Recursive Space–Time Trellis Codes Using Differential Encoding

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***Abstract*—Differential space–time modulation (DSTM) has been recently proposed by Hughes, and Hochwald and Sweldens when the channel information is not known at the receiver, where the demodulation is in fact the same as the coherent demodulation of space–time block coding by replacing the channel matrix with the previously received signal matrix. On the other hand, the DSTM also needs a recursive memory of a matrix block at the encoder and therefore provides a trellis structure when the channel information is known at the receiver, which is the interest of this paper. This recursive structure of the DSTM has been adopted lately by Schlegel and Grant in joint with a conventional binary code and joint iterative decoding/demodulation with a superior performance. The number of states of the trellis from the recursive structure depends on both the memory size, which is fixed in this case, and the unitary space–time code (USTC). When a USTC for the DSTM forms a group, the number of states is the same as the size of the USTC, otherwise the number of the states is the size of the semi-group generated by the USTC from all the multiplications of the matrices in the USTC. It is well known in the conventional convolutional coding (CC) or the trellis coded modulation (TCM), the free (Hamming or Euclidean) distance (or the performance) increases when the number of states increases by adding more memory with a properly designed CC or TCM. In this paper, we systematically study and design the USTC/DSTM for the recursive space–time trellis modulation and show that the diversity product increases when the number of states increases, which is not because of the memory size but because of the different USTC designs that generate different sizes of semi-groups. We propose a new USTC design criterion to ensure that the trellis structure improves the diversity product over the USTC as a block code. Based on the new criterion, we propose a new class of USTC design for an arbitrary number of transmit antennas that has an analytical diversity product formula for two transmit antennas. We then follow Schlegel and Grant's approach for joint encoding and iterative decoding of a binary coded DSTM (turbo space–time coding) and numerically show that our new USTC designs for the recursive space–time trellis modulation outperforms the group USTC used by Schlegel and Grant.**

***Index Terms*—Differential encoding, group, iterative decoding, recursive space–time trellis codes, turbo principle.**

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I. Introduction

It is well understood that multiple antennas can be used to effectively combat the fading in wireless links by exploiting the spatial diversity [1]. Telatar [2] and Foschini and Gans [3] have shown that the capacity of a multi-antenna system grows linearly in terms of the minimum between the numbers of transmit and receive antennas. Similar to single antenna systems, to approach the capacity, coding and modulation called space–time coding/modulation is one of the key steps. Most of the current research on the space–time code designs follows two major directions. One is to achieve the diversity-multiplexing tradeoff proposed by Zheng and Tse [7], such as nonvanishing determinant codes and perfect codes [19]–[30]. The other is based on the rank and diversity product criteria proposed in Guey *et al.* [4] and Tarokh *et al.* [5], where not only the full rank is achieved but also the large diversity product, if not the largest, is pursued, see for example, [11]–[19], [27], [28]. In this paper, we are also interested in pursuing large diversity product space–time codes. Some early related references can be found in, for example, [8].

Based on code structures, space–time codes can be categorized into two groups: space–time block codes (STBCs) and space–time trellis codes (STTCs) [9] and some other early references can be also found in [8]. While an STBC is more similar to modulation (called *uncoded*) than to the conventional block coding, such as RS codes, in single antenna systems, an STTC is similar to the single antenna trellis codes by adding memory to the encoding. Recently, an interesting and different space–time trellis code was proposed by Schlegel and Grant in [8] by adopting the differential space–time modulation (DSTM) proposed in [11], [12] where the memory is due to the differential encoding and the trellis decoding rather than block decoding in [11], [12] is due to the assumption of the known channel at the receiver. One of the most interesting characteristics of this space–time trellis coding is that the encoding is recursive and can be well combined as an inner code with an outer binary code to form iterative decoding and achieve a similar performance as a typical turbo code does over an AWGN channel, which has been shown in [8] with a superior performance. This serially concatenated structure can be regarded as a natural generalization of turbo DPSK [31] used in single antenna systems. Different from the conventional trellis codes, the states of the trellis of DSTM are from the different results of the multiplications of the matrices in a unitary space–time code (USTC), where the memory size is fixed to the matrix size in USTC. Different USTC produce different state numbers and also different trellises. In [8], the 2×2 group codes of size 8 proposed in [12], [13] were used. When the USTC forms a group, the number of the states of the trellis code is the same as the number of the codewords of an USTC, which is 8 in [8]. A more general group USTC and subsets of group constructions were proposed in [14]. However, the group structure of a USTC has its limitation on the diversity product property. By relaxing the group structure, other designs of USTC were proposed in [15]–[17]. When a USTC is not a group, the number of states of DSTM is determined by the number of all possible different product matrices of multiplications of any code matrices in the USTC, where the set of all such product matrices forms a semi-group that is called *the semi-group generated by the USTC*. The size of the semi-group generated from a nongroup USTC is larger than the size of the USTC itself, i.e., the number of states of DSTM using a nongroup USTC is increased over the one of a group USTC of the same size as the nongroup USTC. Since a nongroup USTC may have a better diversity product than a group USTC, increasing the number of states in the above sense may produce a better trellis code of a DSTM, which is similar to the conventional TCM schemes. What we should emphasize here is that the increase of a state number is not related to the memory size but due to the increase of a number of different product matrices of matrices in a USTC, which is, however, essentially different from single antenna TCM schemes and the STTC mentioned before in [5], [32]–[38].

In this paper, we systematically study USTC designs for recursive space–time modulation from differential encoding. We first show that, for the space time trellis codes proposed in [8], their diversity products cannot be greater than that of the corresponding unitary codes [12] because there exist error events with length $L = 2$, i.e., two typical paths diverge from one state and reemerge to one state after two trellis transitions. If there is no error event with length $L = 2$, a space time trellis code from the differential encoding may have larger diversity product over the corresponding space–time block code. Based on this observation, we propose a new design criterion for a USTC for the recursive space time trellis code (RSTTC) from differential encoding. Considering the input symbols carrying R-bit information, $R \geq 1$, then the size of input symbols is $M = 2^R$. Let $\mathcal{G}_M = \{G_0, G_1, \ldots, G_{M-1}\}$ be the information symbols and $\mathcal{S} = \langle \mathcal{G}_M \rangle$ be the semigroup generated by $\mathcal{G}_M$, i.e., all product matrices of any combinations (repeats are allowed) of $G_0, \ldots, G_{M-1}$. Obviously, the differential encoding with $\mathcal{G}_M$ can be represented as a trellis diagram whose state set is $\mathcal{S}$ and input symbols are $G_0, \ldots, G_{M-1}$. If $G_i G_j \neq G_m G_n$ whenever $(i,j) \neq (m,n)$, $\forall\, i,\ j,\ m,\ n \in [0, M-1]$, we can show that the error event lengths of the trellis code are greater than 2, which makes it possible for the STTC to produce larger diversity product than $\mathcal{G}_M$ itself. If we treat a space–time modulation $\mathcal{G}_M$ as an uncoded modulation (without memory), then this new space time trellis construction is a coded modulation (with memory) similar to the conventional TCM. Note that due to the data rate reduction of a binary convolutional code in a TCM, the expansion of a signal constellation is used to maintain the same bandwidth efficiency as an uncoded modulation. However, here the expansion of a signal constellation (since the size of $\mathcal{S}$ is always greater than that of a nongroup $\mathcal{G}_M$) is for the increase of the number of states and therefore possibly the increase of error event lengths, which corresponds to the increase of the memory size in a conventional TCM. Interestingly, the new criterion is unique for the design of a USTC for multiple antennas in the sense that it can not be applied to a single antenna system because in a single antenna system $G_i \in \mathcal{G}_M$, $i = 0, \ldots, M-1$, is always a scalar and therefore it is always true that $G_i G_j = G_j G_i$, for any $G_i, G_j \in \mathcal{G}_M$. Thus, it is impossible to design a trellis code with error event lengths greater than 2 through differential encoding for a classical single antenna system. Under our newly proposed design criterion, we propose a class of USTC for RSTTC from differential encoding for any size of constellation M and arbitrary number of transmit antennas. The closed form diversity product analysis for a two transmit antenna system is also presented. For $M = 2, 4, 8$, the diversity products of the RSTTC are $1, 1, 0.8040$, while that of the optimal and best-known uncoded block codes are $1,\ \sqrt{2/3}$, and 0.7071, respectively. The new class RSTTC not only shows how to design the recursive space time trellis codes using differential encoding but also confirms our findings on the design criterion to achieve larger diversity products.

Related works on iterative decoding to achieve “turbo gain” include [39]–[42]. In [39], the outputs of a turbo encoder are directly mapped to QPSK symbols and transmitted over multiple antennas. At the decoder, the loop for soft information exchange is within the turbo decoders. Further improvement is achieved in [41], where soft information exchange is implemented between a space–time modulation and an inner turbo decoder. In this sense, the space–time coding functions as both the modulator and the inner encoder for the serial concatenated system. While there is no trellis structure available to extract soft information with Bahl–Cocke–Jelinek–Raviv (BCJR) algorithm [43], the authors derive the soft information from sphere decoding. The larger the loop of the iterative process takes, the better the advantage of received space–time signals is taken, and therefore the better performance it may result in. Similar work can be found in [42] where the designs of interleaver and precoder are discussed to improve the performance of the concatenated MIMO systems in terms of outage probability. In [40], a simple space–time interleaved coding scheme with iterative decoding of good performance is proposed. While these works focus on space–time block codes (with size $t \times 1$ where t is the number of transmit antennas) as inner codes and large diversity product may not be guaranteed, in this paper, we discuss space–time trellis codes as inner codes with large diversity product. One advantage of STTCs is that the BCJR algorithm can be applied to obtain the soft information through the trellis structure. The motivation for the study of recursive STTC is that, to better exploit turbo gain, an inner encoder needs to be recursive, which is shown in [44]. In [45], [46], the effectiveness of recursive structures for iterative decoding in MIMO systems is also demonstrated.

This paper is organized as follows. In Section II, we describe the problem of interest and the motivation of the study in details. In Section II, we also present a new USTC design criterion. In Section III, we provide a new class of USTC for RSTTC from differential encoding. The analysis of the diversity product for this class of RSTTC is also presented in this section. In Section IV, we present some simulation results. Finally, some concluding remarks are provided in Section V.

II. Problem Description and New Design Criterion

In this section, we present some necessary preliminaries about the space–time trellis modulation, the motivation, and a new design criterion for the construction of recursive space–time trellis code from differential encoding.

A. Background

Consider a system with t transmit and r receive antennas over a Rayleigh-fading channel. Let h_{ij} be the fading coefficient of the channel between the ith transmit and the jth receive antenna. As in [5], it is modeled as an independent complex Gaussian variable with zero mean and 0.5 variance per dimension. The transmitted signal matrix for a frame of length L can be represented as

$$\mathcal{C} = [C_0 \quad C_1 \quad \cdots \quad C_L]^T \tag{1}$$

where C_l, $0 \le l \le L$, is the codeword matrix transmitted at the lth trellis transition and ^{T} denotes the transpose of a matrix. At the receiver, we have

$$Y = \sqrt{\rho}\mathcal{C}H + W \tag{2}$$

where ρ is the signal-to-noise ratio (SNR) at each receive antenna, Y is the received signal, $H = (h_{ij})$ is the channel coefficient matrix, and W is the complex Gaussian noise with zero mean and unit variance. Let us first consider the error event of $\mathcal{C}$ and $\tilde{\mathcal{C}}$, i.e., if $\mathcal{C}$ is transmitted but the decoder makes an erroneous decision and chooses

$$\tilde{\mathcal{C}} = [\tilde{C}_0 \quad \tilde{C}_1 \quad \cdots \quad \tilde{C}_L]^T \tag{3}$$

as the most likely transmitted signal. It has been shown in [5] that the determinant of $(\mathcal{C} - \tilde{\mathcal{C}})^\dagger(\mathcal{C} - \tilde{\mathcal{C}})$ determines the space–time code performance in terms of pairwise error probability $P(\mathcal{C} \to \tilde{\mathcal{C}})$, where $\dagger$ denotes the Hermitian transpose. This determinant criterion facilitates the evaluation of a given space–time code and also provides a design criterion: For a space–time block code $\{G_i : 0 \le i \le M-1\}$ of size M of square matrices, since the minimum of $|\det(G_i - G_{i'})|$, $0 \le i \ne i' \le M-1$, taken over all pairs of distinct codewords dominates the performance, the design objective is to maximize this minimum value (called diversity product or distance product or coding advantage) [11]

$$\zeta_B = \frac{1}{2} \min_{0 \le i \ne i' \le M-1} |\det(G_i - G_{i'})|^{1/t}. \tag{4}$$

For a space–time trellis code, we need to consider the minimum of the determinants of all the error events, which is defined as

$$\zeta_T = \frac{1}{2} \min_{\mathcal{C} \ne \tilde{\mathcal{C}}} |\det((\mathcal{C} - \tilde{\mathcal{C}})^\dagger(\mathcal{C} - \tilde{\mathcal{C}}))|^{1/2t} \tag{5}$$

where $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are a pair of distinct codewords of an arbitrary length L on the trellis, i.e., they diverge from a common state and re-merge at a common state after L steps. In a TCM scheme of single antenna systems, the Euclidean distance of an error event is the sum of all distances of all branches of the error event. This distance additivity does not hold anymore for matrix determinants for space–time trellis codes: let us rewrite the difference matrix as

$$\mathcal{C} - \tilde{\mathcal{C}} = [B_0^T \quad B_1^T \quad \cdots \quad B_L^T]^T \tag{6}$$

then, in general

$$\det\left((\mathcal{C} - \tilde{\mathcal{C}})^\dagger(\mathcal{C} - \tilde{\mathcal{C}})\right) = \det\left(\sum_{l=0}^{L}(B_l^\dagger B_l)\right) \ne \sum_{l=0}^{L}\det\left(B_l^\dagger B_l\right). \tag{7}$$

The above nonadditivity makes it more difficult to analyze the diversity product properties for a space–time trellis code. Although the additivity does not hold for matrix determinants, we have the following *semiadditivity* [47]:

$$\det\left(\sum_{l=0}^{L}(B_l^\dagger B_l)\right) \ge \sum_{l=0}^{L}\det\left(B_l^\dagger B_l\right). \tag{8}$$

Both the Euclidean distance additivity and the matrix determinant semi-additivity suggest that, to increase error event lengths L leads to increase Euclidean distance and diversity product and therefore improve the performance. In conventional TCM schemes, to increase error event lengths one needs to either increase the number of states, i.e., memory size, or decrease the coding rate, i.e., decrease the number of branches from each state, and it is usually independent of a symbol mapping. Therefore, for a fixed rate, the only way to increase error event lengths is to increase memory size as we also explained in Introduction. This is essentially different for space–time coding for multiple antenna systems as we shall see in more details in the next subsection, where it is possible to increase the number of states and therefore error event lengths by choosing different space–time block/matrix modulations when differential coding is used as we explained in Introduction. Since differential coding is recursive, it can be naturally combined in a turbo type coding as an inner code as proposed in [8].

B. Problem Description of Designing RSTTC From DSTM

As a generalization of DPSK in single antenna systems, differential space–time modulation (DSTM) has been recently proposed in [11], [12] by using unitary space–time codes (USTC) for multiple antenna systems when the channel information is not known at the receiver. Let $\mathcal{G} = \{G_0, G_1, \ldots, G_{M-1}\}$ be a USTC of size M where $G_i G_i^\dagger = I$. Binary information bits are mapped to unitary matrices G in $\mathcal{G}$ and the transmitter transmits $\sqrt{\rho}S_l$ at the lth time slot and S_l is from the following differential encoding:

$$S_l = S_{l-1}G_{i_l} \tag{9}$$

and $S_0 = I$, where $G_{i_l} \in \mathcal{G}$. At the receiver, at the lth time slot

$$\begin{aligned} Y_l &= \sqrt{\rho}S_l^T H + W_l \\ &= \sqrt{\rho}G_{i_l}^T S_{l-1}^T H + W_l \\ &= G_{i_l}^T Y_{l-1} + \bar{W}_l \end{aligned} \tag{10}$$

where $\bar{W}_l = W_l - G_{i_l}^T W_{l-1}$. From the above equation, the channel information H is not necessary for decoding the information matrix G_{i_l} at the receiver if the previous received signal matrix Y_{l-1} is used as an approximation of the channel H, which is basically the same as the coherent detection $Y = G^T H + W$ of the space–time block code $\mathcal{G}$. However, when the channel information H is known at the receiver as what is commonly studied in space–time coding as coherent detection in the above differentially encoded system $Y_l = \sqrt{\rho} S_l^T H + W_l$, the information matrix sequence $\{G_{i_l}\}$ can be decoded from the trellis built upon the recursive structure of the differential encoding as recently proposed in [8]. Since the inner space–time coding is recursive, by combining it with an outer binary code, an iterative decoding is proposed in [8] where a superior performance is achieved. In [8], USTC with group structure, i.e., any product matrix of any number of matrices of any powers of matrices in code $\mathcal{G}$ is still in code $\mathcal{G}$, are used. The group structure guarantees that the number of states of the trellis from the differential encoding is the same as the size of USTC $\mathcal{G}$. This group structure property has both advantage and disadvantage. The advantage is that it prevents from that the number of states of the trellis being too large. The disadvantage is that, since the number of states is fixed for a fixed size of USTC $\mathcal{G}$ (or a fixed diversity product or performance of $\mathcal{G}$), it is not possible to increase their error event lengths, i.e., limits the performance. In fact, since the number of states is the same as the size of a group code $\mathcal{G}$, in the trellis of the differential coding, each state reaches each state and therefore, the minimum error event length is always 2 and furthermore, for any state and any two branches leaving the state there exists an error event of length 2 and containing the two branches and thus, the diversity product of the trellis code is the same as the one of $\mathcal{G}$ that is limited by the size M, which is shown in more details below. Let us see the example studied in [8] where the group code $\mathcal{G}$ of size 8 in [12], [13] was used

$$G_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ G_2 = \begin{pmatrix} \mathbf{j} & 0 \\ 0 & -\mathbf{j} \end{pmatrix}$$
$$G_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ G_6 = \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix} \tag{11}$$

and, correspondingly, $G_1 = -G_0, G_3 = -G_2, G_5 = -G_4$ and $G_7 = -G_6$, where $\mathbf{j} = \sqrt{-1}$. Let

$$\mathcal{G} = \{G_0, \quad G_1, \quad \ldots, \quad G_7\}. \tag{12}$$

Then, $\mathcal{G}$ is a group and the trellis of the differential coding is shown in Fig. 1.

Theorem 1: The diversity product of a recursive space–time trellis code from the differential encoding with a group space–time block modulation $\mathcal{G}$ is the same as the diversity product of $\mathcal{G}$ itself.

Proof: When $\mathcal{G}$ is a group, the number of states of the trellis of the differential coding is the same as the size of $\mathcal{G}$. Thus, each state from the trellis reaches each state. Let G_i and G_j be any two distinct code matrices in $\mathcal{G}$. Consider an arbitrary state S and the two branches leaving S due to the transitions of $G_i^1 = G_i$ and $G_j^1 = G_j$, respectively, as shown in Fig. 2. Since $G_i \neq G_j$, the next states are different, i.e., $SG_i^1 \neq SG_j^1$, but both of them

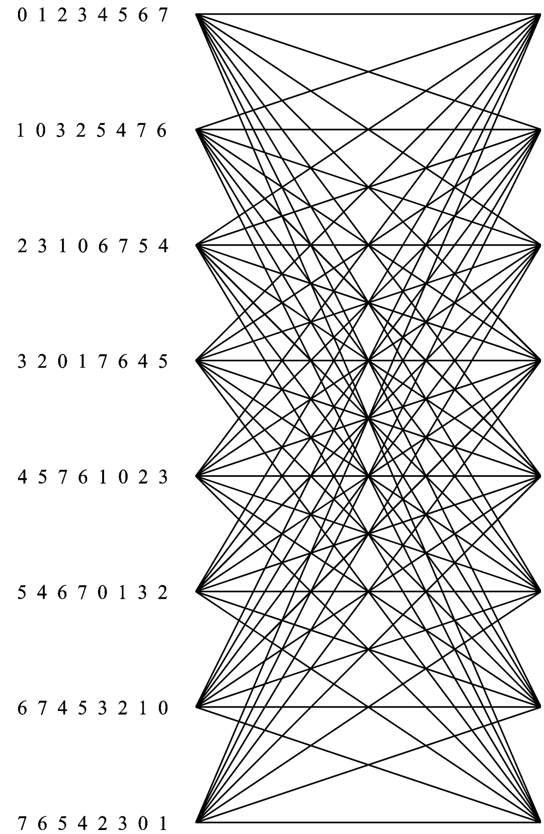

Fig. 1. Trellis representation for (12).

can reach another state $SG_i^1 G_i^2 = SG_j^1 G_j^2$ simultaneously after another transition. Thus, the two paths can be represented as

$$\mathcal{C} = [S \quad SG_i^1 \quad SG_i^1 G_i^2]^T \tag{13}$$
$$\tilde{\mathcal{C}} = [S \quad SG_j^1 \quad SG_j^1 G_j^2]^T \tag{14}$$

and they form an error event of length 2 that is also the minimum error event length. Since

$$G_i^1 \neq G_j^1, \tag{15}$$
$$SG_i^1 G_i^2 = SG_j^1 G_j^2 \quad \text{implies} \quad G_i^1 G_i^2 = G_j^1 G_j^2 \tag{16}$$

and substituting (15), (16) into (5), we have

$$\begin{aligned}\zeta_T &= \frac{1}{2} \min_{\mathcal{C} \neq \tilde{\mathcal{C}}} |\det((\mathcal{C} - \tilde{\mathcal{C}})^\dagger (\mathcal{C} - \tilde{\mathcal{C}}))|^{1/(2t)} \\ &= \frac{1}{2} \min_{0 \le i \neq j \le M-1} |\det(G_i - G_j)|^{1/t}\end{aligned} \tag{17}$$

which is the diversity product ζ_B of the group code $\mathcal{G}$. □

From the above proof, one can see that the main reason why the diversity product of the trellis code is limited to the one of the block code/modulation $\mathcal{G}$ is due to the fact that for any pair of branches G_i^1 and G_j^1 in $\mathcal{G}$ there exist G_i^2 and G_j^2 that satisfy (16). This can also be illustrated in Fig. 1, where we can see that any pair of states are connected to any state after one transition. Also, one can see that the reason why the minimum error event length is 2 is again due to (16). This motivates us to use a new design criterion on USTC for recursive trellis codes

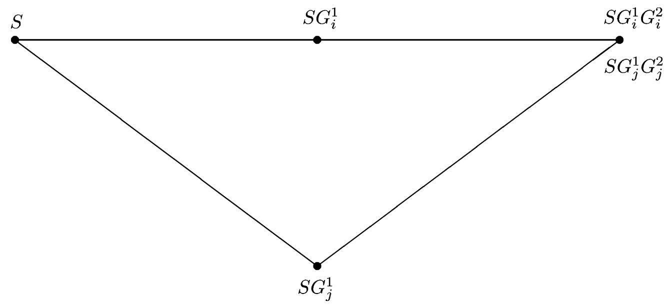

Fig. 2. An error event of length 2 ($L = 2$).

from differential coding in the next subsection by eliminating (16).

C. A New Design Criterion for RSTTC From DSTM

From (8) in the preceding subsection, one can see that, to increase the diversity product of a space–time trellis code, it is important to increase error event lengths, and to increase error event lengths, it is necessary to use a space–time block code/modulation $\mathcal{G}$ that does not satisfy (16) for any two distinct code matrices in $\mathcal{G}$. As we see from Theorem 1, clearly, to do so, $\mathcal{G}$ can not be a group. A code $\mathcal{G}$ is called a *group code* if $(G_{i_1})^{l_1} \cdots (G_{i_p})^{l_p} \in \mathcal{G}$ for any integers $l_1, \ldots, l_p$ and any $G_{i_1}, \ldots, G_{i_p} \in \mathcal{G}$. A code $\mathcal{G}$ is called a *semigroup code* if $(G_{i_1})^{l_1} \cdots (G_{i_p})^{l_p} \in \mathcal{G}$ for any nonnegative integers $l_1, \ldots, l_p$ and any $G_{i_1}, \ldots, G_{i_p} \in \mathcal{G}$. One can see that Theorem 1 also holds for a semi-group code since none of the inverse matrices of G_i in $\mathcal{G}$ is involved in the trellis code and the difference between semi-group and group is whether it is closed for the matrix inversion. Therefore, to increase error event lengths to be above 2, a code $\mathcal{G}$ can not be a semigroup code. In order to study recursive space–time trellis codes (RSTTC) from differential encoding and a general USTM, let us first see some notations and properties.

Let $\langle G_0, \ldots, G_{M-1} \rangle$ be the semigroup generated by $G_0, \ldots, G_{M-1}$, i.e., it consists of all products of nonnegative powers of $G_0, \ldots, G_{M-1}$ with possible repeats. The following proposition is obvious.

Proposition 1: Let $\mathcal{G} = \{G_0, \ldots, G_{M-1}\}$ be a unitary space–time block code and $\langle G_0, \ldots, G_{M-1} \rangle$ be its generated semigroup. If the size of $\langle G_0, \ldots, G_{M-1} \rangle$ is finite, then the recursive trellis code from the differential encoding and the USTC $\mathcal{G}$ has and only has its states in $\mathcal{S} = \langle G_0, \ldots, G_{M-1} \rangle$ and each branch of the trellis code carries an information symbol $G \in \{G_0, \ldots, G_{M-1}\}$.

The recursive trellis code from differential encoding and a USTC $\mathcal{G} = \{G_0, \ldots, G_{M-1}\}$ is called the *trellis representation* of $\langle G_0, \ldots, G_{M-1} \rangle$ and denoted by $\mathcal{T}(\mathcal{G})$, if $\langle G_0, \ldots, G_{M-1} \rangle$ has finite size.

Theorem 2: Let $\mathcal{T}(G_0, \ldots, G_{M-1})$ be the trellis representation of $\langle G_0, \ldots, G_{M-1} \rangle$. If $G_iG_j \neq G_mG_n$ whenever $(i,j) \neq (m,n)$, $\forall\, 0 \leq i,j,m,n \leq M-1$, then, any error event length of trellis representation $\mathcal{T}(G_0, \ldots, G_{M-1})$ is greater than 2.

Proof: From (13) and (14) we can see that for any pair $G_i^1, G_j^1 \in \{G_0, \ldots, G_{M-1}\}$ and any state $S \in \langle G_0, \ldots, G_{M-1} \rangle$, if $G_i^1 \neq G_j^1$, then it is impossible to find G_i^2 and G_j^2 in $\{G_0, \ldots, G_{M-1}\}$ such that $SG_i^1G_i^2 = SG_j^1G_j^2$ because $G_iG_j \neq G_mG_n, \forall\, 0 \leq i,j,m,n \leq M-1$, and $(i,j) \neq (m,n)$. This means that it takes at least 3 trellis transitions for $\mathcal{C}$ and $\tilde{\mathcal{C}}$ to reemerge after diverging from any state. □

Based on the result in Theorem 2, we propose the following *new design criterion* for designing a USTC $\mathcal{G} = \{G_0, \ldots, G_{M-1}\}$ of size M for a recursive space–time trellis code from differential encoding and the USTC:

i) The semigroup $\langle G_0, \ldots, G_{M-1} \rangle$ generated by $\{G_0, \ldots, G_{M-1}\}$ has finite size; $G_iG_j \neq G_mG_n$;
ii) $G_iG_j \neq G_mG_n$, whenever $(i,j) \neq (m,n)$, $\forall\, 0 \leq i,j,m,n \leq M-1$.

From ii), one can see that $G_iG_j \neq G_jG_i$ for all $G_i \neq G_j \in \mathcal{G}$. This means that all elements in $\mathcal{G}$ do not commute. Criterion ii) also implies that the number of states of the trellis, i.e., the size $\mathcal{S} = \langle G_0, \ldots, G_{M-1} \rangle$, is at least M^2 that is certainly much higher than the one, M, of a semigroup code. Since in single antenna systems, all symbols G_i are scalars and therefore they commute, i.e., $G_iG_j = G_jG_i$ for any i and j, and therefore, the noncommutativity ii) does not hold. This tells us that, in single antenna case, the number of states of a recursive trellis code from DPSK may not be increased by choosing different modulation constellations. In other words, the above design criterion has the *essential difference* between single and multiple antenna systems.

As a remark. it is not hard to see that, to further increase the minimum error event length from 2 to $\nu > 2$, the above criterion can be easily generalized to

$$G_{i_1} \cdots G_{i_l} \neq G_{j_1} \cdots G_{j_l}$$

when $(i_1, \ldots, i_l) \neq (j_1, \ldots, j_l)$ for $2 \leq l \leq \nu$. In this case, the number of states is, however, at least M^ν and may be too large to deal with.

The major difference between the above systematic scheme and the scheme in [8] is that the states in our scheme are generated by a nongroup USTC while the states in the latter are from a group USTC itself and a group USTC limits the number of states and its diversity product that may affect the inner code performance. From our designs and simulations shown in next sections, we shall see that by relaxing the group requirement of a USTC and using the above new design criterion, our newly designed RSTTC with higher diversity product by avoiding the error events of length 2 may have improved performance over the existing ones from group codes in [8].

Before concluding this section, we emphasize that the two conditions for the design of RSTTCs provide a way to increase the diversity product ζ_T through the increase of error event length. However, the increase of error event length itself can not guarantee the increase of the diversity product ζ_T. As shown in (5) the diversity product ζ_T depends on both the codewords and the length associated with an error event. An error event with larger length may not lead to a larger diversity product. Therefore, to construct a USTC $\mathcal{G}$ which satisfies the above criteria with larger diversity product is the major challenge for this design.

III. A Class of Recursive Space-Time Trellis Codes

In this section, we first propose a class of USTC for two transmit antennas and then generalize it to any number of antennas.

A. Design for Two Transmit Antennas, i.e., $t = 2$

We first present a design and then its diversity product calculation.

1) Design: Consider the input symbols carrying R-bit information. Then, the size of the input symbols is $M = 2^R, R \geq 1$. Let $\mathbb{Z}_M = \{0, 1, \ldots, M-1\}$. For any given two integers $p, q \in \mathbb{Z}_M$, we define 2×2 unitary matrix $G(p,q)$ as follows:

$$G(p,q) = \begin{bmatrix} 0 & e^{\mathrm{j}2\pi p/M} \\ e^{\mathrm{j}2\pi q/M} & 0 \end{bmatrix} \tag{18}$$

and then we construct the following M constellations for the R-bit input signals:

$$\mathcal{G}_M = \{G_m = G(p_m, q_m) \mid m = 0, 1, \ldots, M-1\} \tag{19}$$

where $p_m, q_m \in \mathbb{Z}_M$, and they are chosen as

$$q_m = m, \; m = 0, 1, \ldots, M-1 \tag{20}$$
$$p_m = 2q_m = 2m \bmod M, \; m = 0, 1, \ldots, M-1. \tag{21}$$

We next show that the above class $\mathcal{G}_M$ does satisfy the criterion i)-ii). To do so, let us first define diagonal 2×2 unitary matrix $D(p,q)$ for any integers $p, q \in \mathbb{Z}_M$:

$$D(p,q) = \begin{bmatrix} e^{\mathrm{j}2\pi p/M} & 0 \\ 0 & e^{\mathrm{j}2\pi q/M} \end{bmatrix} \tag{22}$$

which is in fact a form of a product matrix of two matrices in $\mathcal{G}_M$.

Theorem 3: Let $\mathcal{S} = \langle G_0, \ldots, G_{M-1} \rangle$ be the semi-group generated by $\{G_0, \ldots, G_{M-1}\}$. Then

$$\mathcal{S} = \{G(i,j), D(i,j) | i, j \in \{0, 1, \ldots, M-1\}\} \tag{23}$$

and $\mathcal{S}$ is also a group and the size of $\mathcal{S}$ is $|\mathcal{S}| = 2M^2$.

Proof: See Appendix I.

From the result in Theorem 3, the size of $\mathcal{S}$ is finite and therefore, $\mathcal{G}_M$ has a trellis representation $\mathcal{T}(\mathcal{G}_M)$ with finite states $\mathcal{S}$. We next check Criterion ii).

Theorem 4: For any $G_i, G_j, G_m,$ and $G_n \in \mathcal{G}_M$, where $\mathcal{G}_M$ is defined in (19), if $(i,j) \neq (m,n)$, then $G_i G_j \neq G_m G_n$.

Proof: It is easy to check $G_i G_j = D(p_i + q_j, q_i + p_j)$ and $G_m G_n = D(p_m + q_n, q_m + p_n)$. Suppose that $G_i G_j = G_m G_n$ for some $(i,j) \neq (m,n)$. Then

$$p_i + q_j = p_m + q_n \bmod M \tag{24}$$
$$q_i + p_j = q_m + p_n \bmod M. \tag{25}$$

From (21), we have

$$2q_i + q_j = 2q_m + q_n \bmod M \tag{26}$$

$$q_i + 2q_j = q_m + 2q_n \bmod M. \tag{27}$$

Solving for q_i and q_j we have

$$q_i = q_m \text{ and } q_j = q_n. \tag{28}$$

Combining with (20), we have $i = m$ and $j = n$, which contradicts with the assumption $(i,j) \neq (m,n)$. This contradiction completes the proof. □

From Theorems 2 and 4, we immediately have the following property.

Corollary 1: Let $\mathcal{T}(\mathcal{G}_M)$ be the trellis representation (or the recursive trellis code) of $\mathcal{G}_M$, where $\mathcal{G}_M$ is defined in (19), then any error event length of $\mathcal{T}(\mathcal{G}_M)$ is greater than 2.

An additional property of $\mathcal{G}_M$ is as follows.

Proposition 2: If $G_i, G_j \in \mathcal{G}_M$, then $G(p_i + p_j, q_i + q_j) \in \mathcal{G}_M$ and $G(p_i - p_j, q_i - q_j) \in \mathcal{G}_M$.

Proof: From (20) and (21), $p_i + p_j = 2(q_i + q_j)$, and $q_i + q_j \in \mathbb{Z}_M$, thus we have $G(p_i + p_j, q_i + q_j) \in \mathcal{G}_M$. Similarly, we have $G(p_i - p_j, q_i - q_j) \in \mathcal{G}_M$. □

Proposition 2 will be used for the following diversity product analysis for this class of recursive space–time trellis codes.

2) Diversity Product Formula: In this subsection, we calculate the diversity product of the above class of differential space–time trellis codes $\mathcal{T}(\mathcal{G}_M)$. We first calculate the determinant of difference matrix for any error event. We then determine the minimum among all the error events, which is also the diversity product of the trellis codes $\mathcal{T}(\mathcal{G}_M)$.

Consider an error event with length $L, L > 2$, the two paths $\mathcal{C}$ and $\tilde{\mathcal{C}}$, which start from the same state S and reemerge after L trellis transition, are given by

$$\mathcal{C} = [C_0 \quad C_1 \quad C_2 \quad \cdots \quad C_L]^T \tag{29}$$
$$\tilde{\mathcal{C}} = [\tilde{C}_0 \quad \tilde{C}_1 \quad \tilde{C}_2 \quad \cdots \quad \tilde{C}_L]^T \tag{30}$$

where

$$C_l = C_{l-1} G_{u_l}, \; 1 \leq l \leq L, \; G_{u_l} \in \mathcal{G}_M \tag{31}$$
$$\tilde{C}_l = \tilde{C}_{l-1} G_{v_l}, \; 1 \leq l \leq L, \; G_{v_l} \in \mathcal{G}_M \tag{32}$$
$$C_0 = \tilde{C}_0 = S \tag{33}$$
$$C_L = \tilde{C}_L \tag{34}$$
$$C_l \neq \tilde{C}_l, \; 0 < l < L. \tag{35}$$

Note that (31) and (32) are just the encoding procedure of the differential space–time trellis codes, and (33) to (35) are the requirements for the two paths that diverge from one state and re-converge to another state after L transitions. We have the following proposition for the determinant of the difference matrix between $\mathcal{C}$ and $\tilde{\mathcal{C}}$.

Proposition 3: Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be the two paths for an error event of length L, where $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are defined by (29) and (30), then

$$\det\left((\mathcal{C} - \tilde{\mathcal{C}})^\dagger (\mathcal{C} - \tilde{\mathcal{C}})\right) = \left(\sum_{l=1}^{L-1} |\Phi_l|^2\right)\left(\sum_{l=1}^{L-1} |\Psi_l|^2\right) \tag{36}$$

where Φ_l and Ψ_l are defined as follows:

$$\Phi_l \triangleq \begin{cases} \exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d} q_{u_{2k-1}} + \sum_{k=1}^{d-1} p_{u_{2k}}\right)\right) \\ -\exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d} q_{v_{2k-1}} + \sum_{k=1}^{d-1} p_{v_{2k}}\right)\right), \text{if } l = 2d-1 \\ \exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d} p_{u_{2k-1}} + \sum_{k=1}^{d} q_{u_{2k}}\right)\right) \\ -\exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d} p_{v_{2k-1}} + \sum_{k=1}^{d} q_{v_{2k}}\right)\right), \text{if } l = 2d \end{cases} \tag{37}$$

and

$$\Psi_l \triangleq \begin{cases} \exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d} p_{u_{2k-1}} + \sum_{k=1}^{d-1} q_{u_{2k}}\right)\right) \\ -\exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d} p_{v_{2k-1}} + \sum_{k=1}^{d-1} q_{v_{2k}}\right)\right), \text{if } l = 2d-1 \\ \exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d} q_{u_{2k-1}} + \sum_{k=1}^{d} p_{u_{2k}}\right)\right) \\ -\exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d} q_{v_{2k-1}} + \sum_{k=1}^{d} p_{v_{2k}}\right)\right), \text{if } l = 2d \end{cases} \tag{38}$$

where $1 \le d \le \lfloor L/2 \rfloor$.

Proof: See Appendix II.

From Proposition 3, we can observe that the determinant of difference matrix for any error event does not depend on the initial state. Since the identity matrix $I \in \mathcal{S}$, we only need to examine the error events originated from the state I. This property will greatly simplify the derivation of the diversity product.

Although for a general space–time trellis code, there is no all-zero path that can be exploited for the calculation of diversity product as in a conventional linear binary code, for the trellis code $\mathcal{T}(\mathcal{G}_M)$ defined in the previous subsection, there does exist an all-zero path corresponding to any error event in the sense of equal-determinants of the difference matrices as follows.

Proposition 4: For any error event of two paths $\mathcal{C}$ and $\tilde{\mathcal{C}}$ with length L, where the information symbols carried by $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are:

$$\mathcal{C} \to [G_{u_1}, G_{u_2}, \ldots, G_{u_L}], \tag{39}$$
$$\tilde{\mathcal{C}} \to [G_{v_1}, G_{v_2}, \ldots, G_{v_L}] \tag{40}$$

where $G_{u_i}, G_{v_i} \in \mathcal{G}_M, 1 \le i \le L$, then, there exists an error event of two paths $\mathcal{C}_0$ and $\tilde{\mathcal{C}}_0$ such that

$$\det\left((\mathcal{C}_0 - \tilde{\mathcal{C}}_0)^\dagger(\mathcal{C}_0 - \tilde{\mathcal{C}}_0)\right) = \det\left((\mathcal{C} - \tilde{\mathcal{C}})^\dagger(\mathcal{C} - \tilde{\mathcal{C}})\right) \tag{41}$$

where $\mathcal{C}_0$ and $\tilde{\mathcal{C}}_0$ are given by

$$\mathcal{C}_0 \to [G_0, G_0, \ldots, G_0] \tag{42}$$
$$\begin{aligned}\tilde{\mathcal{C}}_0 \to [&G(p_{u_1} - p_{v_1}, q_{u_1} - q_{v_1}), \\ &G(p_{u_2} - p_{v_2}, q_{u_2} - p_{v_2}) \\ &\ldots, G(p_{u_L} - p_{v_L}, q_{u_L} - p_{v_L})].\end{aligned} \tag{43}$$

Proof: See Appendix III.

From Propositions 3 and 4, we have the following theorem for the diversity product of the trellis code $\mathcal{T}(\mathcal{G}_M)$.

Theorem 5: Let $\mathcal{T}(\mathcal{G}_M)$ be the trellis representation of $\mathcal{G}_M$. Then, the diversity product of the trellis code $\mathcal{T}(\mathcal{G}_M)$ is given by

$$\zeta_T(\mathcal{T}(\mathcal{G}_M)) = \begin{cases} \frac{1}{2}\sqrt{4 - 2\cos(\frac{2\pi}{M}) - 2\cos(\frac{4\pi}{M})}, & \text{if } M \ne 4 \\ \frac{1}{2}\sqrt{4 - 4\cos(\frac{2\pi}{M})} = 1, & \text{if } M = 4. \end{cases} \tag{44}$$

Proof: See Appendix IV.

B. Design for General Number of Antennas, i.e., $t > 2$

The RSTTC for $t = 2$ can be easily generalized to an arbitrary number of transmit antenna system such that it satisfies the criterion i)–ii) and therefore has error event length greater than 2.

Consider the constellation size $M = 2^R$, $R \ge 1$. For any integers $k_1, \ldots, k_t \in \mathbb{Z}_M$, we define a $t \times t$ off-diagonal matrix

$$G(k_1, \ldots, k_t) = \begin{bmatrix} 0 & \cdots & 0 & e^{\mathbf{j}2\pi k_1/M} \\ 0 & \cdots & e^{\mathbf{j}2\pi k_2/M} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ e^{\mathbf{j}2\pi k_t/M} & \cdots & 0 & 0 \end{bmatrix}. \tag{45}$$

Then, we have the following construction for M codewords:

$$\mathcal{G}_M = \{G_m = G(p_m, r_m^1, \ldots, r_m^{t-2}, q_m) | m = 0, 1, \ldots, M-1\} \tag{46}$$

where $r_m^1, \ldots, r_m^{t-2} \in \mathbb{Z}_M$ arbitrarily, and

$$q_m = m, \; m = 0, 1, \ldots, M-1 \tag{47}$$
$$p_m = 2q_m = 2m \bmod M, \; m = 0, 1, \ldots, M-1. \tag{48}$$

For any integers $k_1, \ldots, k_t \in \mathbb{Z}_M$, we define a $t \times t$ diagonal matrix

$$D(k_1, \ldots, k_t) = \begin{bmatrix} e^{\mathbf{j}2\pi k_1/M} & 0 & \cdots & 0 \\ 0 & e^{\mathbf{j}2\pi k_2/M} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\mathbf{j}2\pi k_t/M} \end{bmatrix}. \tag{49}$$

The same as Theorem 4, it can be easily shown that for any G_i, G_j, G_m and $G_n \in \mathcal{G}_M$, if $(i, j) \ne (m, n)$, then $G_iG_j \ne G_mG_n$. Similar to Theorem 3, for the semi-group $\mathcal{S} = \langle G_0, \ldots, G_{M-1}\rangle$ generated by $\{G_0, \ldots, G_{M-1}\}$, we have

$$\mathcal{S} \subseteq \{G(k_1, \ldots, k_t), D(k_1, \ldots, k_t)\} \tag{50}$$

where $k_1, \ldots, k_t \in \mathbb{Z}_M$. Thus, the number of states satisfies $|\mathcal{S}| \le 2M^t$. Note that here we present a family of RSTTC from DSTM for a t transmit antenna system that satisfies the newly proposed criterion i)–ii). In (46), r_m^i, $1 \le i \le t-2, 0 \le m \le M-1$, can be any integers in $\mathbb{Z}_M$. It is interesting to note that the diversity product of this construction is always greater than 0, when M is odd and all $r_{m_1}^i \ne r_{m_2}^i$ for $0 \le m_1 \ne m_2 \le M-1$ for all $1 \le i \le t-2$ in (46). In fact, when M is odd, it is not hard to check that $p_{m_1} \ne p_{m_2}$ for $0 \le m_1 \ne m_2 \le$

TABLE I
DIVERSITY PRODUCT ζ FOR TWO TRANSMIT ANTENNAS

M	ζ	No. of States	codes and comments
2	1	2	parametric code of $(k_1,k_2,k_3)=(1,0,0)$ [15]
2	1	8	RSTTC
4	0.7071	4	cyclic code [11], [12], [13], quaternion code [12], [13]
4	0.7071	8	orthogonal design [6], [10]
4	$\sqrt{2/3}$	Infinite	optimal code [15]
4	1	32	RSTTC
8	0.5946	8	cyclic code [11], [12], [13]
8	0.7071	8	quaternion code [12], [13], parametric code $(k_1,k_2,k_3)=(7,2,0)$ [15]
8	0.8040	128	RSTTC
16	0.3827	16	cyclic code [11], [12], [13], quaternion code [12], [13]
16	0.4295	512	RSTTC
16	0.5000	48	orthogonal design [6], [10]
16	$\sqrt[4]{2}/2$	32	parametric code of $(k_1,k_2,k_3)=(3,4,2)$ [15]
32	0.1951	32	quaternion code [12], [13]
32	0.2183	2048	RSTTC
32	0.2494	32	cyclic code [11], [12], [13]
32	0.3827	64	parametric code of $(k_1,k_2,k_3)=(7,8,2)$ [15]
32	0.4461	Infinite	parametric code of $(k_1,k_2,k_3)=(30,6,0)$ [15]

$M-1$ in (48). Thus, the USTC $\mathcal{G}_M$ in (46) has full diversity, i.e., its diversity product $\zeta_B > 0$. As we have shown before, in this case the diversity product $\zeta_T \geq \zeta_B > 0$. When M is even, one only needs to consider $M+1$ and take M elements from $\mathcal{G}_{M+1}$ to form $\mathcal{G}'_M$ that has nonzero diversity product and thus, its corresponding RSTTC diversity product ζ_T is nonzero too. Note that while this design may provide nonzero diversity product and error event length great than 2, it does not mean that better diversity product can be always achieved. How to optimally determine the values of $r_m^1, \ldots, r_m^{t-2}$ with respect to the diversity product (besides a possible computer searching) is open.

IV. SOME DESIGN EXAMPLES AND SIMULATION RESULTS

Let us first consider the case of $M=4$. From the design in Section III, we have $\mathcal{G}_4 = \{G_0 = G(0,0), G_1 = G(2,1), G_2 = G(0,2), G_3 = G(2,3)\}$. From Theorem 3, the size of state set $\mathcal{S}$ of the trellis $\mathcal{T}(\mathcal{G}_4)$ is 32 and the 32 states are

$$S_i = \begin{cases} D(0,i) & \text{for } 0 \leq i \leq 3 \\ D(1,i-4) & \text{for } 4 \leq i \leq 7 \\ D(2,i-8) & \text{for } 8 \leq i \leq 11 \\ D(3,i-12) & \text{for } 12 \leq i \leq 15 \\ G(0,i-16) & \text{for } 16 \leq i \leq 19 \\ G(1,i-20) & \text{for } 20 \leq i \leq 23 \\ G(2,i-24) & \text{for } 24 \leq i \leq 27 \\ G(3,i-28) & \text{for } 28 \leq i \leq 31. \end{cases} \quad (51)$$

Although the diversity product of the space–time block code/modulation $\mathcal{G}_4$ is 0 by itself, the diversity product of the recursive space–time trellis code $\mathcal{T}(\mathcal{G}_4)$ is 1 from Theorem 5, which is better than the optimal diversity product, $\sqrt{2/3}$, of 2×2 unitary differential space–time block codes of size 4 [15].

In Table I we list the diversity products and the numbers of states for constellation sizes $M = 2, 4, 8, 16$, and 32, respectively. They are also compared with some of the existing results on space–time block codes. Note that the designs shown in Table I are for systems with two transmit antennas. It can be seen from Table I that for $M = 2,\ 4,\ 8$ the RSTTC have equal or better diversity products than those of the known uncoded

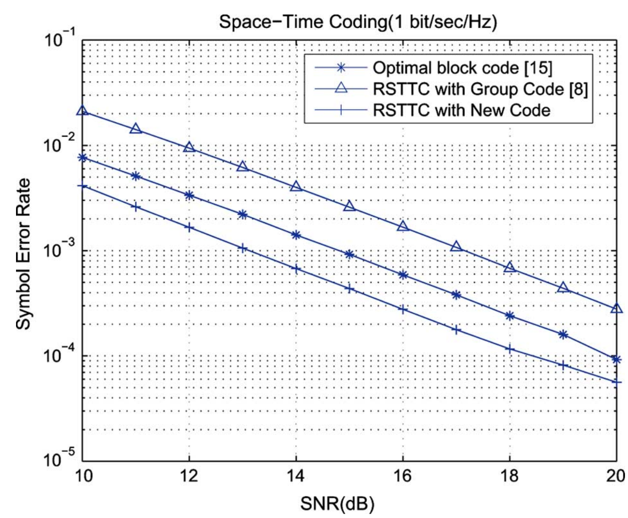

Fig. 3. Simulation results for $M=4$ with two transmit and one receive antennas.

cyclic codes, quaternion codes, orthogonal designs and parametric codes, where, however, the number of states of RSTTC increases with M^2. When M is large, the diversity products of the RSTTC in the design are not as good as other block codes. Other designs with better diversity products are certainly interesting for future investigations.

We next show some simulation results on symbol error rates (SER) vs. SNR. In our simulations, two transmit antennas are used and the channel is assumed fast Rayleigh fading as in [8] and the channel is known at the receiver. Two sets of simulations are presented: one is on RSTTC shown in Figs. 3 and 4 as trellis codes without involving with the turbo principle; the other is on the joint binary error correction coding and RSTTC with the turbo principle, i.e., joint iterative decoding shown in Fig. 5. In Figs. 3 and 4, one receive antenna is used while in Fig. 5, two receive antennas are used in order to be consistent with [8].

In Figs. 3 and 4, the RSTTC with our new unitary space–time block code designs shown in Table I of $M=4$ and 8 are compared with the optimal and some of existing space–time block codes of size $M=4$ and $M=8$ with bandwidth efficiencies

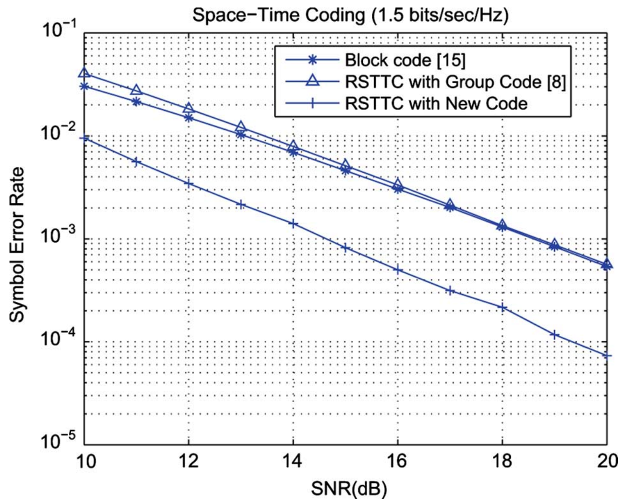

Fig. 4. Simulation results for $M = 8$ with two transmit and one receive antennas.

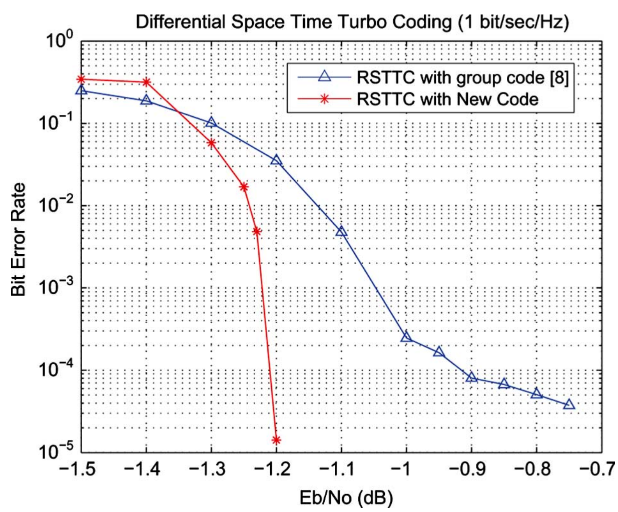

Fig. 5. Simulation results for the serial concatenated systems with two transmit and two receive antennas and 30 iterations.

1 bit/s/Hz and 1.5 bits/s/Hz, respectively. They are also compared with the RSTTC with the group space–time block codes in [12], [13]. From these two figures, one can clearly see the improvement.

In Fig. 5, we compare the RSTTC using our newly proposed unitary space–time block code with the one using the existing group code when the joint turbo encoding and decoding are used as in [8]. Comparing with the result presented in [8], the scheme is the same but only the inner recursive space–time code with the unitary group space–time block code of size 8 is replaced by our newly proposed nongroup code of the same size. The outer binary coder is the $(3, 2)$ parity-check code and the bit interleaver length is 24288. The number of iterations is 30. The number of receive antennas is 2. It is clear to see that at the BER of 10^{-3}, the performance of the new scheme is about 0.2 dB better.

V. Conclusion

In this paper, we proposed a new design criterion and method for unitary space–time block codes used in the recursive space time trellis codes (RSTTC) from differential modulation. With the new design criterion, it is possible to design an RSTTC with different number of states for a fixed bandwidth efficiency. The increase of the number of states is purely due to the unitary space–time block codes and no more memory is needed, which is essentially different from the conventional TCM schemes in single antenna systems. With our newly proposed design criterion, we presented a family of unitary space–time block codes of any size M for any transmit antennas. A closed form diversity product analysis was also presented when the number of transmit antennas is two. For $M = 2, 4$, and 8, the product diversities of the new RSTTC are $1, 1, 0.8040$, while that of the existing optimal and best-known unitary space–time block codes in the literature are 1, $\sqrt{2/3}$, and 0.7071, respectively. The RSTTC with our new unitary space–time block code was compared with the one with the group code presented by Schlegel and Grant in [8] when the turbo principle is used. Finally, we believe that this paper only initiates the study on the design issues of unitary space–time block codes for RSTTC from differential modulation. Many interesting problems remain to be further investigated, such as the problem of the optimal designs of a unitary space–time block code for the RSTTC for a fixed bandwidth efficiency and a fixed number of states. In a recent study [46], unitary space–time codes used in the differential encoding to generate trellis have been generalized and relaxed.

Appendix I
Proof of Theorem 3

Because

$$
\begin{aligned}
D(i_1, j_1)D(i_2, j_2) &= D(i_1 + i_2, j_1 + j_2)\\
G(i_1, j_1)G(i_2, j_2) &= D(i_1 + j_2, j_1 + i_2)\\
D(i_1, j_1)G(i_2, j_2) &= G(i_1 + i_2, j_1 + j_2)\\
G(i_1, j_1)D(i_2, j_2) &= G(i_1 + j_2, j_1 + i_2)\\
D^{-1}(i, j) &= D(M - i, M - j)\\
G^{-1}(i, j) &= G(M - j, M - i)
\end{aligned}
$$

and $I = D(0, 0)$, where i, j, i_1, i_2, j_1, $j_2 \in \mathbb{Z}_M$, the right-hand side of (23) is a group. Hence, $\mathcal{S} \subseteq \{G(i, j), D(i, j) | (i, j) \in \{0, \ldots, M - 1\}\}$. Thus, to prove the theorem, it is enough to show the opposite inclusion. To do so, we want to show that it is enough to prove $D(0, 1) \in \mathcal{S}$. In fact, we have

$$
\begin{aligned}
G(1, 0) &= G(0, 0)D(0, 1)\\
D(1, 0) &= G(1, 0)G(0, 0)\\
G(0, 1) &= G(0, 0)D(1, 0).
\end{aligned}
$$

If $D(0, 1) \in \mathcal{S}$, since $\mathcal{S}$ is a semigroup, we have $G(1, 0), G(0, 1), D(1, 0) \in \mathcal{S}$. Furthermore, for any $0 \le p, q \le M - 1$, we have

$$
\begin{aligned}
D(p, q) &= D^p(1, 0)D^q(0, 1) \in \mathcal{S}\\
G(p, q) &= D^p(1, 0)G(0, 0)D^q(1, 0) \in \mathcal{S}.
\end{aligned}
$$

Therefore

$$\{G(i, j), D(i, j) | i, j \in \{0, 1, \ldots, M - 1\}\} \subseteq \mathcal{S}.$$

To prove $D(0,1) \in \mathcal{S}$, it is enough to consider the following two cases: $3 \mid M+1$ and $3 \mid M+2$ since when $M = 2^R$, $M+1$ or $M+2$ divides 3.

Assume that $3 \mid (M+2)$. Let $q_2 = M + 2/3, q_1 = M - 1/3$. Then

$$D(0,1) = G(2q_1, q_1)G(2q_2, q_2) = G_{q_1}G_{q_2} \in \mathcal{S}.$$

Assume that $3 \mid (M+1)$. Let $q_2 = 2(M+1)/3$, $q_1 = 2M + 1 - 2q_2$. Then

$$D(0,1) = G(2q_1, q_1)G(2q_2, q_2) = G_{q_1}G_{q_2} \in \mathcal{S}.$$

Note G_q is given by (19). Therefore, we have $D(0,1) \in \mathcal{S}$. □

Appendix II
Proof of Proposition 3

Consider an error event of two paths $\mathcal{C}$ and $\tilde{\mathcal{C}}$ with length L and both of them leaving state S, where the information symbols carried by $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are

$$\mathcal{C} \rightarrow [G_{u_1}, G_{u_2}, \ldots, G_{u_L}] \quad (52)$$

$$\tilde{\mathcal{C}} \rightarrow [G_{v_1}, G_{v_2}, \ldots, G_{v_L}] \quad (53)$$

where $G_{u_i}, G_{v_i} \in \mathcal{G}_M, 1 \le i \le L$. From (31) and (32) it is easy to show by induction C_l as in (54) at the bottom of the page, and $\tilde{C}_l$ in (55), also shown at the bottom of the page, where $1 \le d \le \lfloor L/2 \rfloor$. Since $S^\dagger S = I$, we have

$$(C_l - \tilde{C}_l)^\dagger (C_l - \tilde{C}_l) = \begin{bmatrix} |\Phi_l|^2, & 0 \\ 0, & |\Psi_l|^2 \end{bmatrix} \quad (56)$$

where Φ_l and Ψ_l are given by (37) and (38). By summing up all the indices and noting $C_0 = \tilde{C}_0$ and $C_L = \tilde{C}_L$, we have proved Proposition 3. □

Appendix III
Proof of Proposition 4

We first show that $\mathcal{C}_0$ and $\tilde{\mathcal{C}}_0$ are a pair of paths in an error event, and we then show (41).

From Proposition 2, we have $G(p_{u_l} - p_{v_l}, q_{u_l} - q_{v_l}) \in \mathcal{G}_M$, $1 \le l \le L$, which means that $\tilde{\mathcal{C}}_0$ is a valid path, i.e., codeword sequence, on the trellis. Let $\mathcal{C}_0 = [C_0^0\ C_1^0\ C_2^0\ \cdots\ C_L^0]^T$ and $\tilde{\mathcal{C}}_0 = [\tilde{C}_0^0\ \tilde{C}_1^0\ \tilde{C}_2^0 \cdots \tilde{C}_L^0]^T$. To show the first part, we need to show

$$\begin{cases} C_l^0 \ne \tilde{C}_l^0, & \text{if } 1 \le l \le L-1 \\ C_l^0 = \tilde{C}_l^0, & \text{if } l = 0,\ L. \end{cases} \quad (57)$$

From Proposition 3, we only need to consider paths starting from the identity matrix I. Thus, we have $C_0^0 = \tilde{C}_0^0 = I$. Note that $C_l^0 = C_{l-1}^0 G_0$, from (31) and (42), we have

$$C_l^0 = \begin{cases} G_0 = G(0,0), & \text{if } l = 2d-1 \\ I, & \text{if } l = 2d \end{cases} \quad (58)$$

where $1 \le d \le \lfloor L/2 \rfloor$.

If $l = 2d-1$, from (43) and (54), we have

$$\tilde{C}_l^0 = G\left(\sum_{k=1}^{d}(p_{u_{2k-1}} - p_{v_{2k-1}}) + \sum_{k=1}^{d-1}(q_{u_{2k}} - q_{v_{2k}}) \right. \\ \left. \sum_{k=1}^{d}(q_{u_{2k-1}} - q_{v_{2k-1}}) + \sum_{k=1}^{d-1}(p_{u_{2k}} - p_{v_{2k}})\right). \quad (59)$$

Since $C_l \ne \tilde{C}_l$, $1 \le l \le L-1$ from (54) and (55), we have

$$\sum_{k=1}^{d} p_{u_{2k-1}} + \sum_{k=1}^{d-1} q_{u_{2k}} \ne \sum_{k=1}^{d} p_{v_{2k-1}} + \sum_{k=1}^{d-1} q_{v_{2k}} \mod M \quad (60)$$

or

$$\sum_{k=1}^{d} q_{u_{2k-1}} + \sum_{k=1}^{d-1} p_{u_{2k}} \ne \sum_{k=1}^{d} q_{v_{2k-1}} + \sum_{k=1}^{d-1} p_{v_{2k}} \mod M \quad (61)$$

which implies that $\tilde{C}_l^0 \ne C_l^0$.

Similarly we can show $\tilde{C}_l^0 \ne C_l^0$, $l = 2d$ and $\tilde{C}_L^0 = C_L^0$, which completes the proof of the first part.

Let

$$(C_l^0 - \tilde{C}_l^0)^\dagger (C_l^0 - \tilde{C}_l^0) = \begin{bmatrix} |\Phi_l^0|^2, & 0 \\ 0, & |\Psi_l^0|^2 \end{bmatrix} \quad (62)$$

then, we have (63) shown at the top of the following page, and (64), also at the top of the following page, where $1 \le d \le \lfloor L/2 \rfloor$. Comparing (63)–(64) with (37)–(38) we have $|\Phi_l|^2 = |\Phi_l^0|^2$ and $|\Psi_l|^2 = |\Psi_l^0|^2$, which completes the proof of (41). □

$$C_l = \begin{cases} SG\left(\sum_{k=1}^{d} p_{u_{2k-1}} + \sum_{k=1}^{d-1} q_{u_{2k}}, \sum_{k=1}^{d} q_{u_{2k-1}} + \sum_{k=1}^{d-1} p_{u_{2k}}\right), & \text{if } l = 2d-1 \\ SD\left(\sum_{k=1}^{d} p_{u_{2k-1}} + \sum_{k=1}^{d} q_{u_{2k}}, \sum_{k=1}^{d} q_{u_{2k-1}} + \sum_{k=1}^{d} p_{u_{2k}}\right), & \text{if } l = 2d \end{cases} \quad (54)$$

$$\tilde{C}_l = \begin{cases} SG\left(\sum_{k=1}^{d} p_{v_{2k-1}} + \sum_{k=1}^{d-1} q_{v_{2k}}, \sum_{k=1}^{d} q_{v_{2k-1}} + \sum_{k=1}^{d-1} p_{v_{2k}}\right), & \text{if } l = 2d-1 \\ SD\left(\sum_{k=1}^{d} p_{v_{2k-1}} + \sum_{k=1}^{d} q_{v_{2k}}, \sum_{k=1}^{d} q_{v_{2k-1}} + \sum_{k=1}^{d} p_{v_{2k}}\right), & \text{if } l = 2d \end{cases} \quad (55)$$

$$\Phi_l^0 = \begin{cases} 1-\exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d}(q_{u_{2k-1}}-q_{v_{2k-1}})+\sum_{k=1}^{d-1}(p_{u_{2k}}-p_{v_{2k}})\right)\right), & \text{if } l=2d-1 \\ 1-\exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d}(p_{u_{2k-1}}-p_{v_{2k-1}})+\sum_{k=1}^{d}(q_{u_{2k}}-q_{v_{2k}})\right)\right), & \text{if } l=2d \end{cases} \tag{63}$$

$$\Psi_l^0 = \begin{cases} 1-\exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d}(p_{u_{2k-1}}-p_{v_{2k-1}})+\sum_{k=1}^{d-1}(q_{u_{2k}}-q_{v_{2k}})\right)\right), & \text{if } l=2d-1 \\ 1-\exp\left(\frac{\mathbf{j}2\pi}{M}\left(\sum_{k=1}^{d}(q_{u_{2k-1}}-q_{v_{2k-1}})+\sum_{k=1}^{d}(p_{u_{2k}}-p_{v_{2k}})\right)\right), & \text{if } l=2d \end{cases} \tag{64}$$

Appendix IV
Proof of Theorem 5

We prove this theorem in two steps. First, we show (44) for $L=3$, and then we show (44) for $L>3$.

From Propositions 3 and 4, it can be concluded that to determine the diversity product of trellis $\mathcal{T}(\mathcal{G}_M)$ we only need to examine all-zero error events $\mathcal{C}$ and $\tilde{\mathcal{C}}$ with length L. The information sequences carried by $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are

$$\mathcal{C} = [C_0\, C_1\, C_2\, \cdots\, C_L]^T \rightarrow [G_0, G_0, \ldots, G_0] \tag{65}$$
$$\tilde{\mathcal{C}} = [\tilde{C}_0\, \tilde{C}_1\, \tilde{C}_2\, \cdots\, \tilde{C}_L]^T \rightarrow [G_{n_1}, G_{n_2}, \ldots, G_{n_L}] \tag{66}$$

where $0\le n_l \le M-1$, $1\le l\le L$, $C_0=\tilde{C}_0=I$, $C_l\ne\tilde{C}_l$ for $1\le l\le L-1$, and $C_L=\tilde{C}_L$. Comparing (65)–(66) with (42)–(43) and using (63)–(64), we have $\Phi_l = 1-\exp \mathbf{j}2\pi\alpha_l/M$ with

$$\alpha_l = \begin{cases} \sum_{k=1}^{d} q_{n_{2k-1}} + \sum_{k=1}^{d-1} p_{n_{2k}}, & \text{if } l=2d-1 \\ \sum_{k=1}^{d} p_{n_{2k-1}} + \sum_{k=1}^{d} q_{n_{2k}}, & \text{if } l=2d \end{cases} \tag{67}$$

and $\Psi_l = 1-\exp \mathbf{j}2\pi\beta_l/M$ with

$$\beta_l = \begin{cases} \sum_{k=1}^{d} p_{n_{2k-1}} + \sum_{k=1}^{d-1} q_{n_{2k}}, & \text{if } l=2d-1 \\ \sum_{k=1}^{d} q_{n_{2k-1}} + \sum_{k=1}^{d} p_{n_{2k}}, & \text{if } l=2d \end{cases} \tag{68}$$

and

$$\det\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right) = \left(\sum_{l=1}^{L-1}|1-\exp\frac{\mathbf{j}2\pi\alpha_l}{M}|^2\right)\left(\sum_{l=1}^{L-1}|1-\exp\frac{\mathbf{j}2\pi\beta_l}{M}|^2\right) \tag{69}$$
$$= \left(\sum_{l=1}^{L-1}(2-2\cos\frac{2\pi\alpha_l}{M})\right)\left(\sum_{l=1}^{L-1}(2-2\cos\frac{2\pi\beta_l}{M})\right). \tag{70}$$

Also we have the following lemmas.

Lemma 1: For the pair α_l and β_l defined in (67) and (68), $1\le l\le L$, we have

$$\alpha_{l-1}+p_{n_l}=\beta_l \quad \text{and} \quad \beta_{l-1}+q_{n_l}=\alpha_l. \tag{71}$$

Proof: We just give the proof for $\alpha_{l-1}+p_{n_l}=\beta_l$ when L is even and the others are similar. When $l=2d$, we have

$$\alpha_{l-1}+p_{n_l} = \sum_{k=1}^{d} q_{n_{2k-1}} + \sum_{k=1}^{d-1} p_{n_{2k}} + p_{n_l}$$
$$= \sum_{k=1}^{d} q_{n_{2k-1}} + \sum_{k=1}^{d} p_{n_{2k}}$$
$$= \beta_l.$$ □

Lemma 2: For the pair α_{L-1} and β_{L-1} defined in (67) and (68), we have

$$\alpha_{L-1} = M - p_{n_L} \bmod M$$

and

$$\beta_{L-1} = M - q_{n_L} \bmod M. \tag{72}$$

Proof: Comparing (66) with (53) and using (55) and $C_0=\tilde{C}_0=I$, we have

$$\tilde{C}_L = \begin{cases} G(\sum_{k=1}^{(L+1)/2} p_{n_{2k-1}} + \sum_{k=1}^{(L-1)/2} q_{n_{2k}}, \\ \quad \sum_{k=1}^{(L+1)/2} q_{n_{2k-1}} + \sum_{k=1}^{(L-1)/2} p_{n_{2k}}), & \text{if } L \text{ is odd} \\ D(\sum_{k=1}^{L/2} p_{n_{2k-1}} + \sum_{k=1}^{L/2} q_{n_{2k}}, \\ \quad \sum_{k=1}^{L/2} q_{n_{2k-1}} + \sum_{k=1}^{L/2} p_{n_{2k}}), & \text{if } L \text{ is even.} \end{cases} \tag{73}$$

Since

$$\tilde{C}_L = C_L = \begin{cases} G(0,0), & \text{if } L \text{ is odd} \\ D(0,0), & \text{if } L \text{ is even} \end{cases}$$

we have

$$\begin{cases} \sum_{k=1}^{(L+1)/2} p_{n_{2k-1}} + \sum_{k=1}^{(L-1)/2} q_{n_{2k}} = 0 \bmod M \\ \sum_{k=1}^{(L+1)/2} q_{n_{2k-1}} + \sum_{k=1}^{(L-1)/2} p_{n_{2k}} = 0 \bmod M \end{cases}$$

if L is odd and

$$\begin{cases} \sum_{k=1}^{L/2} p_{n_{2k-1}} + \sum_{k=1}^{L/2} q_{n_{2k}} = 0 \bmod M \\ \sum_{k=1}^{L/2} q_{n_{2k-1}} + \sum_{k=1}^{L/2} p_{n_{2k}} = 0 \bmod M \end{cases}$$

if L is even. Comparing with (67) and (68) we have $\alpha_L = 0 \bmod M$ and $\beta_L = 0 \bmod M$. From Lemma 1, we have

$$\alpha_{L-1}+p_{n_L}=\beta_L=0 \bmod M$$

and

$$\beta_{L-1}+q_{n_L}=\alpha_L=0 \bmod M$$

which completes the proof. □

From Lemma 2, we have

$$\begin{cases} |\Phi_{L-1}|^2=\left|1-\exp\left(\frac{\mathbf{j}2\pi\alpha_{L-1}}{M}\right)\right|^2=\left|1-\exp\left(\frac{\mathbf{j}2\pi p_{n_L}}{M}\right)\right|^2 \\ |\Psi_{L-1}|^2=\left|1-\exp\left(\frac{\mathbf{j}2\pi\beta_{L-1}}{M}\right)\right|^2=\left|1-\exp\left(\frac{\mathbf{j}2\pi q_{n_L}}{M}\right)\right|^2. \end{cases} \tag{74}$$

Thus, we can rewrite (70) as

$$\begin{aligned}\det&\left(\left(\mathcal{C}-\tilde{\mathcal{C}}\right)^\dagger\left(\mathcal{C}-\tilde{\mathcal{C}}\right)\right)\\ &=\left(\left(2-2\cos\frac{2\pi q_{n_1}}{M}\right)+\sum_{l=2}^{L-2}\left(2-2\cos\frac{2\pi\alpha_l}{M}\right)\right.\\ &\qquad\left.+\left(2-2\cos\frac{2\pi p_{n_L}}{M}\right)\right)\\ &\quad\cdot\left(\left(2-2\cos\frac{2\pi p_{n_1}}{M}\right)+\sum_{l=2}^{L-2}\left(2-2\cos\frac{2\pi\beta_l}{M}\right)\right.\\ &\qquad\left.+\left(2-2\cos\frac{2\pi q_{n_L}}{M}\right)\right).\end{aligned} \tag{75}$$

Lemma 3: For any pair α_l and β_l, $1\le l<L$, defined in (67)–(68), if $\alpha_l=0 \bmod M$, then $\beta_l\neq 0 \bmod M$; if $\beta_l=0 \bmod M$ then $\alpha_l\neq 0 \bmod M$.

Proof: From (54) and (55), we can easily check that if $\alpha_l=0$ and $\beta_l=0 \bmod M$, then $C_l=\tilde{C}_l$, which contradicts with $C_l\neq\tilde{C}_l$ for $1\le l\le L-1$. □

Lemma 4: For α_1 and β_{L-1} defined in (67)–(68), we have $\alpha_1\neq 0$ and $\beta_{L-1}\neq 0$.

Proof: If $\alpha_1=q_{n_1}=0$, then $p_{n_1}=0$, which contradicts with $C_1\neq\tilde{C}_1$. Similarly, if $\beta_{L-1}=M-q_{n_L}=0$, then $q_{n_L}=0$ and $p_{n_L}=0$, which contradicts with $C_{L-1}\neq\tilde{C}_{L-1}$. □

Lemma 5: For an error event of two paths $\mathcal{C}$ and $\tilde{\mathcal{C}}$ with length L, Let

$$\Lambda_L=\det\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right).$$

Then

$$\min\Lambda_3=\begin{cases}(4-2\cos\frac{2\pi}{M}-2\cos\frac{4\pi}{M})^2, & \text{if } M\neq 4\\ (4-4\cos\frac{2\pi}{M})^2, & \text{if } M=4.\end{cases} \tag{76}$$

Proof: For any integer $k\neq 0$ and $M=2^R\ge 8$, we will use the following inequalities frequently:

$$\begin{aligned}&\left|1-\exp\frac{\mathbf{j}2\pi k}{M}\right|^2\ge\left|1-\exp\frac{\mathbf{j}2\pi 2}{M}\right|^2\\ &\text{i.e., } 2-2\cos\frac{2\pi k}{M}\ge 2-2\cos\frac{4\pi}{M},\quad \text{if } k \text{ is even}\end{aligned} \tag{77}$$

$$\begin{aligned}&\left|1-\exp\frac{\mathbf{j}2\pi k}{M}\right|^2\ge\left|1-\exp\frac{\mathbf{j}2\pi}{M}\right|^2\\ &\text{i.e., } 2-2\cos\frac{2\pi k}{M}\ge 2-2\cos\frac{2\pi}{M},\quad \text{if } k \text{ is odd.}\end{aligned} \tag{78}$$

When $L=3$, from (75) we have

$$\Lambda_3=\det\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right) \tag{79}$$

$$\begin{aligned}&=\left(4-2\cos\frac{2\pi q_{n_1}}{M}-2\cos\frac{2\pi p_{n_3}}{M}\right)\\ &\quad\cdot\left(4-2\cos\frac{2\pi p_{n_1}}{M}-2\cos\frac{2\pi q_{n_3}}{M}\right).\end{aligned} \tag{80}$$

When $M=2$, (p,q) is chosen from $\{(0,0),(0,1)\}$. Because $q_{n_1}\neq 0$, then $q_{n_1}=1$. By $q_{n_1}+p_{n_2}+q_{n_3}=0 \bmod M$, we have $q_{n_3}=1$. Thus

$$\begin{aligned}\Lambda_3&=\left(4-2\cos\frac{2\pi q_{n_1}}{M}-2\cos\frac{2\pi p_{n_3}}{M}\right)\\ &\quad\cdot\left(4-2\cos\frac{2\pi p_{n_1}}{M}-2\cos\frac{2\pi q_{n_3}}{M}\right)\\ &=(4-2\cos\pi-2)(4-2-2\cos\pi)\\ &=4^2=\min\Lambda_3 \text{ in (76).}\end{aligned}$$

When $M=4$, (p,q) is chosen from $\{(0,0),(2,1),(0,2),(2,3)\}$. Because $q_{n_1}\neq 0$, $q_{n_1}=1,2,$ or 3.

1) If $q_{n_1}=1$, by $q_{n_1}+p_{n_2}+q_{n_3}=0 \bmod M$ and p_{n_2} is 0 or 2, then q_{n_3} is odd. So

$$\begin{aligned}\Lambda_3&=\left(4-2\cos\frac{2\pi q_{n_1}}{M}-2\cos\frac{2\pi p_{n_3}}{M}\right)\\ &\quad\cdot\left(4-2\cos\frac{2\pi p_{n_1}}{M}-2\cos\frac{2\pi q_{n_3}}{M}\right)\\ &\ge(4-2\cos\frac{\pi}{2}-2\cos(q_{n_3}\pi))\\ &\quad\cdot(4-2\cos\pi-2\cos\frac{q_{n_3}\pi}{2})\\ &\ge 36>\min\Lambda_3 \text{ in (76).}\end{aligned}$$

2) If $q_{n_1}=2$, then $p_{n_1}=0$, again by $q_{n_1}+p_{n_2}+q_{n_3}=0 \bmod M$ we have $q_{n_3}=0$ or 2. If $q_{n_3}=0$, by $p_{n_1}+q_{n_2}+p_{n_3}=0 \bmod M$, we get $q_{n_2}=0$, then $p_{n_2}=0$, which contradicts with $q_{n_1}+p_{n_2}+q_{n_3}=0 \bmod M$. Therefore $q_{n_3}=2$ and

$$\begin{aligned}\Lambda_3&=(4-2\cos\pi-2)(4-2-2\cos\pi)\\ &=\min\Lambda_3 \text{ in (76).}\end{aligned}$$

3) If $q_{n_1}=3$, it is the same as the case $q_{n_1}=1$.

When $M\ge 8$, by (77) and (78), we have

$$\begin{aligned}\Lambda_3&=\left(4-2\cos\frac{2\pi q_{n_1}}{M}-2\cos\frac{2\pi p_{n_3}}{M}\right)\\ &\quad\cdot\left(4-2\cos\frac{2\pi p_{n_1}}{M}-2\cos\frac{2\pi q_{n_3}}{M}\right)\\ &\ge(4-2\cos\frac{2\pi}{M}-2\cos\frac{4\pi}{M})^2=\min\Lambda_3 \text{ in (76).}\end{aligned}$$

Lemma 5 presents the minimum determinant of error events with length 3. Next we show that $\min\Lambda_3$ in (76) is also the minimum value for all the error events with length $L>3$, i.e., $\min\Lambda_3\le\min\Lambda_L$ for $L>3$. We first show the following inequalities.

Lemma 6: For $M=2^R$ with $R\geq 3$, we have the following:

1) $(6-2\cos 4\pi/M)(2-2\cos 4\pi/M)\geq(4-2\cos 2\pi/M-2\cos 4\pi/M)^2$;
2) $(2-2\cos 6\pi/M)(8-2\cos 2\pi/M-2\cos 4\pi/M)\geq(4-2\cos 2\pi/M-2\cos 4\pi/M)^2$;
3) $(4-4\cos 2\pi/M)(6-2\cos 4\pi/M)\geq(4-2\cos 2\pi/M-2\cos 4\pi/M)^2$;
4) $(2-2\cos 2\pi/M)(10-2\cos 4\pi/M-2\cos 6\pi/M)\geq(4-2\cos 2\pi/M-2\cos 4\pi/M)^2$.

Proof: We just give the proof for the first inequality and the others can be shown similarly as given in the equation at the bottom of the page. □

From Lemma 4, we have $q_{n_1}=\alpha_1\neq 0 \bmod M$ and $q_{n_L}=M-\beta_{L-1}\neq 0 \bmod M$. Then, to prove Theorem 5, we have the following four cases.

Case 1) q_{n_1} and q_{n_L} are both odd.

In this case, from (75), we have

$$
\begin{aligned}
&\det\left((\mathcal{C}-\tilde{\mathcal{C}})^{\dagger}(\mathcal{C}-\tilde{\mathcal{C}})\right)\\
&\geq\left((2-2\cos\frac{2\pi q_{n_1}}{M})+(2-2\cos\frac{2\pi p_{n_L}}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi p_{n_1}}{M})+(2-2\cos\frac{2\pi q_{n_L}}{M})\right)\\
&\geq\left((2-2\cos\frac{2\pi}{M})+(2-2\cos\frac{2\pi 2}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi 2}{M})+(2-2\cos\frac{2\pi}{M})\right)\\
&\geq\min\Lambda_3 \text{ in (76).}
\end{aligned}
$$

Case 2) q_{n_1} and q_{n_L} are both even.

In this case, there are four subcases as follows.

a) $q_{n_1}=M/2$ and $q_{n_L}=M/2$

Then, (75) is

$$
\begin{aligned}
&\det\left((\mathcal{C}-\tilde{\mathcal{C}})^{\dagger}(\mathcal{C}-\tilde{\mathcal{C}})\right)\\
&\geq\left((2-2\cos\frac{2\pi q_{n_1}}{M})+(2-2\cos\frac{2\pi p_{n_L}}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi p_{n_1}}{M})+(2-2\cos\frac{2\pi q_{n_L}}{M})\right)\\
&=(2-2\cos\frac{2\pi\frac{M}{2}}{M})^2=16\\
&\geq\min\Lambda_3 \text{ in (76).}
\end{aligned}
$$

b) $q_{n_1}=M/2$ and $q_{n_L}\neq M/2$

Since by the assumptions q_{n_L} is even and $q_{n_L}\neq 0$, this case is not possible for $M=2$ or $M=4$. Thus, we only need to consider $M\geq 8$

$$
\begin{aligned}
&\det\left((\mathcal{C}-\tilde{\mathcal{C}})^{\dagger}(\mathcal{C}-\tilde{\mathcal{C}})\right)\\
&\geq\left((2-2\cos\frac{2\pi q_{n_1}}{M})+(2-2\cos\frac{2\pi p_{n_L}}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi p_{n_1}}{M})+(2-2\cos\frac{2\pi q_{n_L}}{M})\right)\\
&\geq\left((2-2\cos\frac{2\pi\frac{M}{2}}{M})+(2-2\cos\frac{2\pi 2}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi 2}{M})\right)\\
&\geq\min\Lambda_3 \text{ in (76).}
\end{aligned}
$$

where the last inequality is from Lemma 6.

c) $q_{n_1}\neq M/2$ and $q_{n_L}=M/2$

This case is the same as b).

d) $q_{n_1}\neq M/2$ and $q_{n_L}\neq M/2$

Similar to a), we only need to consider $M\geq 8$. Thus, from (77)–(78)

$$
\begin{aligned}
&\det\left((\mathcal{C}-\tilde{\mathcal{C}})^{\dagger}(\mathcal{C}-\tilde{\mathcal{C}})\right)\\
&\geq\left((2-2\cos\frac{2\pi q_{n_1}}{M})+(2-2\cos\frac{2\pi p_{n_L}}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi p_{n_1}}{M})+(2-2\cos\frac{2\pi q_{n_L}}{M})\right)\\
&\geq\left((2-2\cos\frac{2\pi 2}{M})+(2-2\cos\frac{2\pi 2}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi 2}{M})+(2-2\cos\frac{2\pi 2}{M})\right)\\
&\geq\min\Lambda_3 \text{ in (76).}
\end{aligned}
$$

Case 3) q_{n_1} is odd and q_{n_L} is even.

This case has the following two subcases.

a) $q_{n_L}\neq M/2$

Similar to b) in Case 2), we only need to consider $M\geq 8$. Thus, from (77)–(78)

$$
\begin{aligned}
&\det\left((\mathcal{C}-\tilde{\mathcal{C}})^{\dagger}(\mathcal{C}-\tilde{\mathcal{C}})\right)\\
&\geq\left((2-2\cos\frac{2\pi q_{n_1}}{M})+(2-2\cos\frac{2\pi p_{n_L}}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi p_{n_1}}{M})+(2-2\cos\frac{2\pi q_{n_L}}{M})\right)\\
&\geq\left((2-2\cos\frac{2\pi}{M})+(2-2\cos\frac{2\pi 2}{M})\right)\\
&\quad\cdot\left((2-2\cos\frac{2\pi 2}{M})+(2-2\cos\frac{2\pi 2}{M})\right)\\
&\geq\min\Lambda_3 \text{ in (76).}
\end{aligned}
$$

$$
\left(6-2\cos\frac{4\pi}{M}\right)\left(2-2\cos\frac{4\pi}{M}\right)-\left(4-2\cos\frac{2\pi}{M}-2\cos\frac{4\pi}{M}\right)^2=4\left(1-\cos\frac{2\pi}{M}\right)\left(\left(1+\cos\frac{2\pi}{M}\right)\left(4\cos\frac{2\pi}{M}-1\right)+2\cos\frac{2\pi}{M}\right)
$$
$$
\geq 0,\quad \text{for } M\geq 8. \qquad \square
$$

b) $q_{n_L} = M/2$
When $M = 2$, we have $q_{n_1} = 1$, $q_{n_L} = M/2 = 1$, then $p_{n_1} = 0$ and $p_{n_L} = 0$, it is easy to see

$\det\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right) \geq \min\Lambda_3$ in (76).

When $M = 4$, since q_{n_1} is odd, we have $p_{n_1} = 2$.

i) If $\alpha_2 = p_{n_1} + q_{n_2} = 0 \bmod M$ then $q_{n_2} = 2$ and $q_{n_1} + p_{n_2} = 1$ or 3. Thus, from (77)–(78),

$$\begin{aligned}\det&\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right)\\ &\geq (2-2\cos\frac{2\pi q_{n_1}}{M})\Bigg((2-2\cos\frac{2\pi p_{n_1}}{M})\\ &\quad + (2-2\cos\frac{2\pi(q_{n_1}+p_{n_2})}{M}) + (2-2\cos\frac{2\pi q_{n_L}}{M})\Bigg)\\ &\geq (2-2\cos\frac{2\pi}{M})\Bigg((2-2\cos\frac{2\pi 2}{M}) + (2-2\cos\frac{2\pi}{M})\\ &\quad +(2-2\cos\frac{2\pi 2}{M})\Bigg)\\ &= 20 \geq \min\Lambda_3 \text{ in (76).}\end{aligned}$$

ii) If $\alpha_2 = p_{n_1}+q_{n_2} \neq 0 \bmod M$, from (77)–(78) we have

$$\begin{aligned}\det&\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right)\\ &\geq \Bigg((2-2\cos\frac{2\pi q_{n_1}}{M}) + (2-2\cos\frac{2\pi(p_{n_1}+q_{n_2})}{M})\Bigg)\\ &\quad\cdot\Bigg((2-2\cos\frac{2\pi p_{n_1}}{M}) + (2-2\cos\frac{2\pi q_{n_L}}{M})\Bigg)\\ &\geq \Bigg((2-2\cos\frac{2\pi}{M}) + (2-2\cos\frac{2\pi}{M})\Bigg)\\ &\quad\cdot\Bigg((2-2\cos\frac{2\pi 2}{M}) + (2-2\cos\frac{2\pi 2}{M})\Bigg)\\ &= 32 \geq \min\Lambda_3 \text{ in (76).}\end{aligned}$$

When $M \geq 8$, since q_{n_1} is odd, there are two subcases:

i) If $q_{n_1} \neq 1, M-1$
Then, $3 \leq q_{n_1} < M-1$. If $\alpha_2 = p_{n_1} + q_{n_2} = 0 \bmod M$, then $\beta_2 = q_{n_1} + p_{n_2} \neq 0$ from Lemma 3. Thus,

$$\begin{aligned}\det&\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right)\\ &\geq (2-2\cos\frac{2\pi q_{n_1}}{M})\Bigg((2-2\cos\frac{2\pi p_{n_1}}{M})\\ &\quad + (2-2\cos\frac{2\pi(q_{n_1}+p_{n_2})}{M}) + (2-2\cos\frac{2\pi q_{n_L}}{M})\Bigg)\\ &\geq (2-2\cos\frac{2\pi 3}{M})\Bigg((2-2\cos\frac{2\pi 2}{M}) + (2-2\cos\frac{2\pi}{M})\\ &\quad + (2-2\cos\frac{2\pi\frac{M}{2}}{M})\Bigg)\\ &\geq \min\Lambda_3 \text{ in (76)}\end{aligned}$$

where the last inequality is from Lemma 6.

If $\alpha_2 = p_{n_1}+q_{n_2} \neq 0 \bmod M$, from (77)–(78) we have

$$\begin{aligned}\det&\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right)\\ &\geq \Bigg((2-2\cos\frac{2\pi q_{n_1}}{M}) + (2-2\cos\frac{2\pi(p_{n_1}+q_{n_2})}{M})\Bigg)\\ &\quad\cdot\Bigg((2-2\cos\frac{2\pi p_{n_1}}{M}) + (2-2\cos\frac{2\pi q_{n_L}}{M})\Bigg)\\ &\geq \Bigg((2-2\cos\frac{2\pi 3}{M}) + (2-2\cos\frac{2\pi}{M})\Bigg)\\ &\quad\cdot\Bigg((2-2\cos\frac{2\pi 2}{M}) + (2-2\cos\frac{2\pi\frac{M}{2}}{M})\Bigg)\\ &\geq \min\Lambda_3 \text{ in (76).}\end{aligned}$$

ii) If $q_{n_1} = 1$ or $M-1$
In this case, we have $p_{n_1} = 2q_{n_1} = 2$ or $M - 2 \bmod M$ and $p_{n_L} = 2q_{n_L} = 2M/2 = M = 0 \bmod M$. There are two subcases as follows.
If there exists $\alpha_l \neq 0 \bmod M$ for some l with $2 \leq l \leq L-2$, $L \geq 4$, then

$$\begin{aligned}\det&\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right)\\ &\geq \Bigg((2-2\cos\frac{2\pi q_{n_1}}{M}) + (2-2\cos\frac{2\pi\alpha_l}{M})\Bigg)\\ &\quad\cdot\Bigg((2-2\cos\frac{2\pi p_{n_1}}{M}) + (2-2\cos\frac{2\pi q_{n_L}}{M})\Bigg)\\ &\geq \Bigg((2-2\cos\frac{2\pi}{M}) + (2-2\cos\frac{2\pi}{M})\Bigg)\\ &\quad\cdot\Bigg((2-2\cos\frac{2\pi 2}{M}) + (2-2\cos\frac{2\pi\frac{M}{2}}{M})\Bigg)\\ &\geq \min\Lambda_3 \text{ in (76)}\end{aligned}$$

where the last inequality is from Lemma 6.
If $\alpha_l = 0 \bmod M$ for $2 \leq l \leq L-2$, $L \geq 4$, from Lemma 3 we have $\beta_l \neq 0 \bmod M$ for $2 \leq l \leq L-2$. We first show that in this case $L \neq 4$. If $L = 4$, from $\alpha_2 = p_{n_1} + q_{n_2} = 0 \bmod M$, $\alpha_3 = q_{n_1}+p_{n_2}+q_{n_3} = 0 \bmod M$, and $\beta_3 = p_{n_1}+q_{n_2}+p_{n_3} = M/2 \bmod M$, we have $q_{n_3} = M/4$ and $q_{n_1} + p_{n_2}$ is even, which contradicts with that q_{n_1} is odd.
When $L > 4$, from $q_{n_1} = 1$ or $M-1$, $\alpha_2 = p_{n_1} + q_{n_2} = 0 \bmod M$, and $p_{n_1} = 2$ or $M-2$, we have $\beta_2 = q_{n_1} + p_{n_2} = M-3$ or $M+3 \bmod M$. From Lemmas 1 and 2, we have $\alpha_{L-2} + p_{n_{L-1}} = \beta_{L-1} = M - q_{n_L} = M/2 \bmod M$ and $\beta_{L-2}+q_{n_{L-1}} = \alpha_{L_1} = M - p_{n_L} = 0 \bmod M$, then $\beta_{L-2} = 3M/4 \bmod M$. Thus

$$\begin{aligned}\det&\left((\mathcal{C}-\tilde{\mathcal{C}})^\dagger(\mathcal{C}-\tilde{\mathcal{C}})\right)\\ &\geq (2-2\cos\frac{2\pi q_{n_1}}{M})\Bigg((2-2\cos\frac{2\pi p_{n_1}}{M})\end{aligned}$$

$$
\begin{aligned}
&+(2-2\cos\frac{2\pi\beta_2}{M})+(2-2\cos\frac{2\pi\beta_{L-2}}{M})\\
&+(2-2\cos\frac{2\pi q_{n_L}}{M})\Big)\\
&=(2-2\cos\frac{2\pi}{M})\Big((2-2\cos\frac{2\pi 2}{M})+(2-2\cos\frac{2\pi 3}{M})\\
&+(2-2\cos\frac{2\pi\frac{3M}{4}}{M})+(2-2\cos\frac{2\pi\frac{M}{2}}{M})\Big)\\
&\geq \min\Lambda_3 \text{ in } (76)
\end{aligned}
$$

where the last inequality is from Lemma 6.

Case 4) q_{n_1} is even and q_{n_L} is odd.

This case is the same as Case 3). □

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