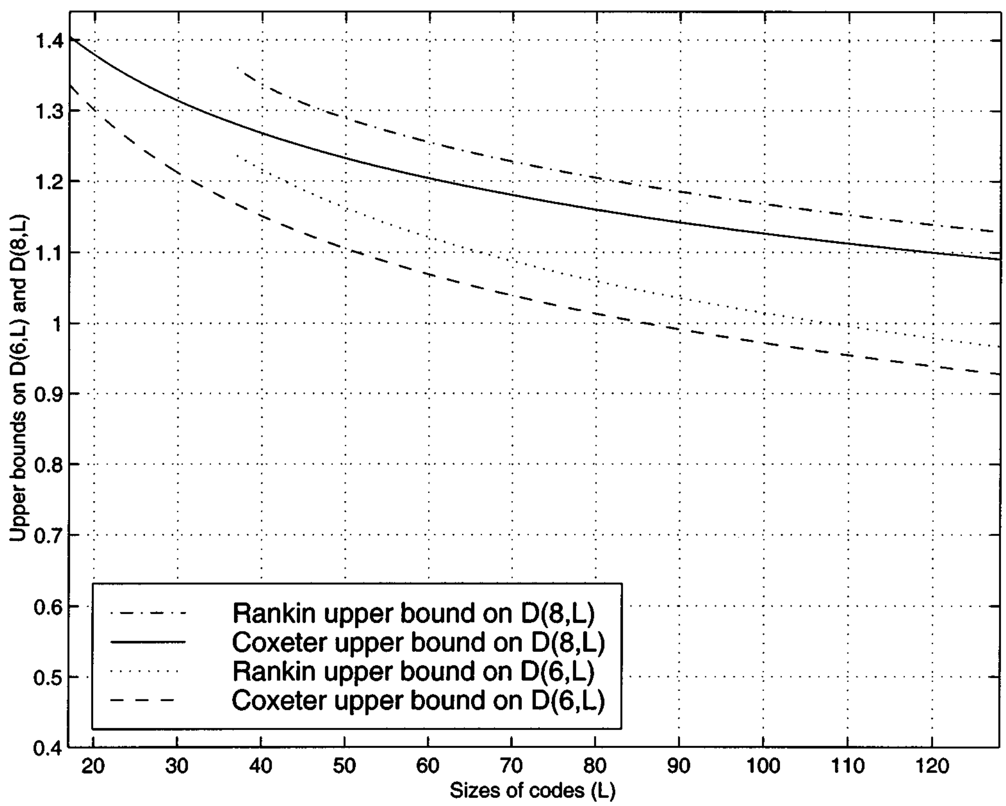

Fig. 5. Comparison of Coxeter and Rankin's upper bounds on $D(6,\ L)$ and $D(8,\ L)$.

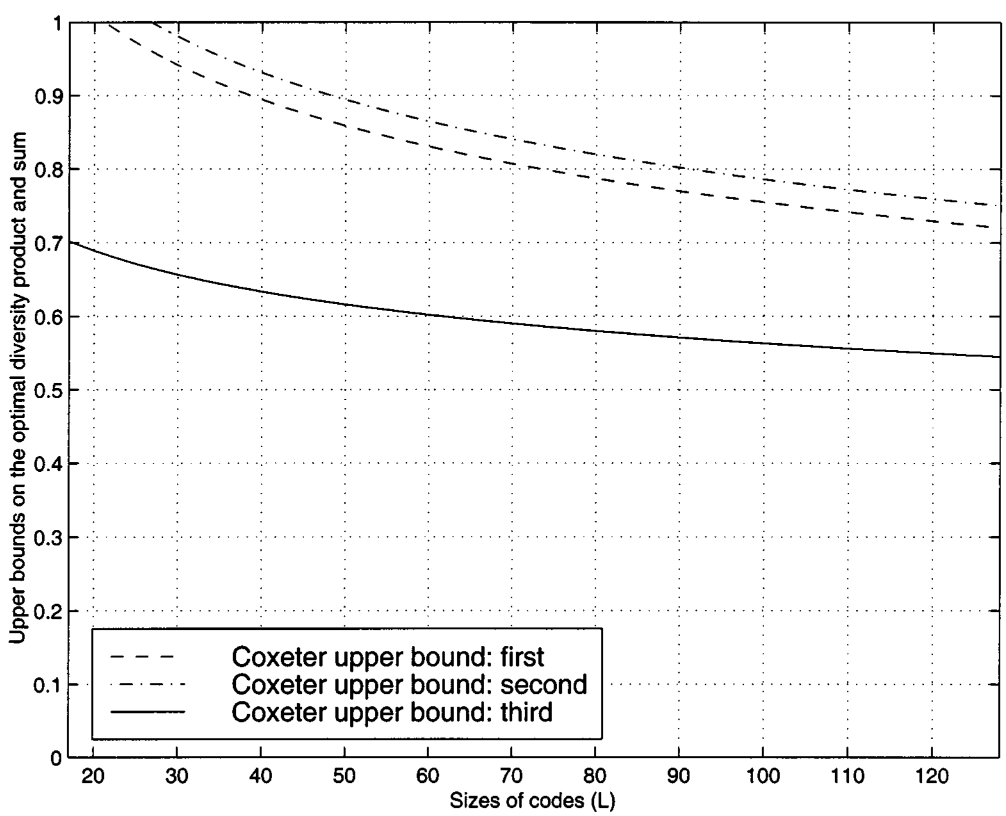

Fig. 6. Comparison of Coxeter's three upper bounds on $\Xi(L)$ and $\Delta(L)$, where the first, second, and third bounds are, respectively, given by (34)–(36).

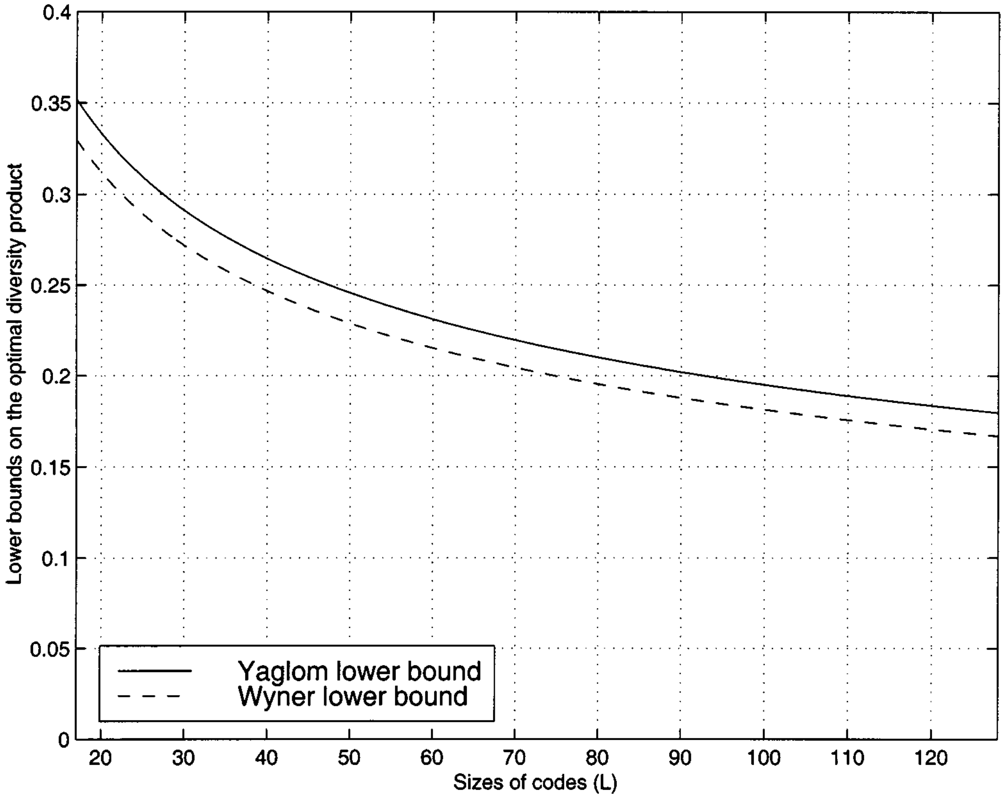

Fig. 7. Comparison of Yaglom's and Wyner's lower bounds on the optimal diversity products $\Xi(L)$.

Wyner's lower bound can apply to the unitary signal constellations with size $L \geq 17$ and be given by

$$\begin{aligned}\Delta(L) &\geq \Xi(L)\\ &\geq \frac{1}{2}\inf\{d \,|\, A_{\rm wyner}(4,\, \phi) < L,\ \phi = 2\arcsin(d/2)\},\\ &\qquad\qquad\text{for } L \geq 17 \qquad (41)\end{aligned}$$

where $A_{\rm wyner}(4, \phi)$ defined in Lemma 5 can be written as $A_{\rm wyner}(4,\, \phi) = 2\pi[2\phi - \sin(2\phi)]^{-1}$.

For $17 \leq L \leq 128$, the numerical simulation results of Yaglom's lower bound given by (33) and Wyner's lower bound given by (41) are presented in Fig. 7, which shows that Yaglom's lower bound on the optimal diversity product and sum is better than Wyner's lower bound. Furthermore, we can prove analytically that Yaglom's lower bound given by (33) is actually better than Wyner's lower bound in (41) for all $L \geq 128$. The following lemma is needed for our proof.

Lemma 6: For $x \in \mathbb{R}$ and $x > 0$, we have

$$\frac{1}{3!}\,x^3 - \frac{1}{5!}\,x^5 < x - \sin x < \frac{1}{6}\,x^3.$$

Proof: Let the functions

$$h_1(x) = \frac{1}{6}\,x^3 - x + \sin x$$

and

$$h_2(x) = x - \sin x - \frac{1}{6}\,x^3 + \frac{1}{120}\,x^5, \qquad \text{for } x \geq 0.$$

Then, the first-order derivative of $h_1(x)$ is

$$h_1'(x) = \frac{1}{2}\,x^2 - 1 + \cos\, x = \frac{1}{2}\,x^2 - 2\sin^2\left(\frac{x}{2}\right).$$

Therefore, $h_1'(x) > 0$ for $x > 0$. By applying Taylor's theorem, we know that $h_1(x) > 0$ for $x > 0$.

The first- to fourth-order derivatives of $h_2(x)$ are, respectively

$$\begin{aligned}h_2'(x) &= 1 - \cos x - \frac{1}{2}\,x^2 + \frac{1}{4!}\,x^4\\ h_2''(x) &= \sin x - x + \frac{1}{3!}\,x^3\\ h_2'''(x) &= \cos x - 1 + \frac{1}{2}\,x^2\end{aligned}$$

and

$$h_2''''(x) = x - \sin x$$

which is larger than zero for $x > 0$. By applying Taylor's theorem again, we obtain $h_2(x) > 0$ for $x > 0$ as well. The proof is completed. □

Proposition 10: For all $L \geq 128$, we have

$$\begin{aligned}0.834404\, L^{-\frac{1}{3}} &\leq \frac{1}{2}\inf\{d \,|\, A_{\rm wyner}(4,\, \phi) < L\}\\ &\leq 0.844644\, L^{-\frac{1}{3}}.\end{aligned}$$

Proof: Let $\phi_0 \stackrel{\rm def}{=} 0.33517$, which satisfies

$$2\phi_0 - \sin(2\phi_0) > 0.490875 > 2\pi/128.$$

Then, for $L \geq 128$, we have

$$\begin{aligned}\inf\{\phi \,|\, A_{\rm wyner}(4,\, \phi) < L\} &= \inf\{\phi \,|\, 2\pi/L < 2\phi - \sin(2\phi)\}\\ &\leq \inf\{\phi \,|\, 2\pi/128 < 2\phi - \sin(2\phi)\}\\ &< \phi_0.\end{aligned}$$

From this and Lemma 6, we can obtain, for $L \geq 128$

$$\begin{aligned}
&\inf\{\phi \,|\, A_{\text{wyner}}(4,\,\phi) < L\} \\
&\quad = \inf\{\phi < \phi_0 \,|\, 2\pi/L < 2\phi - \sin(2\phi)\} \\
&\quad \leq \inf\left\{\phi < \phi_0 \left| \frac{2\pi}{L} < \frac{1}{3!}\,(2\phi)^3 - \frac{1}{5!}\,(2\phi)^5 \right.\right\} \\
&\quad \leq \inf\left\{\phi < \phi_0 \left| \frac{2\pi}{L} < \frac{4}{3}\,\phi^3\left(1 - \frac{1}{5}\,\phi_0^2\right) \right.\right\} \\
&\quad = \left(\frac{15\pi}{10 - 2\phi_0^2}\right)^{\frac{1}{3}} L^{-\frac{1}{3}} < \phi_0.
\end{aligned}$$

Therefore, by $d = 2\sin(\phi/2) \leq \phi$

$$\begin{aligned}
\frac{1}{2}\inf\{d \,|\, A_{\text{wyner}}(4,\,\phi) < L\} &\leq \frac{1}{2}\left(\frac{15\pi}{10 - 2\phi_0^2}\right)^{\frac{1}{3}} L^{-\frac{1}{3}} \\
&\leq 0.844644\, L^{-\frac{1}{3}}.
\end{aligned}$$

On the other hand, by Lemma 6, we have, for $L \geq 128$

$$\begin{aligned}
&\inf\{\phi \,|\, A_{\text{wyner}}(4,\,\phi) < L\} \\
&\quad = \inf\{\phi \,|\, 2\pi/L < 2\phi - \sin(2\phi)\} \\
&\quad \geq \inf\{\phi \,|\, 2\pi/L < 4\phi^3/3\} = \left(\frac{3}{2}\,\pi\right)^{\frac{1}{3}} L^{-\frac{1}{3}}.
\end{aligned}$$

Then, by $d = 2\sin(\phi/2) \geq \phi - \frac{1}{24}\,\phi^3$, we obtain, for $L \geq 128$

$$\begin{aligned}
&\frac{1}{2}\inf\{d \,|\, A_{\text{wyner}}(4,\,\phi) < L\} \\
&\quad \geq \left(\frac{3}{16}\,\pi\right)^{\frac{1}{3}} L^{-\frac{1}{3}} \left(1 - \frac{1}{24}\left(\frac{3}{2}\,\pi\right)^{\frac{2}{3}} 128^{-\frac{2}{3}}\right) \\
&\quad \geq 0.834404\, L^{-\frac{1}{3}}.
\end{aligned}$$

The proof of Proposition 10 is thus completed. □

According to Proposition 10 and Fig. 7, Yaglom's lower bound on the optimal diversity product and sum, given by (33), is better than Wyner's lower bound given by (41) for all $L \geq 17$.

Now, we give some asymptotic bounds on the optimal diversity product and sum or their decaying rates for unitary signal constellations with large code sizes. A notation in asymptotic analysis is needed and described in what follows. For any two nonnegative real-number sequences $f(L)$ and $g(L)$ in terms of $L \in \mathbb{N}$, the notation $f(L) = O(g(L))$ means that there exist a positive constant $C > 0$ independent of L and some positive integer L_0 such that $f(L) \leq Cg(L)$ for all $L \geq L_0$.

Theorem 4: For large constellation size L, the optimal diversity product and sum satisfy the following conditions.

1) In the case $M = 2$, we have

$$\begin{aligned}
\Delta(L) \geq \Xi(L) &\geq \frac{\pi^{\frac{2}{3}}}{\sqrt{2}}\, L^{-\frac{1}{3}}(1 + o_L(1)) \\
&\gtrsim 1.516764\, L^{-\frac{1}{3}}(1 + o_L(1)).
\end{aligned}$$

2) For all $M \geq 2$, we have

$$\Xi_M(L) \leq \Delta_M(L) = O\left(L^{-\frac{1}{M^2}}\right).$$

Proof: See Appendix H. □

The numerical values of the diversity product and sum of the parametric codes and cyclic codes are plotted in Fig. 8 for $17 \leq L \leq 128$. The best known Coxeter's upper bound given by (36) for $17 \leq L \leq 128$ and the best known Yaglom's lower bound in an *analytical* form given by (33) are also shown in Fig. 8. It can be seen from Fig. 8 that the parametric codes have a large improvement in both diversity product and diversity sum over the cyclic codes. The numerical values of Coxeter's upper bounds on the optimal diversity product and sum in the case $M = 2$ for some constellation sizes L are listed in Table VI.

VI. Conclusion

In this paper, by making use of the parameterization of unitary groups, we have proposed a new class of unitary space–time codes, called parametric codes, for the differential modulation with double transmit antennas across a Rayleigh-fading channel whose fading coefficients are unknown to both the transmitter and the receiver. The parametric codes have been shown to have a large improvement in diversity product and diversity sum over the diagonal cyclic codes. It has been shown that the parametric codes can lead to a five-signal constellation which has the largest possible diversity product and sum and a 16-signal constellation which possesses the largest known diversity product and the largest possible diversity sum. Although the parametric code of size 16 is not a group by itself, it is a subset of a group of order 32. Computer simulation results have demonstrated that, compared with the existing unitary space–time codes, the above 16-signal constellation has an improvement in terms of the block error rate up to 1 dB at SNR 22 dB in the case of two receive antennas and at SNR 10 dB in the case of five receive antennas. Furthermore, the unitary space–time codes of sizes $32,\ 64,\ 128$, and 256 as the subsets taken from the parametric codes of sizes $37,\ 75,\ 135$, and 273, respectively, have the largest known diversity products in the literature. These 2×2 unitary space–time codes may be useful not only in two-transmit-antenna systems but also in single-transmit-antenna systems with frequency-selective fading as described in [43] where a precoded and vector orthogonal frequency-division multiplexing (OFDM) was introduced.

For the differential modulation with double transmit antennas, we have presented unitary signal constellations with the optimal diversity products for sizes up to 5 and the unitary signal constellations with the optimal diversity sums for sizes up to 16. Considering the theoretical and practical significance of the upper and lower bounds on the optimal diversity product and sum that unitary signal constellations of any given size can achieve, we have investigated these bounds by resorting to the existing numerous results in sphere packing and spherical codes. A main conclusion is that for the 2×2 unitary signal constellations, the optimal diversity product and sum are of an order between $L^{-1/3}$ and $L^{-1/4}$ for large constellation size L. For the general $M \times M$ $(M \geq 1)$ unitary signal constellations, the optimal diversity product and sum are of an order of $O(L^{-1/M^2})$ for a large constellation size L.

Appendix A
Proof of Proposition 1

We define the function

$$H(\omega) = \prod_{m=1}^{M}\left[1 + \frac{\rho^2\sigma_m^2}{1 + 2\rho}\left(\omega^2 + \frac{1}{4}\right)\right]^{-N}, \qquad \omega \in \mathbb{R}$$

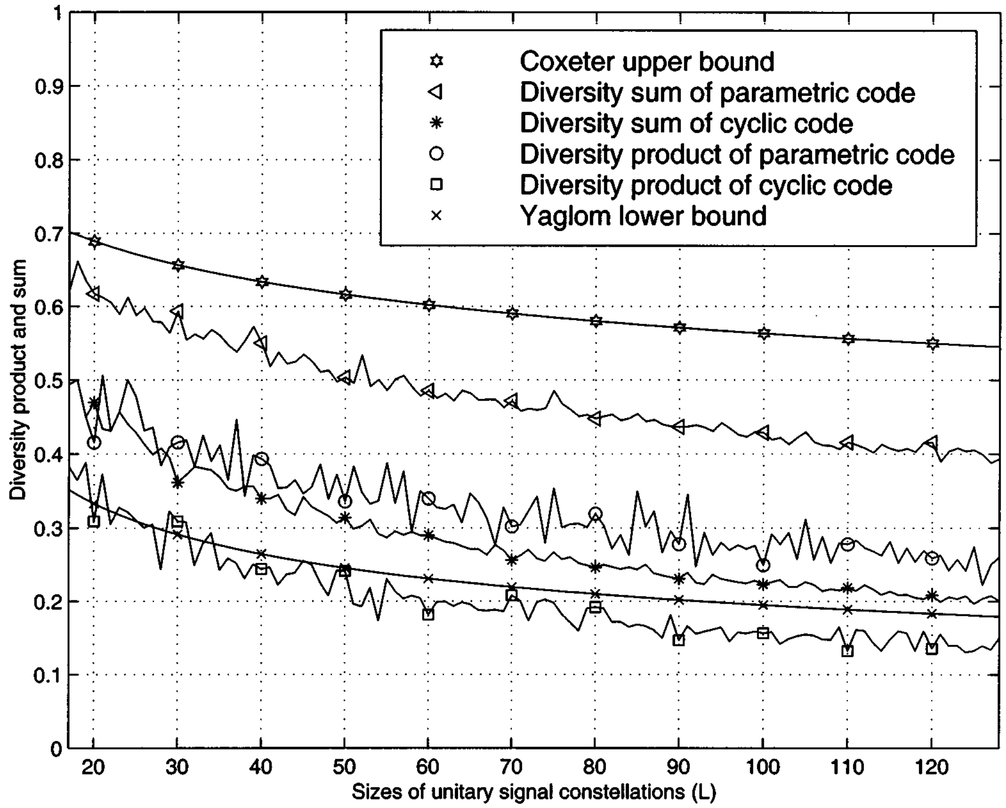

Fig. 8. Diversity products and sums of parametric and cyclic codes with lower and upper bounds.

TABLE VI
COXETER'S UPPER BOUNDS ON THE OPTIMAL DIVERSITY PRODUCTS AND SUMS FOR TWO TRANSMIT ANTENNAS

L	17	18	24	32	48	64	80	100	120	128
UBs	0.7023	0.6977	0.6746	0.6515	0.6193	0.5969	0.5799	0.5632	0.5499	0.5452

which is even and monotonically decreasing with respect to $\omega \geq 0$. Then the right-hand side of (8) is

$$F(a) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^a \frac{1}{\omega^2 + \frac{1}{4}} H(\omega)\, d\omega + \left(\frac{1}{2} - \frac{\arctan(2a)}{\pi}\right) H(a)$$
$$= P_{\ell,\ell'} + \frac{1}{2\pi} \int_a^{+\infty} \frac{1}{\omega^2 + \frac{1}{4}} (H(a) - H(\omega))\, d\omega$$

for $a \in [0, +\infty)$.

1) Let $0 \leq a < b < +\infty$, then we have

$$\begin{aligned} F(a) - F(b) &= \frac{1}{2\pi} \left[\int_a^b \frac{1}{\omega^2 + \frac{1}{4}} (H(a) - H(\omega))\, d\omega \right. \\ &\qquad \left. + \int_b^{+\infty} \frac{1}{\omega^2 + \frac{1}{4}} (H(a) - H(b))\, d\omega \right] \\ &\geq \frac{H(a) - H(b)}{2\pi} \int_b^{+\infty} \frac{1}{\omega^2 + \frac{1}{4}}\, d\omega \\ &= \left(\frac{1}{2} - \frac{\arctan(2b)}{\pi}\right) (H(a) - H(b)) \geq 0 \end{aligned}$$

which means that $F(a)$ is monotonically decreasing with respect to $a \geq 0$. Moreover, if $\rho > 0$ and there is at least one singular value of $V_\ell - V_{\ell'}$ which is nonzero (i.e., $V_\ell \neq V_{\ell'}$), then the above function $H(\omega)$ is *strictly* monotonically decreasing with respect to $\omega \geq 0$. Hence, the function $F(a)$ in term of $a \geq 0$ is *strictly* monotonically decreasing as well.

It is clear that $F(a)$ approaches $P_{\ell,\ell'}$ as a tends to infinity. That is,

$$\lim_{a \to +\infty} F(a) = \frac{1}{2\pi} \int_0^\infty \frac{1}{\omega^2 + \frac{1}{4}} H(\omega)\, d\omega = P_{\ell,\ell'}.$$

2) Now, we derive an upper bound on the nonnegative relative error of $F(a)$ for $a > 0$ when used to numerically evaluate the pairwise error probability $P_{\ell,\ell'}$ as follows:

$$\begin{aligned} \frac{F(a) - P_{\ell,\ell'}}{P_{\ell,\ell'}} &= \frac{\int_a^{+\infty} \left(\omega^2 + \frac{1}{4}\right)^{-1} (H(a) - H(\omega))\, d\omega}{\int_0^{+\infty} \left(\omega^2 + \frac{1}{4}\right)^{-1} H(\omega)\, d\omega} \\ &\leq \frac{H(a) \int_a^{+\infty} \left(\omega^2 + \frac{1}{4}\right)^{-1} d\omega}{H(a) \int_0^a \left(\omega^2 + \frac{1}{4}\right)^{-1} d\omega} \\ &= \frac{\pi}{2 \arctan(2a)} - 1. \end{aligned}$$

3) Let $\tilde{M} \geq 1$ be the number of nonzero singular values of the $M \times M$ matrix $V_\ell - V_{\ell'}$. We want to prove that, for large-SNR ρ, the pairwise error probability $P_{\ell,\ell'}$ and its Chernoff bound $F(0)$ decay at a rate of the same order. Without loss of generality, we assume $\tilde{M} = M$ in the sequel.

By use of the relation $1/[1+\rho] = (1+o_\rho(1))/\rho$ when $\rho \to +\infty$, we can see that for large ρ

$$H(\omega) = \prod_{m=1}^{M} \left[\frac{1+2\rho}{\rho^2 \sigma_m^2 \left(\omega^2 + \frac{1}{4}\right)} (1+o_\rho(1)) \right]^N = (1+o_\rho(1)) \prod_{m=1}^{M} \left(\frac{1+2\rho}{\rho^2 \sigma_m^2 \left(\omega^2 + \frac{1}{4}\right)} \right)^N$$

where $\sigma_m = \sigma_m(V_\ell - V_{\ell'})$ is the mth singular value of the $M \times M$ difference matrix $V_\ell - V_{\ell'}$ for $m = 1, 2, \ldots, M$. Then, for large ρ

$$\begin{aligned} P_{\ell,\ell'} &= \frac{1}{2\pi} \int_0^\infty \frac{1}{\omega^2 + \frac{1}{4}} \times \left[(1+o_\rho(1)) \prod_{m=1}^{M} \left(\frac{1+2\rho}{\rho^2 \sigma_m^2 \left(\omega^2 + \frac{1}{4}\right)} \right)^N \right] d\omega \\ &= \frac{1+o_\rho(1)}{2\pi} \left(\frac{1+2\rho}{\rho^2} \right)^{MN} \times \prod_{m=1}^{M} \frac{1}{\sigma_m^{2N}} \int_0^\infty \left(\omega^2 + \frac{1}{4} \right)^{-(MN+1)} d\omega. \end{aligned}$$

On the other hand, for large ρ, the Chernoff bound of the pairwise error probability $P_{\ell,\ell'}$ is

$$F(0) = \frac{1}{2} H(0) = (1+o_\rho(1)) \frac{4^{MN}}{2} \left(\frac{1+2\rho}{\rho^2} \right)^{MN} \prod_{m=1}^{M} \frac{1}{\sigma_m^{2N}}.$$

Therefore, for large ρ

$$\begin{aligned} \frac{P_{\ell,\ell'}}{F(0)} &= \frac{1}{4^{MN}\pi} (1+o_\rho(1)) \int_0^\infty \left(\omega^2 + \frac{1}{4} \right)^{-(MN+1)} d\omega \\ &= \frac{1}{4^{MN}\pi} \int_0^\infty \left(\omega^2 + \frac{1}{4} \right)^{-(MN+1)} d\omega + o_\rho(1) \\ &\overset{\omega=\frac{1}{2}t}{=} \frac{2}{\pi} \int_0^\infty (t^2+1)^{-(MN+1)} dt + o_\rho(1) \\ &\overset{t=\tan\theta}{=} \frac{2}{\pi} \int_0^{\pi/2} (\cos\theta)^{2MN} d\theta + o_\rho(1) \\ &= \prod_{k=1}^{MN} \left(1 - \frac{1}{2k} \right) + o_\rho(1). \end{aligned}$$

The proof of Proposition 1 is thus completed. □

APPENDIX B
PROOF OF PROPOSITION 2

We first introduce two inequalities which play a key role in our subsequent proof.

Let $a_1, a_2, \ldots, a_M$ be M positive-real numbers and

$$a \overset{\text{def}}{=} (a_1, a_2, \ldots, a_M)^\top \in \mathbb{R}^M.$$

We define the *elementary symmetric function* of degree m of the M elements $a_1, a_2, \ldots, a_M$ as follows:

$$E_m = E_m(a) = E_m(a_1, a_2, \ldots, a_M) = \sum_{1 \le i_1 < \cdots < i_m \le M} \prod_{k=1}^{m} a_{i_k}, \qquad m = 1, 2, \ldots, M.$$

For the mth elementary symmetric function E_m, we call

$$P_m = \frac{E_m}{\binom{M}{m}}$$

the mth *weighted elementary symmetric function* of the M elements $a_1, a_2, \ldots, a_M$. Then, for the weighted elementary symmetric functions $P_1, P_2, \ldots, P_M$, we have the following two inequalities [21, p. 106]:

$$P_1 \ge P_2^{1/2} \ge P_3^{1/3} \ge \cdots \ge P_M^{1/M}$$

and $P_m^2 \ge P_{m+1} P_{m-1}$ for $1 < m < M$. By (11), we have

$$P_m = (2D_m)^{2m} = (4D_m^2)^m = (2^m D_m^m)^2$$

where $D_m \overset{\text{def}}{=} D_m(V_1, V_2)$ for $m = 1, 2, \ldots, M$. By virtue of the above two inequalities, we obtain

$$D_1(V_1, V_2) \ge D_2(V_1, V_2) \ge \cdots \ge D_M(V_1, V_2) \qquad (42)$$

and

$$D_m^{2m}(V_1, V_2) \ge D_{m+1}^{m+1}(V_1, V_2) D_{m-1}^{m-1}(V_1, V_2)$$

for $1 < m < M$. Therefore, according to the definition of $\xi_m(L, \mathcal{V})$ given in (12), the conclusions in Propoerty 1) can immediately be derived. In the sequel, we give a proof of Properties 2)–4). By Property 1), it suffices to prove these inequalities in the case $m = 1$.

For any $M \times M$ unitary matrix V, the M^2 many complex numbers in the $M \times M$ complex matrix $V/\sqrt{M}$ can be regarded as $2M^2$ many real numbers whose squared sum is unity. Hence, a finite set of the $M \times M$ matrices $V_\ell/\sqrt{M}$ for $\ell \in \mathbb{Z}_L$ is equivalent to a spherical code lying on the surface of the unit-radius sphere Ω_{2M^2} defined by (26). Therefore, $\xi_1(L, \mathcal{V})$ in (13) cannot exceed half of the largest possible minimum distance of a spherical code with L code points on the surface of Ω_{2M^2}. That is,

$$\xi_1(L, \mathcal{V}) \le \frac{1}{2} D(2M^2, L)$$

where $D(\cdot, \cdot)$ is defined in (28). According to (31) in Proposition 9, the three inequalities about $\xi_1(L, \mathcal{V})$ given in Properties 2)–4) are true.

We now prove that the equality in (15) holds in the case $m = 1$ if and only if the Euclidean distance between any two distinct matrices in $\mathcal{V}$ is the same and the sum of all the L matrices in $\mathcal{V}$ is an all-zero matrix.

We define

$$\Sigma_L = \Sigma_L(V_0, V_1, \ldots, V_{L-1}) = \sum_{0 \le \ell < \ell' \le L-1} \|V_\ell - V_{\ell'}\|_F^2.$$

It is easy to see that

$$\left\| \sum_{\ell=0}^{L-1} V_\ell \right\|_F^2 = ML + \sum_{0 \le \ell < \ell' \le L-1} \operatorname{Tr}\left(V_\ell^{\mathrm{H}} V_{\ell'} + V_{\ell'}^{\mathrm{H}} V_\ell \right).$$

Therefore, by a simple calculation, we obtain

$$\begin{aligned}\Sigma_L &= \sum_{0\le \ell<\ell'\le L-1} \mathrm{Tr}\left((V_\ell^{\mathrm{H}} - V_{\ell'}^{\mathrm{H}})(V_\ell - V_{\ell'})\right)\\ &= ML(L-1) - \sum_{0\le \ell<\ell'\le L-1} \mathrm{Tr}\left(V_\ell^{\mathrm{H}} V_{\ell'} + V_{\ell'}^{\mathrm{H}} V_\ell\right)\\ &= ML^2 - \left\|\sum_{\ell=0}^{L-1} V_\ell\right\|_F^2.\end{aligned}$$

On the other hand, we have

$$\Sigma_L \ge 2ML(L-1)\xi_1^2(L, \mathcal{V})$$

with equality if and only if the Euclidean distance between any two distinct matrices in $\mathcal{V}$ is the same.

Thus, we can get

$$\begin{aligned}\xi_1^2(L, \mathcal{V}) &\le \frac{1}{2ML(L-1)}\Sigma_L\\ &= \frac{L}{2(L-1)} - \frac{1}{2ML(L-1)}\left\|\sum_{\ell=0}^{L-1} V_\ell\right\|_F^2\\ &\le \frac{L}{2(L-1)}.\end{aligned}$$

From this, it is clear that

$$\xi_1(L, \mathcal{V}) = \sqrt{\frac{L}{2(L-1)}}$$

if and only if the Euclidean distance between any two distinct matrices in $\mathcal{V}$ is the same and

$$\sum_{\ell=0}^{L-1} V_\ell = 0 \in \mathbb{R}^{M\times M}.$$

Hence, the proof of Proposition 2 is completed. □

Appendix C
Proof of Theorem 1

The signal constellation $\mathcal{V}(k_1, k_2, k_3)$ for $k_1, k_2, k_3 \in \mathbb{Z}_L$ has a positive diversity sum if and only if any two signal matrices in the constellation are mutually different.

We first give two lemmas upon which our proof of Theorem 1 is established.

Lemma 7: For any fixed $m \in \mathbb{Z}_{L-1} = \{0, 1, \ldots, L-2\}$, $\ell \in \{1, 2, \ldots, L-1-m\}$, and $k_1, k_2, k_3 \in \mathbb{Z}_L$, the equality

$$A_m(k_1, k_2, k_3) = A_{m+\ell}(k_1, k_2, k_3)$$

holds if and only if the following two conditions of A) and B) are met.

- A) $\ell(k_1+1) \equiv 0 \pmod L$.
- B) At least one of the following twelve conditions is satisfied:
- B1) $mk_2 \equiv L/4 \pmod L$, $\ell k_2 \equiv 0 \pmod L$, and $\ell(k_3-1) \equiv 0 \pmod L$.
- B2) $mk_2 \equiv L/4 \pmod L$, $\ell k_2 \equiv L/2 \pmod L$, and $\ell(k_3-1) \equiv L/2 \pmod L$.
- B3) $mk_2 \equiv 3L/4 \pmod L$, $\ell k_2 \equiv L/2 \pmod L$, and $\ell(k_3-1) \equiv L/2 \pmod L$.
- B4) $mk_2 \equiv 3L/4 \pmod L$, $\ell k_2 \equiv 0 \pmod L$, and $\ell(k_3-1) \equiv 0 \pmod L$.
- B5) $mk_2 \equiv 0 \pmod L$, $\ell k_2 \equiv 0 \pmod L$, and $\ell(k_3+1) \equiv 0 \pmod L$.
- B6) $mk_2 \equiv 0 \pmod L$, $\ell k_2 \equiv L/2 \pmod L$, and $\ell(k_3+1) \equiv L/2 \pmod L$.
- B7) $mk_2 \equiv L/2 \pmod L$, $\ell k_2 \equiv L/2 \pmod L$, and $\ell(k_3+1) \equiv L/2 \pmod L$.
- B8) $mk_2 \equiv L/2 \pmod L$, $\ell k_2 \equiv 0 \pmod L$, and $\ell(k_3+1) \equiv 0 \pmod L$.
- B9) $\ell k_2 \equiv 0 \pmod L$, $\ell(k_3+1) \equiv 0 \pmod L$, and $\ell(k_3-1) \equiv 0 \pmod L$.
- B10) $(2m+\ell)k_2 \equiv 0 \pmod L$, $\ell(k_3+1) \equiv 0 \pmod L$, and $\ell(k_3-1) \equiv L/2 \pmod L$.
- B11) $(2m+\ell)k_2 \equiv L/2 \pmod L$, $\ell(k_3+1) \equiv L/2 \pmod L$, and $\ell(k_3-1) \equiv 0 \pmod L$.
- B12) $\ell k_2 \equiv L/2 \pmod L$, $\ell(k_3+1) \equiv L/2 \pmod L$, and $\ell(k_3-1) \equiv L/2 \pmod L$.

For compactness, the proof of Lemma 7 will be given at the end of the Appendix, following the proof of Theorem 1.

For any $p \in \mathbb{N}$, we define two nonnegative integers $\tilde{m} = \tilde{m}(p)$ and $\tilde{n} = \tilde{n}(p)$ which satisfy

$$p = 2^{\tilde{n}}(2\tilde{m}+1).$$

It is apparent that $\tilde{m} = \tilde{m}(p)$ and $\tilde{n} = \tilde{n}(p)$ with the above property are uniquely determined by $p \in \mathbb{N}$. In addition, when $p = 0$, we define $\tilde{m} = -1/2$ and $\tilde{n} = +\infty$.

According to the above definition, we know that $\tilde{n}(p) = 0$ if $p \in \mathbb{N}$ is odd, and that for any two nonnegative integers p and q

$$\tilde{n}(pq) = \tilde{n}(p) + \tilde{n}(q).$$

The following lemma can be easily verified.

Lemma 8: Let $p \in \mathbb{N}$, $L = 2^p$, $k \in \mathbb{Z}_L$, and

$$\ell \in \mathbb{Z}_L \backslash \{0\} = \{1, 2, \ldots, L-1\}.$$

Then, we have

1) $\tilde{n}(\ell) \in \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$;
2) $\ell k \equiv 0 \pmod L$ if and only if $\tilde{n}(\ell) + \tilde{n}(k) \ge p$.

According to Lemma 8, we obtain the following three rules which play an important role in the ensuing proof.

Under the notations and assumptions in Lemma 8, we have

Rule 1) If $\ell k \equiv 0 \pmod L$, then k is even.

Rule 2) $\ell \cdot \mathrm{odd} \equiv 0 \pmod L$ and $\ell \cdot \mathrm{even} \equiv 0 \pmod L$ cannot hold simultaneously.

Rule 3) $\ell \cdot \mathrm{odd} \equiv L/2\, (\mathrm{mod}\, L$ and $\ell \cdot \mathrm{even} \equiv L/2 \pmod L$ cannot hold simultaneously.

Now, we can give the proof of Theorem 1. According to Lemma 7, for any fixed $k_1, k_2, k_3 \in \mathbb{Z}_L$, the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ has a positive diversity sum if and only if for *all* $m \in \mathbb{Z}_{L-1} = \{0, 1, \ldots, L-2\}$ and for *all* $\ell \in \{1, 2, \ldots, L-1-m\}$, *either* condition A) in Lemma 7 is not satisfied *or* none of the 12 conditions of B1)–B12) in Lemma 7 are met. Equivalently, for any fixed $k_1, k_2, k_3 \in \mathbb{Z}_L$, the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ has a diversity sum of zero if and only if there are *some* $m \in \mathbb{Z}_{L-1} = \{0, 1, \ldots, L-2\}$ and *some* $\ell \in \{1, 2, \ldots, L-1-m\}$ which satisfy condition A) *and* at least one of the 12 conditions of B1)–B12) in Lemma 7.

If $p=1$, i.e., $L=2$, then the constellation $\mathcal{V}(k_1, k_2, k_3)$ has exactly two matrices one of which is the 2×2 identity matrix for $\ell=0$ and the other one for $\ell=1$ is $A_1(k_1, k_2, k_3)$ which equals the 2×2 identity matrix if and only if $(k_1, k_2, k_3)=(1, 0, 1)$ or $(k_1, k_2, k_3)=(1, 1, 0)$. That is, in the case $L=2$, the constellation $\mathcal{V}(k_1, k_2, k_3)$ has a diversity sum of zero if and only if $(k_1, k_2, k_3)=(1, 0, 1)$ or $(k_1, k_2, k_3)=(1, 1, 0)$.

In the sequel, we set $p\geq 2$, i.e., $L\geq 4$.

If $k_1\in\mathbb{Z}_L$ is even, then, by virtue of Rule 1), condition A) in Lemma 7 cannot be satisfied for all $\ell\in\{1, 2, \ldots, L-1-m\}$ where m is any number belonging to $\mathbb{Z}_{L-1}$. Hence, the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ has a positive diversity sum.

Now, we assume that $k_1\in\mathbb{Z}_L$ is odd and make the discussion in the following four mutually exclusive cases.

1) $(k_1, k_2, k_3)=(\text{odd, even, even})$.
According to the previously mentioned Rules 1)–3), we can see that for *all* $m\in\mathbb{Z}_{L-1}=\{0, 1, \ldots, L-2\}$ and for *all* $\ell\in\{1, 2, \ldots, L-1-m\}$, none of the 12 conditions of B1)–B12) in Lemma 7 is met. Hence, the constellation $\mathcal{V}(k_1, k_2, k_3)$ has a *positive* diversity sum.

2) $(k_1, k_2, k_3)=(\text{odd, even, odd})$.
We take $m=L/4$ and $\ell=L/2$ which satisfy condition A) of $\ell(k_1+1)\equiv 0 \pmod L$ in Lemma 7. In the case of $k_2\equiv 0 \pmod 4$, condition B5) in Lemma 7 is met, while in the case of $k_2\equiv 2 \pmod 4$, condition B8) in Lemma 7 is satisfied. Hence, the constellation $\mathcal{V}(k_1, k_2, k_3)$ has a diversity sum of *zero*.

3) $(k_1, k_2, k_3)=(\text{odd, odd, even})$.
We still take $m=L/4$ and $\ell=L/2$, then condition A) of $\ell(k_1+1)\equiv 0 \pmod L$ in Lemma 7 is met and that in the case of $k_2\equiv 1 \pmod 4$ condition B2) in Lemma 7 is satisfied while in the case of $k_2\equiv 3 \pmod 4$ condition B3) in Lemma 7 is met. Therefore, the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ has a diversity sum of *zero*.

4) $(k_1, k_2, k_3)=(\text{odd, odd, odd})$.
According to the preceding Rules 1)–3), it can be seen that for *all*
$$m\in\mathbb{Z}_{L-1}=\{0, 1, \ldots, L-2\}$$
and for *all*
$$\ell\in\{1, 2, \ldots, L-1-m\}$$
conditions of B1)–B9) and condition B12) in Lemma 7 cannot be satisfied. Therefore, we can focus on conditions of B10) and B11) and condition A) in Lemma 7.

If condition B10) *or* B11) in Lemma 7 is met, then there should be $2\ell\equiv L/2 \pmod L$ which means that
$$\ell\equiv L/4 \pmod L$$
or
$$\ell\equiv 3L/4 \pmod L.$$
Recall that we have assumed
$$m\in\mathbb{Z}_{L-1}=\{0, 1, \ldots, L-2\}$$
and
$$\ell\in\{1, 2, \ldots, L-1-m\}.$$

If $L=4$, then, in order to satisfy condition B10) or B11), there are only two possibilities of $\ell=1$ and $\ell=3$. Then, the equivalence relations of $(2m+\ell)k_2\equiv 0 \pmod L$ in condition B10) and $(2m+\ell)k_2\equiv L/2 \pmod L$ in condition B11) in Lemma 7 cannot hold, since odd $\cdot$ odd $\equiv$ even $\pmod L$ is impossible. Hence, none of the 12 conditions of B1)–B12) in Lemma 7 can be met, which implies that the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ of size $L=4$ has a *positive* diversity sum.

Now, we assume that $p\geq 3$, i.e., $L\geq 8$.

In the case of $k_1\equiv 1 \ (\mathrm{mod}, 4)$, if condition B10) *or* B11) in Lemma 7 is met, then $\ell\equiv L/4 \pmod L$ or $\ell\equiv 3L/4 \pmod L$, and consequently, condition A) of $\ell(k_1+1)\equiv 0 \pmod L$ in Lemma 7 cannot hold. On the other hand, if neither conditions B10) nor B11) is met, then, combining with the above results, we know that none of the 12 conditions B1)–B12) in Lemma 7 can be satisfied. Therefore, in the case of $k_1\equiv 1 \pmod 4$, the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ always has a *positive* diversity sum.

In the case of $k_1\equiv 3 \pmod 4$, if $k_3\equiv 1 \pmod 4$, then we can take $m=L/8$ and $\ell=L/4$ which satisfy conditions A) and B11) in Lemma 7. If $k_3\equiv 3 \ (\mathrm{mod}, 4)$, then we can take $m=3L/8$ and $\ell=L/4$ which satisfy conditions A) and B10) in Lemma 7. Hence, in the case of $k_1\equiv 3 \pmod 4$, the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ always has a diversity sum of *zero*.

Integrating the preceding results in all situations, we see that all the cases in which the signal constellation $\mathcal{V}(k_1, k_2, k_3)$ has a diversity sum of zero are exactly those stated in Theorem 1. The proof of Theorem 1 is thus completed. □

At the end of the Appendix, we briefly present a proof of Lemma 7.

We can expand the 2×2 matrix equation
$$A_m(k_1, k_2, k_3)=A_{m+\ell}(k_1, k_2, k_3)$$
into the following equivalent system of four equations:
$$\begin{cases}\cos((m+\ell)k_2\theta_L)e^{j\ell(k_3+1)\theta_L}=\cos(mk_2\theta_L)\\ \sin((m+\ell)k_2\theta_L)e^{j\ell(k_3-1)\theta_L}=\sin(mk_2\theta_L)\\ \sin((m+\ell)k_2\theta_L)e^{j\ell(k_1+k_3)\theta_L}=\sin(mk_2\theta_L)\\ \cos((m+\ell)k_2\theta_L)e^{j\ell(k_1-k_3)\theta_L}=\cos(mk_2\theta_L)\end{cases}$$
which can be reduced to the three equations of
$$\begin{cases}e^{j\ell(k_1+1)\theta_L}=1\\ \cos((m+\ell)k_2\theta_L)e^{j\ell(k_3+1)\theta_L}=\cos(mk_2\theta_L)\\ \sin((m+\ell)k_2\theta_L)e^{j\ell(k_3-1)\theta_L}=\sin(mk_2\theta_L).\end{cases}\tag{43}$$
In the derivation of the conditions in Lemma 7, the following two facts are frequently used.

1) We have $\cos(k\theta_L)=0$ if and only if $k\equiv L/4 \pmod L$ or $k\equiv 3L/4 \pmod L$, which corresponds to $\sin(k\theta_L)=1$ and $\sin(k\theta_L)=-1$, respectively. Equivalently, $e^{jk\theta_L}=j$ if and only if $k\equiv L/4 \pmod L$, and that $e^{jk\theta_L}=-j$ if and only if $k\equiv 3L/4 \pmod L$.

2) We have $\sin(k\theta_L)=0$ if and only if $k\equiv 0 \pmod L$ or $k\equiv L/2 \pmod L$, which corresponds to $\cos(k\theta_L)=1$ and $\cos(k\theta_L)=-1$, respectively. Equivalently, $e^{jk\theta_L}=$

1 if and only if $k \equiv 0 \pmod L$, and that $e^{jk\theta_L} = -1$ if and only if $k \equiv L/2 \pmod L$.

The first equation in (43) is equivalent to condition A) in Lemma 7, i.e., $\ell(k_1+1) \equiv 0 \pmod L$. The second and third equations in (43) can be handled in the following three mutually exclusive cases.

1) $\cos(mk_2\theta_L) = \cos((m+\ell)k_2\theta_L) = 0$.
2) $\sin(mk_2\theta_L) = \sin((m+\ell)k_2\theta_L) = 0$.
3) $\sin(mk_2\theta_L)\cos(mk_2\theta_L) \neq 0$.

Each of the preceding three cases can be further discussed separately in the four situations as described in the following.

For case 1), we have that $mk_2 \equiv L/4 \pmod L$ or $mk_2 \equiv 3L/4 \pmod L$ and that $(m+\ell)k_2 \equiv L/4 \pmod L$ or $(m+\ell)k_2 \equiv 3L/4 \pmod L$. Therefore, there are exactly four situations in which the second and third equations in (43) can be equivalently reduced to conditions B1)–B4) in Lemma 7. For example, in the case that $mk_2 \equiv L/4 \pmod L$ and $(m+\ell)k_2 \equiv L/4 \pmod L$ are satisfied, the third equation in (43) is equivalent to $e^{j\ell(k_3-1)\theta_L} = 1$. Therefore, condition B1) in Lemma 7 is deduced.

For case 2), we have that $mk_2 \equiv 0 \pmod L$ or $mk_2 \equiv L/2 \pmod L$ and that $(m+\ell)k_2 \equiv 0 \pmod L$ or $(m+\ell)k_2 \equiv L/2 \pmod L$. The second and third equations in (43) in the corresponding four situations are equivalent to conditions B5)–B8) in Lemma 7. For example, in the case that $mk_2 \equiv 0 \pmod L$ and $(m+\ell)k_2 \equiv 0 \pmod L$ are satisfied, the second equation in (43) is equivalent to $e^{j\ell(k_3+1)\theta_L} = 1$. Therefore, condition B5) in Lemma 7 is derived.

For case 3), we have

$$\cos((m+\ell)k_2\theta_L) = r_1 \cos(mk_2\theta_L)$$

and

$$\sin((m+\ell)k_2\theta_L) = r_2 \sin(mk_2\theta_L)$$

where $r_1, r_2 \in \{-1, 1\}$. Thus, there are also four situations in which the equivalent conditions B9) to B12) in Lemma 7 can be deduced from the second and third equations in (43). For example, in the case that $(r_1, r_2) = (1, 1)$, the second and third equations in (43) are equivalent to $e^{j\ell(k_3+1)\theta_L} = 1$, $e^{j\ell(k_3-1)\theta_L} = 1$, and $e^{j(m+\ell)k_2\theta_L} = e^{jmk_2\theta_L}$. Therefore, condition B9) in Lemma 7 is obtained.

The proof of Lemma 7 is thus completed. □

Appendix D
Proof of Proposition 3

Let $A = A_{14}$ and $B = A_7$. Then, it is easy to verify that the signal constellation $\mathcal{V} = \{A_\ell \,|\, \ell \in \mathbb{Z}_{16}\}$ given in Table III can be written as

$$\mathcal{V} = \{A^k, A^kB \,|\, k \in \mathbb{Z}_8\}.$$

Since $A_1 A_5 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \notin \mathcal{V}$, the constellation $\mathcal{V}$ itself is not a group.

By a simple calculation, we have the following relations in terms of A and B:

$$\begin{aligned} A^4 = B^4 = -I_2, \quad B^2 = jI_2, \quad A^8 = B^8 = I_2, \\ AB = BA^3, \quad BA = A^3B. \end{aligned} \tag{44}$$

Hence,

$$\begin{aligned} G_{32} &\stackrel{\text{def}}{=} \{j^m A_\ell \,|\, \ell \in \mathbb{Z}_{16} \text{ and } m = 0, 1\} \\ &= \{A^k, A^kB, A^kB^2, A^kB^3 \,|\, k \in \mathbb{Z}_8\}. \end{aligned} \tag{45}$$

By (44), it is easy to verify that the set G_{32} given by (45) is a group on which the binary operation is the usual matrix multiplication [20]. Moreover, every element in G_{32} can be factorized as a product of some matrices in $\mathcal{V}$. Hence, the conclusion in Proposition 3 is true. □

Appendix E
Proof of Proposition 7

The equivalence between the condition that $|f(\theta_i)| \leq 1$ and $|\lambda a_i| \leq 1$ and the condition that

$$\cos\theta_i \in \left[-1/3 - 4\sqrt{7}/21, \, -1/7\right] \cup \left[-1/3 + 4\sqrt{7}/21, \, 1\right]$$

for $i = 1, 2, 3$ can be easily checked by some trigonometric and algebraic manipulation.

The unitarity of the matrices of A_i and B_i for $i = 1, 2, 3$ follows from their parametric forms of unitary matrices.

Now, we examine the claimed identical relations in terms of the Frobenius norms. It is easy to verify that

$$\|A_i - A_k\|_F = \|B_i - B_k\|_F$$

and

$$\|A_i - B_k\|_F = \|A_k - B_i\|_F, \qquad \text{for } 1 \leq i < k \leq 3.$$

Moreover, the check of the equality $\|C - D\|_F = \sqrt{32/7}$ is of a simple calculation.

In what follows, we prove that

$$\|A_i - B_i\|_F = \|A_i - C\|_F = \|A_i - D\|_F = \sqrt{32/7}, \quad \text{for } i = 1, 2, 3.$$

The proof of $\|B_i - C\|_F = \|B_i - D\|_F = \sqrt{32/7}$ for $i = 1, 2, 3$ can then be obtained in a similar fashion. Therefore, it suffices to give an examination of the following three equalities.

1) $\|A_i - D\|_F^2 = 32/7$ for $i = 1, 2, 3$.
In fact, we have

$$\begin{aligned} \|A_i - D\|_F^2 &= |\lambda a_i - 1|^2 + |\lambda^* a_i^* \Delta_i - 1|^2 + 2|b_i|^2 \\ &= 2(|\lambda|^2|a_i|^2 + r_i^2) + 2 - \lambda a_i(1 + \Delta_i^*) \\ &\quad - \lambda^* a_i^*(1 + \Delta_i) \\ &= 4 - (\lambda + \lambda^*) \\ &= 32/7. \end{aligned}$$

2) $\|A_i - C\|_F^2 = 32/7$ for $i = 1, 2, 3$.
In fact, we have

$$\begin{aligned} \|A_i - C\|_F^2 &= |\lambda a_i - e^{j\phi}|^2 + 2|b_i|^2 + |\lambda^* a_i^* \Delta_i - e^{-j\phi}|^2 \\ &= 2(|\lambda|^2|a_i|^2 + r_i^2) - \lambda a_i e^{-j\phi}(1 + \Delta_i^*) \\ &\quad - \lambda^* a_i^* e^{j\phi}(1 + \Delta_i) \\ &= 4 - \lambda e^{-j\phi} - \lambda^* e^{j\phi} \\ &= 4 + \frac{2j(e^{-j\phi} - e^{j\phi})}{7\sin\phi} \\ &= 32/7. \end{aligned}$$

3) $\|A_i - B_i\|_F^2 = 32/7$ for $i = 1, 2, 3$.

By a simple calculation, we have

$$|a_i^*\Delta_i - a_i\Delta_i^*| = \left|\frac{1}{1+\Delta_i} - \frac{1}{1+\Delta_i^*}\right| = |a_i - a_i^*| = \frac{\sin\theta_i}{1+\cos\theta_i}$$

and

$$\begin{aligned}|b_i - b_i^*|^2 &+ |b_i^*\Delta_i - b_i\Delta_i^*|^2\\ &= 4r_i^2\left(\sin^2\beta_i + \sin^2(\beta_i-\theta_i)\right)\\ &= 4r_i^2\left[\sin^2\beta_i + (\sin\beta_i\cos\theta_i - \cos\beta_i\sin\theta_i)^2\right]\\ &= 4r_i^2\left[1-\cos(2\beta_i)\cos^2\theta_i - \sin(2\beta_i)\sin\theta_i\cos\theta_i\right]\\ &= 4r_i^2[1-\cos\theta_i\cos(2\beta_i-\theta_i)]\\ &= 4r_i^2[1-\cos\theta_i\, f(\theta_i)]\\ &= 4r_i^2\,\frac{22+26\cos\theta_i}{19+21\cos\theta_i}.\end{aligned}$$

Then, by noting that $|\lambda|^2 = 4/21$ and

$$r_i^2 = (19+21\cos\theta_i)/[21(1+\cos\theta_i)]$$

we can obtain

$$\begin{aligned}\|A_i - B_i\|_F^2 &= |\lambda|^2|a_i - a_i^*|^2 + |b_i - b_i^*|^2 + |b_i^*\Delta_i - b_i\Delta_i^*|^2\\ &\quad + |\lambda|^2|a_i^*\Delta_i - a_i\Delta_i^*|^2\\ &= 2|\lambda|^2\,\frac{\sin^2\theta_i}{(1+\cos\theta_i)^2} + 4r_i^2\,\frac{22+26\cos\theta_i}{19+21\cos\theta_i}\\ &= \frac{8\sin^2\theta_i}{21(1+\cos\theta_i)^2} + \frac{8(11+13\cos\theta_i)}{21(1+\cos\theta_i)}\\ &= \frac{8(12+24\cos\theta_i+12\cos^2\theta_i)}{21(1+\cos\theta_i)^2}\\ &= 32/7\end{aligned}$$

as required. □

APPENDIX F
PROOF OF THEOREM 2

In the sequel, the proof is established by the contradiction method.

We assume that there exists a unitary signal constellation $\mathcal{V} = \{V_\ell \,|\, \ell \in \mathbb{Z}_L\}$ with size $L \in \{6, 7, 8, 9\}$ which has a diversity product of

$$\zeta(L, \mathcal{V}) = \sqrt{\frac{L}{2(L-1)}}. \quad (46)$$

It is apparent that any unitary signal constellation remains the unitarity and the same diversity product and sum under the left or right multiplication by a single same unitary matrix and under the transpose or the complex conjugate operation.

Since any unitary matrix can be diagonalized by a unitary similarity transformation [21], without loss of generality, we can assume that the signal constellation $\mathcal{V}$ has the first two matrices with the form of

$$V_0 = I_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad V_1 = \begin{pmatrix}e^{j\phi_1} & 0\\ 0 & e^{j\phi_2}\end{pmatrix}$$

where $\phi_1, \phi_2 \in [0, 2\pi)$.

According to (42), we have for all $0 \le i < k \le L-1$

$$\frac{\sqrt{\det(V_i - V_k)}}{2} \le \frac{\|V_i - V_k\|_F}{2\sqrt{2}}$$

with equality if and only if the two singular values of $V_i - V_k$ are equal. Moreover, by Proposition 2, from (46) we have for all $0 \le i < k \le L-1$

$$\frac{\sqrt{\det(V_i - V_k)}}{2} = \frac{\|V_i - V_k\|_F}{2\sqrt{2}} = \sqrt{\frac{L}{2(L-1)}}.$$

Consequently,

$$\|V_i - V_k\|_F^2 = \frac{4L}{L-1}. \quad (47)$$

From the above facts, under the assumption of (46), we can conclude that for all $0 \le i < k \le L-1$

$$(V_i - V_k)^{\mathrm{H}}(V_i - V_k) = \sigma^2 I_2 = \sigma^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad (48)$$

where

$$\sigma = \sqrt{\frac{2L}{L-1}}.$$

By taking $i = 0$ in (48), we know that the unitary signals V_ℓ for all $\ell = 1, 2, \cdots, L-1$ in the constellation $\mathcal{V}$ can be written in the form

$$V = I_2 + \sigma U$$

where U is a 2×2 unitary matrix and the subscript ℓ is omitted for simplicity.

Let $U = A + jB$, i.e., A and B are the real and imaginary parts of U, respectively. From the two equalities of $V^{\mathrm{H}}V = I_2$ and $U^{\mathrm{H}}U = I_2$, we can deduce the following relations for which A and B must satisfy:

$$A + A^\top = -\sigma I_2, \quad B = B^\top, \quad AA^\top + BB^\top = I_2$$

and

$$BA^\top = AB^\top. \quad (49)$$

It follows from the preceding two relations that B is a real symmetric matrix and A has the form of

$$A = -\frac{\sigma}{2}I_2 + \begin{pmatrix}0 & -\alpha\\ \alpha & 0\end{pmatrix} \quad (50)$$

where α is a real number. By the relation $AA^\top + BB^\top = I_2$ in (49), we know that α satisfies

$$0 \le \alpha^2 \le 1 - \frac{\sigma^2}{4} = \frac{L-2}{2(L-1)} < \frac{1}{2}$$

and that the matrix B satisfies

$$BB^\top = B^2 = \left(1 - \frac{\sigma^2}{4} - \alpha^2\right)I_2.$$

Therefore, the imaginary part of U, i.e., B, can be represented in the form of

$$B = \sqrt{1 - \frac{\sigma^2}{4} - \alpha^2}\,Q \quad (51)$$

where Q is a 2×2 real symmetric and orthogonal matrix; i.e., that Q has three possible forms of $Q = I_2$, $Q = -I_2$, and the following 2×2 reflection matrix:

$$Q = Q(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \quad (52)$$

with $\theta \in [0, 2\pi)$.

In particular, for the second matrix V_1 in $\mathcal{V}$, which is assumed to be diagonal, it is seen from the relation

$$V = I_2 + \sigma U = I_2 + \sigma A + j\sigma B$$

that A and B are both diagonal; i.e., that $\alpha = 0$ and $\sin\theta = 0$ in the representation of A and B, namely, (50)–(52). Therefore, V_1 should have the form of

$$V_1 = \left(1 - \frac{\sigma^2}{2}\right) I_2 + j\sigma\sqrt{1 - \frac{\sigma^2}{4}} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \tag{53}$$

where $r_1, r_2 \in \{-1, 1\}$.

The $L-2$ remaining matrices rather than V_0 and V_1 in $\mathcal{V}$ can be written as

$$\begin{aligned} V &= I_2 + \sigma A + j\sigma B \\ &= \left(1 - \frac{\sigma^2}{2}\right) I_2 + \begin{pmatrix} 0 & -\sigma\alpha \\ \sigma\alpha & 0 \end{pmatrix} + j\sigma\sqrt{1 - \frac{\sigma^2}{4} - \alpha^2}\, Q \end{aligned} \tag{54}$$

where Q is a real symmetric and orthogonal matrix with one of the above-mentioned three forms; i.e., that $Q = I_2$, $Q = -I_2$, and $Q = Q(\theta)$, given by (52), with $\theta \in [0, 2\pi)$.

In the case that $\alpha^2 < 1 - \sigma^2/4$, if $Q = I_2$ or $Q = -I_2$, then the matrix B, given by (51), is a *nonzero* scalar matrix. From this and the relation $BA^\top = AB^\top$ in (49), we know that A is real symmetric. Hence, $\alpha = 0$ in the representation (50) of A. Therefore, the $L-2$ remaining matrices, given by (54), should have three possible forms of

$$V = \left[\left(1 - \frac{\sigma^2}{2}\right) + j\sigma\sqrt{1 - \frac{\sigma^2}{4}}\right] I_2 \tag{55}$$

$$V = \left[\left(1 - \frac{\sigma^2}{2}\right) - j\sigma\sqrt{1 - \frac{\sigma^2}{4}}\right] I_2 \tag{56}$$

and

$$\begin{aligned} V = &\left(1 - \frac{\sigma^2}{2}\right) I_2 + \begin{pmatrix} 0 & -\sigma\alpha \\ \sigma\alpha & 0 \end{pmatrix} \\ &+ j\sigma\sqrt{1 - \frac{\sigma^2}{4} - \alpha^2} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \end{aligned} \tag{57}$$

with $\alpha^2 \le 1 - \sigma^2/4$ and $\theta \in [0, 2\pi)$.

Now, we can claim that in the representation of (53) for the second matrix V_1, the numbers r_1 and r_2 cannot be equal to 1 or -1 simultaneously. Hence, there should be $(r_1, r_2) = (1, -1)$ or $(r_1, r_2) = (-1, 1)$ in (53).

In fact, since $L \ge 6$ under the assumption $L \in \{6, 7, 8, 9\}$, there is at least one of the $L-2$ remaining matrices V which has the form of (57). If $(r_1, r_2) = (1, 1)$ or $(r_1, r_2) = (-1, -1)$ in (53), by letting

$$\lambda \stackrel{\text{def}}{=} \sigma\sqrt{1 - \sigma^2/4}$$

and

$$\mu \stackrel{\text{def}}{=} \sigma\sqrt{1 - \sigma^2/4 - \alpha^2}$$

and noting that V is currently in the form of (57), we can compute that

$$\begin{aligned} &\|V - V_1\|_F^2 \\ &= 2\sigma^2\alpha^2 + (\lambda r_1 + \mu\cos\theta)^2 + (\lambda r_2 - \mu\cos\theta)^2 + 2\mu^2\sin^2\theta \\ &= 2\sigma^2\alpha^2 + 2\lambda^2 + 2\mu^2 \\ &= 2\sigma^2\alpha^2 + 2\sigma^2\left(1 - \frac{\sigma^2}{4}\right) + 2\sigma^2\left(1 - \frac{\sigma^2}{4} - \alpha^2\right) \\ &= 4\sigma^2\left(1 - \frac{\sigma^2}{4}\right) = \frac{4L(L-2)}{(L-1)^2} < \frac{4L}{L-1} \end{aligned}$$

which contradicts (47).

Based on the fact that there should be $(r_1, r_2) = (1, -1)$ or $(r_1, r_2) = (-1, 1)$ in the representation (53) of the second matrix V_1 in $\mathcal{V}$, we can further show that, for those $L-2$ remaining matrices V rather than V_0 and V_1 in $\mathcal{V}$, the matrix Q in the representation (54) of V cannot take the forms of $Q = I_2$ and $Q = -I_2$ in the case that $\alpha^2 < 1 - \sigma^2/4$; i.e., that the $L-2$ remaining matrices V can only take the form of (57) rather than (55) and (56).

In fact, if the matrix V takes the form of (55) or (56), by a simple calculation and noting that $(r_1, r_2) = (1, -1)$ or $(r_1, r_2) = (-1, 1)$ in (53), we can get

$$\begin{aligned} \|V - V_1\|_F^2 &= \sigma^2\left(1 - \frac{\sigma^2}{4}\right)\left[(r_1 \pm 1)^2 + (r_2 \pm 1)^2\right] \\ &= 4\sigma^2\left(1 - \frac{\sigma^2}{4}\right) = \frac{4L(L-2)}{(L-1)^2} < \frac{4L}{L-1} \end{aligned}$$

which is in contradiction to (47). The symbol "$\pm$" in the preceding expression takes "$+$" and "$-$" when V takes the form of (56) or (55), respectively.

Therefore, all the $L-2$ remaining matrices V rather than V_0 and V_1 in $\mathcal{V}$ should take the form of (57). By some algebraic manipulation, we can verify that the 2×2 reflection matrix $Q = Q(\theta)$ in (57) and (52) satisfies

$$Q(\theta)Q(\phi) + Q(\phi)Q(\theta) = 2\cos(\theta - \phi) I_2 \tag{58}$$

for $\theta, \phi \in [0, 2\pi)$.

By the invariance property of the diversity product and the sum of a unitary signal constellation under the operation of complex conjugate and the fact the negative of a 2×2 reflection matrix is also a reflection matrix, we need only handle only one case of either $(r_1, r_2) = (1, -1)$ or $(r_1, r_2) = (-1, 1)$ in the representation (53) of the second matrix V_1 in $\mathcal{V}$. In the following, we take $(r_1, r_2) = (1, -1)$ in (53) as an example. In this case, the signal constellation $\mathcal{V} = \{V_\ell \mid \ell \in \mathbb{Z}_L\}$ with size $L \in \{6, 7, 8, 9\}$ has the following form:

$$V_0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$V_1 = \left(1 - \frac{\sigma^2}{2}\right) I_2 + j\sigma\sqrt{1 - \frac{\sigma^2}{4}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\begin{aligned} V_\ell = &\left(1 - \frac{\sigma^2}{2}\right) I_2 + \begin{pmatrix} 0 & -\sigma\alpha_\ell \\ \sigma\alpha_\ell & 0 \end{pmatrix} \\ &+ j\sigma\sqrt{1 - \frac{\sigma^2}{4} - \alpha_\ell^2} \begin{pmatrix} \cos\theta_\ell & \sin\theta_\ell \\ \sin\theta_\ell & -\cos\theta_\ell \end{pmatrix} \end{aligned} \tag{59}$$

with $\alpha_\ell^2 \le 1 - \sigma^2/4 = (L-2)/[2(L-1)]$ and $\theta_\ell \in [0, 2\pi)$ for $\ell \in \mathbb{Z}_L \backslash \{0, 1\} = \{2, 3, \ldots, L-1\}$. It is noted that V_1 can also be considered with the form of (59) in the case that $\alpha_1 = \theta_1 = 0$.

Then, by using the equalities (48) and (58), we can obtain for all $1 \le i < k \le L-1$

$$(\alpha_i - \alpha_k)^2 + \left(1 - \frac{\sigma^2}{4} - \alpha_i^2\right) + \left(1 - \frac{\sigma^2}{4} - \alpha_k^2\right) -2\sqrt{1 - \frac{\sigma^2}{4} - \alpha_i^2}\sqrt{1 - \frac{\sigma^2}{4} - \alpha_k^2}\cos(\theta_i - \theta_k) = 1.$$

That is,

$$1 - \frac{\sigma^2}{4} - \alpha_i\alpha_k - \sqrt{1 - \frac{\sigma^2}{4} - \alpha_i^2}\sqrt{1 - \frac{\sigma^2}{4} - \alpha_k^2} \times \cos(\theta_i - \theta_k) = \frac{1}{2}, \qquad \text{for } 1 \le i < k \le L-1. \tag{60}$$

We can divide the index set of

$$\mathcal{I} = \mathbb{Z}_L \backslash \{0\} = \{1, 2, \ldots, L-1\}$$

into the following two index subsets:

$$\mathcal{I}_1 = \{k \in \mathcal{I} \,|\, \alpha_k \ge 0\}$$

and

$$\mathcal{I}_2 = \{k \in \mathcal{I} \,|\, \alpha_k \le 0\}.$$

If the index subset $\mathcal{I}_1$ contains at least four elements, then there should be two indexes $i_0 \ne k_0 \in \mathcal{I}_1 \subset \mathcal{I}$ such that

$$0 \le \theta_{i_0} - \theta_{k_0} \le \pi/2.$$

Then, for i_0 and k_0, the left-hand side of (60) is less than or equal to $1 - \sigma^2/4 = (L-2)/[2(L-1)]$, which is smaller than half. Hence, the two indexes i_0 and k_0 cannot satisfy the equality (60). Therefore, there are at most three entries in $\mathcal{I}_1$. By a similar argument, we can conclude that $\mathcal{I}_2$ has at most three indexes as well. Note that the index of $\alpha_1 = 0$ belongs to $\mathcal{I}_1 \cap \mathcal{I}_2$. Therefore, $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I} = \{1, 2, \ldots, L-1\}$ has at most five indexes. Considering the assumption $L \in \{6, 7, 8, 9\}$, we know that we must have $L = 6$.

Now, we want to prove that the equality (60) cannot hold for *some* $1 \le i < k \le L-1$ even in the case of $L = 6$. This is stated as follows.

Lemma 9: There should not exist five pairs of real numbers $(\alpha_\ell, \theta_\ell)$ for $\ell = 1, 2, 3, 4, 5$, where $(\alpha_1, \theta_1) = (0, 0)$, such that

$$1 - \frac{\sigma^2}{4} - \alpha_i\alpha_k - \sqrt{1 - \frac{\sigma^2}{4} - \alpha_i^2}\sqrt{1 - \frac{\sigma^2}{4} - \alpha_k^2} \times \cos(\theta_i - \theta_k) = \frac{1}{2}, \qquad \text{for } 1 \le i < k \le 5 \tag{61}$$

where $1 - \sigma^2/4 = 2/5$ and that $0 \le \alpha_\ell^2 \le 2/5$ and $\theta_\ell \in [0, 2\pi)$ for $2 \le \ell \le 5$.

Therefore, by the contradiction method, the equality (46) cannot hold. Hence, the proof of Theorem 2 is completed. □

In the sequel, we present a proof of Lemma 9 by the contradiction method. We assume the existence of a solution to (61).

According to the preceding discussion, among $\alpha_2, \alpha_3, \alpha_4$ and α_5 there should be two positives and the other two negatives. Without loss of generality, we assume that

$$\alpha_1 = 0, \qquad \alpha_2 > 0, \qquad \alpha_3 > 0, \qquad \alpha_4 < 0, \qquad \alpha_5 < 0.$$

By letting $i = 1$ in (61), we obtain that

$$\sqrt{2/5 - \alpha_k^2}\,\cos\theta_k = -\frac{\sqrt{5}}{10\sqrt{2}}, \qquad \text{for } 2 \le k \le 5. \tag{62}$$

This implies that $\theta_k \in (\pi/2, 3\pi/2)$ for $2 \le k \le 5$. It has been shown that, in order to satisfy (61), there should be $|\theta_i - \theta_k| > \pi/2$ provided that α_i and α_k are nonpositive or nonnegative simultaneously. Therefore, we can assume that without loss of generality

$$\theta_1 = 0, \quad \pi/2 < \theta_2 < \pi, \quad \pi < \theta_3 < 3\pi/2, \quad \pi/2 < \theta_4 < \pi \quad \pi < \theta_5 < 3\pi/2.$$

Moreover, from (62) we get

$$\alpha_k^2 = \frac{2}{5} - \frac{1}{40\cos^2\theta_k} \le \frac{3}{8}, \qquad \text{for } 2 \le k \le 5. \tag{63}$$

By virtue of the equalities of (61)–(63) and noting the ranges of parameters θ_k for $1 \le k \le 5$, we can derive several new equations in terms of α_k for $2 \le k \le 5$. For example, for the cases of $k = 2$ and $k = 3$, it follows from (62) that

$$\sqrt{2/5 - \alpha_2^2}\,\sin\theta_2 = \sqrt{3/8 - \alpha_2^2}$$

and

$$\sqrt{2/5 - \alpha_3^2}\,\sin\theta_3 = -\sqrt{3/8 - \alpha_3^2}.$$

Then, in the case of $(i, k) = (2, 3)$, the equality (61) can yield

$$3/8 - \alpha_2\alpha_3 + \sqrt{3/8 - \alpha_2^2}\,\sqrt{3/8 - \alpha_3^2} = 1/2.$$

From this, it holds that

$$3\alpha_2^2 + 3\alpha_3^2 + 2\alpha_2\alpha_3 - 1 = 0. \tag{64}$$

In a similar manner, we can actually get the following equations:

$$3\alpha_i^2 + 3\alpha_k^2 + 2\alpha_i\alpha_k - 1 = 0, \qquad \text{for } 2 \le i < k \le 5. \tag{65}$$

If $\alpha_2 = \alpha_3 = a > 0$, then from (64) we can obtain the solution $a = \frac{1}{2\sqrt{2}}$. Furthermore, by solving the equations of (65) for the cases of $(i, k) = (2, 4)$ and $(i, k) = (2, 5)$, we can get the result $\alpha_4 = \alpha_5 = -\frac{5}{6\sqrt{2}} < 0$. On the other hand, by using the above identical relation $\alpha_4 = \alpha_5$ and solving (65) for the case $(i, k) = (4, 5)$, we have another result of $\alpha_4 = \alpha_5 = -a = -\frac{1}{2\sqrt{2}} < 0$. Thus, a contradiction results.

If $\alpha_2 \ne \alpha_3$, by the subtraction operation between two equations of (65) in the cases of $(i, k) = (2, 4)$ and $(i, k) = (3, 4)$, we can get $\alpha_4 = -\frac{3}{2}(\alpha_2 + \alpha_3)$. Substituting $k = 5$ for $k = 4$ and repeating the above procedure, we can also obtain

$$\alpha_5 = -\frac{3}{2}(\alpha_2 + \alpha_3).$$

Thus, $\alpha_4 = \alpha_5$. From this identical relation, by solving (65) for the case $(i, k) = (4, 5)$, we have

$$\alpha_4 = \alpha_5 = -\frac{1}{2\sqrt{2}} < 0.$$

Then, by solving (65) for the cases of $(i, k) = (2, 4)$ and $(i, k) = (3, 4)$, we get

$$\alpha_2 = \alpha_3 = \frac{5}{6\sqrt{2}} > 0$$

which contradicts the above assumption $\alpha_2 \ne \alpha_3$. Therefore, the proof of Lemma 9 is completed. □

APPENDIX G
PROOF OF THEOREM 3

The proof is as follows.

1) We consider two special classes of 2×2 unitary matrices. Let the 2×2 signal constellation consist of finitely many unitary matrices all of which are in the form of

$$V_1 = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

or of

$$V_2 = \begin{pmatrix} a & b \\ b^* & -a^* \end{pmatrix}$$

where $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$. It is easily seen that a set of finitely many matrices, all of which are in the form of $V_1/\sqrt{2}$ or of $V_2/\sqrt{2}$, is equivalent to a spherical code on the surface of the four-dimensional unit-radius sphere Ω_4. For the above two classes of signal constellations, it can be verified that the diversity product and the diversity sum of the signal constellations are identical and that their values are both equal to the half of the minimum distance of the above equivalent spherical code. Therefore, the lower bound in 1) can be obtained.

2) For any two 2×2 unitary matrices of

$$\begin{pmatrix} a & b \\ -b^*\Delta_1 & a^*\Delta_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & d \\ -d^*\Delta_2 & c^*\Delta_2 \end{pmatrix}$$

where

$$|a|^2 + |b|^2 = |c|^2 + |d|^2 = |\Delta_1| = |\Delta_2| = 1$$

their difference matrix has an absolute determinant given by

$$\begin{aligned}
\mathcal{D} &\stackrel{\text{def}}{=} \left| \det \begin{pmatrix} a-c & b-d \\ -b^*\Delta_1 + d^*\Delta_2 & a^*\Delta_1 - c^*\Delta_2 \end{pmatrix} \right| \\
&\le |a-c|\,|a^*\Delta_1 - c^*\Delta_2| + |b-d|\,|b^*\Delta_1 - d^*\Delta_2| \\
&\le |a-c|(|a-c| + |a|\,|\Delta_1 - \Delta_2|) \\
&\quad + |b-d|(|b-d| + |b|\,|\Delta_1 - \Delta_2|) \\
&\le |a-c|^2 + |b-d|^2 + \sqrt{|a-c|^2 + |b-d|^2}\,|\Delta_1 - \Delta_2|
\end{aligned}$$

where the inequality

$$|a^*\Delta_1 - c^*\Delta_2| \le |a-c| + |a|\,|\Delta_1 - \Delta_2|$$

is employed.

Let

$$\begin{aligned}
A &\stackrel{\text{def}}{=} \sqrt{|a-c|^2 + |b-d|^2} \\
B &\stackrel{\text{def}}{=} |\Delta_1 - \Delta_2|
\end{aligned}$$

and

$$\epsilon \stackrel{\text{def}}{=} 1 + \sqrt{2}.$$

Then, we have

$$\begin{aligned}
\mathcal{D} &\le A^2 + AB \le A^2 + \frac{A^2}{2\epsilon} + \frac{\epsilon}{2}B^2 = \frac{\epsilon}{2}(A^2 + B^2) \\
&= \frac{1+\sqrt{2}}{2}(|a-c|^2 + |b-d|^2 + |\Delta_1 - \Delta_2|^2).
\end{aligned}$$

Clearly, a finite set of three-dimensional complex vectors $(a, b, \Delta)/\sqrt{2}$ can be reduced to a spherical code on the surface of the six-dimensional unit-radius sphere Ω_6. Therefore, by virtue of (23) in the case $M = 2$, the upper bound in 2) is achieved.

3) The squared Frobenius norm of the difference matrix in the above satisfies

$$\begin{aligned}
\mathcal{F} &\stackrel{\text{def}}{=} |a-c|^2 + |b-d|^2 + |a^*\Delta_1 - c^*\Delta_2|^2 + |b^*\Delta_1 - d^*\Delta_2|^2 \\
&\le 2(|a-c|^2 + |b-d|^2) + |\Delta_1 - \Delta_2|^2 \\
&\quad + 2(|b|\,|b-d| + |a|\,|a-c|)|\Delta_1 - \Delta_2| \\
&\le 2(|a-c|^2 + |b-d|^2) + |\Delta_1 - \Delta_2|^2 \\
&\quad + 2\sqrt{|a-c|^2 + |b-d|^2}\,|\Delta_1 - \Delta_2|.
\end{aligned}$$

Let $\kappa \stackrel{\text{def}}{=} (\sqrt{5} - 1)/2$. Then, we have

$$\begin{aligned}
\mathcal{F} &\le 2A^2 + B^2 + 2AB \le 2A^2 + B^2 + \kappa A^2 + \frac{B^2}{\kappa} \\
&= (2+\kappa)(A^2 + B^2) \\
&= \frac{3+\sqrt{5}}{2}(|a-c|^2 + |b-d|^2 + |\Delta_1 - \Delta_2|^2).
\end{aligned}$$

By the fact that a finite set of three-dimensional complex vector $(a, b, \Delta)/\sqrt{2}$ can be reduced to a spherical code on the surface of the six-dimensional unit-radius sphere Ω_6 and (22) in the case $M = 2$, the upper bound in 3) is derived.

4) In the proof of Proposition 2 given in Appendix B, we have shown that $\xi_1(L, \mathcal{V}) \le \frac{1}{2} D(2M^2, L)$. Therefore, by noting (13) and (22), we have

$$\Xi_M(L) \le \Delta_M(L) = \max_{\mathcal{V}} \xi_1(L, \mathcal{V}) \le \tfrac{1}{2} D(2M^2, L).$$

The proof of Theorem 3 is thus completed. □

APPENDIX H
PROOF OF THEOREM 4

The proof is as follows.

1) We need a fundamental result in sphere packing. Hamkins and Zeger [10], [11] have essentially proved that, for $k \ge 2$

$$A(k, d) = \Delta_{k-1}^{\text{pack}} \frac{S_k}{V_{k-1}} \left(\frac{2}{d}\right)^{k-1} (1 + o_d(1)) \qquad (66)$$

where the notation $o_d(1)$ denotes a variable in terms of d approaching zero as $d \to 0^+$, V_k is the k-dimensional content, or "volume," of the k-dimensional unit-radius sphere Ω_k defined in (26), given by

$$V_k \stackrel{\text{def}}{=} \frac{\pi^{k/2}}{\Gamma\left(\frac{k+2}{2}\right)}$$

and S_k the $(k-1)$-dimensional content, or "surface area," of Ω_k, given by

$$S_k \stackrel{\text{def}}{=} kV_k = \frac{k\,\pi^{k/2}}{\Gamma\left(\frac{k+2}{2}\right)}.$$

By applying Proposition 8, it follows from (66) that for each $k \ge 2$, we have, for large code size L

$$D(k, L) = 2\left(\Delta_{k-1}^{\text{pack}} \frac{S_k}{V_{k-1}}\right)^{\frac{1}{k-1}} L^{-\frac{1}{k-1}}(1 + o_L(1)).$$

By taking $k = 4$ in the preceding equality and making use of Property 1) in Theorem 3 and the fact that $\Delta_3^{\text{pack}} \geq \pi/(3\sqrt{2})$, we can see that 1) is true.

2) Let $V = (V_{mn})$ be an arbitrary $M \times M$ unitary matrix. The unitary matrix V can be written as the form of

$$V = Q_1 D Q_2$$

where Q_1, Q_2 are real orthogonal matrices and D is a diagonal unitary matrix [17]. Therefore, any element V_{mn} in V is a sum of finitely many complex-number terms whose real and imaginary parts are in the form of

$$F(\rho_1, \rho_2, \ldots, \rho_{M^2}) = \prod_{k=1}^{M^2} f_k(\rho_k)$$

where ρ_k for $k = 1, 2, \ldots, M^2$ are the M^2 parameters of the unitary matrix belonging to a compact subset, such as $[0, 2\pi]^{M^2} \subset \mathbb{R}^{M^2}$, and that the functions $f_k(\rho_k)$ for $k = 1, 2, \ldots, M^2$ have three possible forms of $\sin\rho_k$, $\cos\rho_k$, and the constant function 1 [24]. The following lemma is needed in our proof.

Lemma 10: Let $f_k(\rho_k)$ be one of the three functions $\sin\rho_k$, $\cos\rho_k$, and the constant function 1, for $\rho_k \in [0, 2\pi)$ and $1 \leq k \leq K$. Then

$$\left|\prod_{k=1}^{K} f_k(\rho_k) - \prod_{k=1}^{K} f_k(\phi_k)\right| \leq \sum_{k=1}^{K} |f_k(\rho_k) - f_k(\phi_k)| \leq \sum_{k=1}^{K} |\rho_k - \phi_k|.$$

Proof: By the mean value theorem, it is obvious that the above second inequality holds.

We can prove the above first inequality by using a simple induction procedure in terms of $K \in \mathbb{N}$. That the above first inequality holds in the case $K = 1$ is self-evident. We assume that the above first inequality holds for some $K \in \mathbb{N}$ and want to show that it also holds for $K+1$. In fact, noting that $|f_k(\rho_k)| \leq 1$ for $1 \leq k \leq K$, we can obtain

$$\begin{aligned}
&\left|\prod_{k=1}^{K+1} f_k(\rho_k) - \prod_{k=1}^{K+1} f_k(\phi_k)\right| \\
&\leq \left|f_{K+1}(\rho_{K+1}) \prod_{k=1}^{K} f_k(\rho_k) - f_{K+1}(\phi_{K+1}) \prod_{k=1}^{K} f_k(\rho_k)\right| \\
&\quad + \left|f_{K+1}(\phi_{K+1}) \prod_{k=1}^{K} f_k(\rho_k) - f_{K+1}(\phi_{K+1}) \prod_{k=1}^{K} f_k(\phi_k)\right| \\
&\leq |f_{K+1}(\rho_{K+1}) - f_{K+1}(\phi_{K+1})| + \left|\prod_{k=1}^{K} f_k(\rho_k) - \prod_{k=1}^{K} f_k(\phi_k)\right| \\
&\leq \sum_{k=1}^{K+1} |f_k(\rho_k) - f_k(\phi_k)|
\end{aligned}$$

as required. □

We can partition each side of the compact hypercubic subset $[0, 2\pi]^{M^2}$ of parameters for $M \times M$ unitary matrices into K equal sections from 0 to 2π. That is,

$$[0, 2\pi] = \bigcup_{k=1}^{K} \left[(k-1)\frac{2\pi}{K}, k\frac{2\pi}{K}\right].$$

Then, the above hypercubic subset, namely, $[0, 2\pi]^{M^2}$, is partitioned into K^{M^2} equal sections each of which is a hypercube with equal side length of $2\pi/K$. For large L, we can take

$$K = \left\lfloor (L-1)^{1/M^2} \right\rfloor$$

which satisfies $K^{M^2} < L$. Then, for any unitary signal constellation with size L there must be two signal matrices whose parameters belong to a single same hypercubic section. Then, according to Lemma 10, the absolute value of each element of the difference matrix between the above two signal matrices should be of the order

$$O(2\pi/K) = O\left(L^{-\frac{1}{M^2}}\right).$$

Consequently, for large L, the Frobenius norm of the above difference matrix is of the order $O(L^{-\frac{1}{M^2}})$. From this, the upper bound in 2) can be achieved.

The proof of Theorem 4 is thus completed. □

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