On BT-limited Signals

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Abstract: —In this paper, we introduce and characterize a subspace of bandlimited signals. The subspace consists of all bandlimited signals such that the non-zero parts of their Fourier transforms are pieces of some T bandlimited signals. The signals in the subspace are called BT-limited signals and the subspace is named as BT-limited signal space. For BT-limited signals, a signal extrapolation with an analytic error estimate exists outside the interval [−T, T] of given signal values with errors. Some new properties about and applying BT-limited signals are also presented.

Keywords:—Bandlimited signals, BT-limited signals, prolate spheroidal wavefunctions, signal extrapolation

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1. Introduction

Bandlimited signals have played a fundamental role in the digital world in the last decades. The Whittaker-Shannon sampling theorem is the fundamental bridge between analog and digital signal processings/communications, which has brought significant interest in both signal processing and mathematics communities. The sampling theorem is about the reconstruction of a bandlimited signal from its evenly spaced samples and an exact reconstruction is possible if the samples are sampled from the analog signal with a sampling rate not lower than the Nyquist rate.

Another family of bandlimited signal reconstructions is to reconstuct a bandlimited signal from its given segment. It is called bandlimited signal extrapolation and has applications in, for example, CT imaging, where only limited observation angles are available. Bandlimited signal extrapolation has also attracted significant interest in the past, see, for example, [2]-[9].

In this paper, we introduce a subspace of bandlimited signals, which is called *BT-limited* signal space. It consists of all bandlimited signals such that the non-zero parts of their Fourier transforms are pieces of bandlimited signals, which are called BT-limited signals. Note that for a general bandlimited signal, although its Fourier transform has finite support, the non-zero spectrum may not be smooth, while the non-zero spectrum is smooth for a BT-limited signal. It was found in [9] that BT-limited signals can be characterized by using prolate spheroidal wavefunctions [1]. In this paper, a more intuitive and elementary proof for the characterization is given, which may help to better understand BT-limited signals. Some new properties about and applying BT-limited signals are also presented. Interestingly, although there is no any error estimate existed for a general bandlimited signal extrapolation from inaccurate data, an analytic error estimate in the whole time domain was obtained in [9] for a BT-limited signal extrapolation.

The remainder of this paper is organized as follows. In Section II, prolate spheroidal wavefunctions are briefly introduced. In Section III, BT-limited signals are introduced and characterized. In Section IV, a BT-limited signal extrapolation with analytic error estimate is described. In Section V, some simulations are presented to verify the theoretical extrapolation result for BT-limited signals. In Section VI, more properties on BT-limited signals are presented. In Section VII, this paper is concluded.

2. Bandlimited Signal Space

All signals considered in this paper are assumed to have finite energies. A signal $f(t)$ is called bandlimited of bandwidth Ω (or Ω bandlimited), if its Fourier transform $\hat{f}(\omega)$ vanishes when $|\omega| > \Omega$. Let $\mathcal{BL}_\Omega$ denote the space of all Ω bandlimited signals. Let $T > 0$ be a constant and K be the following operator defined on $L^2[-T, T]$:

$$(Kf)(t) = \int_{-T}^{T} \frac{\sin \Omega(t-s)}{\pi(t-s)} f(s)ds, \quad \text{for } f \in L^2[-T, T]. \tag{1}$$

Let ϕ_k and λ_k, $k = 0, 1, 2, ...$, be the eigenfunctions and the corresponding eigenvalues of the operator K with

$$\int_{-T}^{T} \phi_j(t)\phi_k(t)dt = \lambda_k \delta(j-k), \tag{2}$$

where $\delta(n)$ is 1 when $n = 0$ and 0 otherwise, and $1 > \lambda_0 > \lambda_1 > \cdots > 0$ with $\lambda_k \to 0$ as $k \to \infty$.

From (1), for $k = 0, 1, 2, ...,$

$$\phi_k(t) = \frac{1}{\lambda_k} \int_{-T}^{T} \frac{\sin \Omega(t-\tau)}{\pi(t-\tau)} \phi_k(\tau)d\tau, \quad \text{for } t \in [-T, T], \tag{3}$$

which means that $\phi_k(t)$ can be extended from $t \in [-T, T]$ to $t \in (-\infty, \infty)$. Then,

$$\int_{-\infty}^{\infty} \phi_j(t)\phi_k(t)dt = \delta(j-k)$$

and $\{\phi_k(t)\}_{k=0}^{\infty}$ form an orthonormal basis for space $\mathcal{BL}_\Omega$ and every Ω bandlimited signal f can be expanded as

$$f(t) = \sum_{k=0}^{\infty} a_k \phi_k(t) \tag{4}$$

for some constants a_k with

$$\sum_{k=0}^{\infty} |a_k|^2 = \|f\|^2 < \infty.$$

The extended eigenfunctions ϕ_k are called *prolate spheroidal wavefunctions* [1].

3. BT-limited Signal Space

We next define a subspace of Ω bandlimited signals. An Ω bandlimited signal is called *BT-limited* if the non-zero part of its Fourier transform is a piece of a T bandlimited signal. In other words, let $f \in \mathcal{BL}_\Omega$ and its Fourier transform be $\hat{f}$. If there exists $g \in \mathcal{BL}_T$ and $\hat{f}(\omega) = g(\omega)$ for $\omega \in [-\Omega, \Omega]$, then f is called BT-limited. The subspace of all BT-limited signals in $\mathcal{BL}_\Omega$ is denoted as $\mathcal{BL}_\Omega^0$ and called *BT-limited signal space*.

From the above definition, by taking Fourier transform and inverse Fourier transfrom, it is not hard to see that $f(t) \in \mathcal{BL}_\Omega^0$ if and only if there exists $q(t) \in L^2[-T, T]$ such that

$$f(t) = \int_{-T}^{T} \frac{\sin \Omega(t-s)}{\pi(t-s)} q(s) ds, \quad \text{for } t \in (-\infty, \infty). \tag{5}$$

Thus, from (3), we know that every prolate spheroidal wavefunction ϕ_k is BT-limited, $\phi_k \in \mathcal{BL}_\Omega^0$, so is any linear combination of finite many prolate spheroidal wavefunctions.

For a general Ω bandlimited signal, although its Fourier transform has finite suppport, the non-zero part of the Fourier transform is only in $L^2[-\Omega, \Omega]$ and may not be smooth. However, a BT-limited signal is not only smooth (entire function of exponential type [10]) in time domain but also has the same smoothness for the non-zero part in frequency domain. To characterize $\mathcal{BL}_\Omega^0$, the following result was obtained in [9].

Theorem 1: Let $f \in \mathcal{BL}_\Omega$ with the expansion (4). Then, $f \in \mathcal{BL}_\Omega^0$ if and only if

$$\sum_{k=0}^{\infty} \frac{|a_k|^2}{\lambda_k} < \infty. \tag{6}$$

The proof given in [9] is based on a result on operator theory. Below, we provide an elementary and intuitive proof of Theorem 1, which may help to understand BT-limited signals better.

Proof:

We first prove the "if" part. Let $f(t)$ be an Ω bandlimited signal with the expansion (4) and the property (6) hold. Let

$$q(t) = \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k} \phi_k(t). \tag{7}$$

From (2) and (6), we have

$$\int_{-T}^{T} |q(t)|^2 dt = \sum_{k=0}^{\infty} \frac{|a|_k^2}{\lambda_k} < \infty.$$

Thus, $q(t) \in L^2[-T, T]$. Furthermore, from (1), we have

$$(Kq)(t) = \sum_{k=0}^{\infty} a_k \phi_k(t) = f(t), \quad \text{for } t \in (-\infty, \infty),$$

which is (5) and therefore, $f(t) \in \mathcal{BL}_\Omega^0$. This proves the sufficiency.

We next prove the "only if" part. If $f(t) \in \mathcal{BL}_\Omega^0$, then $f(t)$ has the form (5) for some $q(t) \in L^2[-T, T]$. In the meantime, since $f(t)$ is Ω bandlimited, let $f(t)$ have the expansion (4). Since $\{\phi_k(t)\}_{k=0}^\infty$ form an orthogonal basis for $L^2[-T, T]$, [1], and (2), there exist a sequence of constants $\{b_k\}_{k=0}^\infty$ of finite energy, i.e.,

$$\sum_{k=0}^{\infty} |b_k|^2 < \infty, \tag{8}$$

such that

$$q(t) = \sum_{k=0}^{\infty} b_k \frac{\phi_k(t)}{\sqrt{\lambda_k}}, \quad \text{for } t \in [-T, T].$$

Thus, from (5) and (1) , we have

$$f(t) = (Kq)(t) = \sum_{k=0}^{\infty} b_k \sqrt{\lambda_k} \phi_k(t), \quad \text{for } t \in (-\infty, \infty).$$

Therefore, comparing with (4), we obtain $a_k = b_k \sqrt{\lambda_k}$ for $k = 0, 1, 2, ...$ From (8), we then have

$$\sum_{k=0}^{\infty} \frac{|a_k|^2}{\lambda_k} = \sum_{k=0}^{\infty} |b_k|^2 < \infty,$$

which proves (6), i.e., the necessity is proved. **q.e.d.**

The above result characterizes all BT-limited signals. Since any linear combinations of finite many prolate spheroidal wavefunctions are BT-limited, all BT-limited signals are dense in a bandlimited signal space, i.e., any bandlimited signal can be approximated by BT-limited signals.

Since $\{\phi_k\}_{k=0}^\infty$ form an orthonormal basis for space $\mathcal{BL}_\Omega$, [1], for any finite energy sequence $\{a_k\}_{k=0}^\infty$, i.e.,

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty,$$

we know

$$\sum_{k=0}^{\infty} a_k \phi_k(t) \in \mathcal{BL}_\Omega.$$

On the other hand, from (2),

$$\sum_{k=0}^{\infty} a_k \frac{\phi_k(t)}{\sqrt{\lambda_k}} = \sum_{k=0}^{\infty} \frac{a_k}{\sqrt{\lambda_k}} \phi_k(t) \in L^2[-T, T].$$

Since $\lambda_k \to 0$ as $k \to \infty$, we may have

$$\sum_{k=0}^{\infty} \frac{|a_k|^2}{\lambda_k} = \infty.$$

Thus, in general

$$\sum_{k=0}^{\infty} a_k \frac{\phi_k(t)}{\sqrt{\lambda_k}} \notin L^2(-\infty, \infty), \quad \text{or equivalently, } \notin \mathcal{BL}_\Omega.$$

However, if

$$\sum_{k=0}^{\infty} a_k \frac{\phi_k(t)}{\sqrt{\lambda_k}} \in L^2(-\infty, \infty), \quad \text{or equivalently, } \in \mathcal{BL}_\Omega,$$

then, (6) holds, and from Theorem 1, we obtain

$$\sum_{k=0}^{\infty} a_k \frac{\phi_k(t)}{\sqrt{\lambda_k}} \in \mathcal{BL}_\Omega^0.$$

For a general Ω bandlimited signal $f \in \mathcal{BL}_\Omega$ with expansion (4) where $\{a_k\}_{k=0}^\infty$ is a general sequence of finite energy, let

$$f_n(t) = \sum_{k=0}^{n} a_k \phi_k(t). \tag{9}$$

From (1) and (3),

$$f_n(t) = K\left(\sum_{k=0}^{n} \frac{a_k}{\lambda_k}\phi_k(t)\right).$$

Let

$$q_n(t) = \sum_{k=0}^{n} \frac{a_k}{\lambda_k}\phi_k(t).$$

Clearly, we have $f_n(t) = (Kq_n)(t) \in \mathcal{BL}_\Omega^0$. However, if f is not BT-limited, i.e., $f \notin \mathcal{BL}_\Omega^0$, then, from (2) and Theorem 1,

$$\int_{-T}^{T} |q_n(t)|^2 dt = \sum_{k=0}^{n} \frac{|a_k|^2}{\lambda_k} \to \infty, \quad \text{as } n \to \infty.$$

This means that $q_n(t)$ does not converge in $L^2[-T,T]$, although $f_n(t)$ converges to $f(t)$ in $L^2(-\infty,\infty)$, as $n \to \infty$, otherwise f would be BT-limited.

Since Ω bandlimited signal space $\mathcal{BL}_\Omega$ and $L^2[-\Omega,\Omega]$ are isomorphic by using (inverse) Fourier transform, for any signal $g \in L^2[-\Omega,\Omega]$, let it be the Fourier transform $\hat{f}$ of $f \in \mathcal{BL}_\Omega$, i.e., $g = \hat{f}$ on $[-\Omega,\Omega]$. As we can see above, f_n approaches f in $L^2(-\infty,\infty)$, then $\hat{f}_n$ approaches $g = \hat{f}$ in $L^2[-\Omega,\Omega]$. Since $\hat{f}_n = D_\Omega \hat{q}_n$ and $\hat{q}_n$ is T bandlimited as we can see above as well, g can be approximated by a T bandlimited signal $\hat{q}_n$ restricted in $[-\Omega,\Omega]$ in $L^2[-\Omega,\Omega]$, where D_Ω stands for the truncation operator from $(-\infty,\infty)$ to $[-\Omega,\Omega]$. Because T and Ω are both arbitrary, the above analysis proves the following corollary.

Corollary 1: Any finite piece signal on $[a,b]$ can be approximated in $L^2[a,b]$ by a bandlimited signal restricted in $[a,b]$ of bandwidth T, where $-\infty < a < b < \infty$ and $T > 0$ are arbitrary.

Note that the above result does not hold for infinite length signals. Also, as a comparison, the Weierstrass theorem says that any finite piece continuous signal can be approximated by polynomials, which may be thought of as a different perspective of using smooth/simple signals to approximate complicated signals.

4. BT-limited Signal Extrapolation

Bandlimited signal extrapolation had been studied extensively in the 1970s and 1980s, see, for example, [2]- [9]. It is to extrapolate a bandlimited signal f from a given piece of its values, for example, to extrapolate $f(t)$ for t outside $[-T,T]$ when $f(t)$ for $t \in [-T,T]$ is given. It is possible in theory since f is bandlimited and thus it is an entire function [10]. Any entire function is completely determined by its any segment. However, in practice, a given piece signal $f(t)$ for $t \in [-T,T]$ may contain error/noise and in this case, the extrapolation problem becomes a well-known ill-posed inverse problem. Any error in a given segment may cause an arbitrary large error in an extrapolation in general.

However, when f is BT-limited, an extrapolation method was proposed in [9] and an analytic error estimate for the extrapolation over the whole time domain was obtained. It can be described as follows.

Let $f_\epsilon(t)$ be an observation of $f(t)$ for $t \in [-T,T]$ with the maximal error magnitude ϵ, i.e., $|f_\epsilon(t) - f(t)| \le \epsilon$ for $t \in [-T,T]$. Let q_ϵ be the following minimum norm solution (MNS) in space $L^2[-T,T]$:

$$\int_{-T}^{T} |q_\epsilon(t)|^2 dt = \min_{q(t)\in L^2[-T,T]} \left\{ \int_{-T}^{T} |q(t)|^2 dt \; : \right.$$
$$\left. \left| \int_{-T}^{T} \frac{\sin\Omega(t-s)}{\pi(t-s)} q(s)ds - f_\epsilon(t) \right| \le 2\epsilon, \text{ for } t \in [-T,T] \right\}. \quad (10)$$

Let

$$\tilde{f}(t) = \int_{-T}^{T} \frac{\sin\Omega(t-s)}{\pi(t-s)} q_\epsilon(s)ds. \quad (11)$$

One can see that the above $\tilde{f}(t)$ is obtained from the given observation segement $f_\epsilon(t)$ of $f(t)$ on $[-T,T]$ and is called an extrapolation of $f(t)$. Also, from (5), we have $\tilde{f}(t) \in \mathcal{BL}_\Omega^0$. For the above extrapolation of $f(t)$, the following result was obtained in [9].

Theorem 2: If f is BT-limited, i.e., $f \in \mathcal{BL}_\Omega^0$, and $\tilde{f}$ is defined in (11), then

$$|\tilde{f}(t) - f(t)| \le C\epsilon^{1/3}, \quad \text{for all } t \in (-\infty,\infty), \quad (12)$$

for some constant C that is independent of ϵ and t.

This result tells that when signal f is BT-limited, i.e., not only it is bandlimited but also the non-zero part of its Fourier transform is a piece of a bandlimited signal, the above extrapolation (10)-(11) is robust and has an error estimate (12) for time t. To the author's best knowledge, no any other error estimate for a bandlimited signal extrapolation from inaccurate data on the whole time domain exists in the literature.

In practice, a given observation $f_\epsilon(t)$ for $t \in [-T,T]$ is usually discrete in time. A discretization of the above extrapolation (10)-(11) with a proved convergence was also given in [9].

More general subspaces $\mathcal{BL}_\Omega^\gamma$ for $0 \le \gamma < 1/2$ in bandlimited signal space $\mathcal{BL}_\Omega$ than the above $\mathcal{BL}_\Omega^0$ were introduced with the corresponding extrapolation, error estimate and discretization in [9]. It was shown in [9] that, if $f(t)$ is Ω bandlimited with the expansion (4) and, for $0 \le \gamma < 1/2$, the following inequality holds

$$\sum_{k=0}^{\infty} \frac{|a_k|^2}{\lambda_k^{1-2\gamma/3}} < \infty, \quad (13)$$

then, $f(t) \in \mathcal{BL}_\Omega^\gamma$. Clearly, (13) returns to (6) in Theorem 1 when $\gamma = 0$, although when $\gamma \ne 0$, the physical meaning of signals in subspace $\mathcal{BL}_\Omega^\gamma$ is not as clear as signals in subspace $\mathcal{BL}_\Omega^0$ studied in this paper. For more details, we refer the reader to [9].

Another comment we want to make here is that, for a general Ω bandlimited signal f, although it may not be BT-limited, function f_n defined in (9) is BT-limited and approaches f in $L^2(-\infty,\infty)$ as n becomes large. Therefore, for a general Ω bandlimited signal f, from its given segment $\tilde{f}$ with errors, we can still apply the MNS extrapolation (10)-(11). In this case, the analytic error estimate in Theorem 2 may not hold. However, interestingly, it was shown in [9] that the discretization of the above MNS extrapolation and its convergence to the analog solution still hold.

5. Simulations

We next show some simulation results to verify the above MNS extrapolation for BT-limited signals. For simplicity, in this simulation we use $\Omega = \pi$ and $T = 1$. A BT-limited signal $f(t)$ is generated by randomly generating $q(t) \in L^2[-T, T]$ in (5). Its noisy observation $f_\epsilon(t)$ is obtained by adding a random error with uniform distribution to $f(t)$ so that the maximum error magnitude not above ϵ.

We sample a noisy analog BT-limited signal $f_\epsilon(t)$ in $[-1, 1]$ with sampling rate 100 Hz, i.e., 201 samples of $f_\epsilon(t)$ in $[-1, 1]$ are used in the MNS in (10)-(11). In Fig. 1, the case of $\epsilon = 0.0125$ is simulated, where Fig. 1(a) shows the true data of a BT-limited signal $f(t)$ and its noisy data $f_\epsilon(t)$ on $[-1, 1]$, and Fig. 1(b) shows the true signal $f(t)$ and its extrapolation $\tilde{f}(t)$ in (11) using the noisy data $f_\epsilon(t)$ shown in Fig. 1(a). Fig. 2 shows the results when $\epsilon = 0.0031$, where one can see that the error in the extrapolated signal is clearly reduced, comparing to that in Fig. 1.

Fig. 3 shows the curve (dashed) of the maximum error magnitude between the true and the extrapolated signals, i.e., $\max_t |f(t) - \tilde{f}(t)|$, vs. the maximum error magnitude ϵ in the noisy data over $[-1, 1]$, and the curve (dashdot) of the ratio vs. ϵ:

$$R(\epsilon) = \frac{\max_t |f(t) - \tilde{f}(t)|}{\epsilon^{1/3}}.$$

The curves are obtained by using 20 independent trials. From this figure, one can see that the ratio $R(\epsilon)$ is less than a constant as ϵ gets smaller, which verifies the result (12) in Theorem 2. Note that in Fig. 3, the signal magnitudes are similar to those in Figs. 1 and 2.

6. More Properties of BT-limited Signals

The above definition of a BT-limited signal can be easily generalized as follows. Let $\mathcal{BL}_{[A,B]}$ denote the space of all finite energy signals $f(t)$ whose Fourier transforms are supported in the interval $[A, B]$, i.e., $\hat{f}(\omega) = 0$ when $\omega \notin [A, B]$. For real numbers A, B, a, b with $-\infty < A < B < \infty$ and $-\infty < a < b < \infty$, if signal $f(t) \in \mathcal{BL}_{[A,B]}$ and $\hat{f}(\omega) = g(\omega)$ when $\omega \in [A, B]$ for some $g(\omega) \in \mathcal{BL}_{[a,b]}$, then signal $f(t)$ is called BT-limited. The signal space of all the above BT-limited signals is denoted as $\mathcal{BL}_{A,B,a,b}$. Clearly, when $A = -\Omega$, $B = \Omega$, $a = -T$, and $b = T$, the above definition for a BT-limited signal returns to that in Section III and $\mathcal{BL}_{A,B,a,b} = \mathcal{BL}^0_\Omega$.

Let $\Omega = (B - A)/2$ and $T = (b - a)/2$, by some shifts in frequency and time domains, the representation for a BT-limited signal in (5) becomes as follows: $f \in \mathcal{BL}_{A,B,a,b}$ if and only if

$$f(t) = e^{j(A+\Omega)t} \int_{-T}^{T} \frac{\sin \Omega(t + a + T - s)}{\pi(t + a + T - s)} q(s) ds \qquad (14)$$

for any $t \in (-\infty, \infty)$, for some $q(t) \in L^2[-T, T]$.

Since any bandlimited signal is an entire function when t is extended to the complex plane [10], it cannot be 0 in any segment of time domain unless it is all 0 valued. This implies that a bandlimited signal f whose Fourier transform is supported in two separate bands, for example, $\hat{f}(\omega) \neq 0$ for $A_i < \omega < B_i$, $i = 1, 2$, and $\hat{f}(\omega) = 0$ for other ω, where $-\infty < A_1 < B_1 < A_2 < B_2 < \infty$, then, signal f is not BT-limited, i.e., $f \notin \mathcal{BL}_{A_1,B_2,a,b}$ for any $-\infty < a < b < \infty$, although in this case, signal f could be a sum of two BT-limited signals whose Fourier transforms are supported in $[A_1, B_1]$ and $[A_2, B_2]$, respectively, such that, the non-zero supports of the two Fourier transforms are the pieces of two bandlimited signals.

Theorem 3: For two non-zero BT-limited signals $f_i \in \mathcal{BL}_{A_i,B_i,a_i,b_i}$ with $-\infty < A_i < B_i < \infty$ and $-\infty < a_i < b_i < \infty$ for $i = 1, 2$, their linear combination $f = \alpha_1 f_1 + \alpha_2 f_2$ with two non-zero complex coefficients α_1 and α_2 is BT-limited *if and only if* $A_1 = A_2$ and $B_1 = B_2$.

Proof:

The "if" part is easy to see by setting $A = A_1 = A_2$, $B = B_1 = B_2$, $a = \min\{a_1, a_2\}$, and $b = \max\{b_1, b_2\}$. Then, $f \in \mathcal{BL}_{A,B,a,b}$, i.e., f is BT-limited.

We next prove the "only if" part. Without loss of generality, we assume $A_1 < A_2$ and f is BT-limited. Then, $f \in \mathcal{BL}_{A_1,\max\{B_1,B_2\},a,b}$ for some real numbers a, b with $-\infty < a < b < \infty$. From the above definition of BT-limited signals, there exist bandlimited signals $g_i \in \mathcal{BL}_{a_i,b_i}$ such that $\hat{f}_i(\omega) = g_i(\omega)$ for $\omega \in [A_i, B_i]$, $i = 1, 2$, and there exists a bandlimited signal $g \in \mathcal{BL}_{a,b}$ such that $\hat{f}(\omega) = g(\omega)$ for $\omega \in [A_1, \max\{B_1, B_2\}]$. On the other hand, we have $\hat{f} = \alpha_1 \hat{f}_1 + \alpha_2 \hat{f}_2$. This means that $\hat{f}(\omega) = \alpha_1 \hat{f}_1(\omega) = g(\omega) = \alpha_1 g_1(\omega)$ for $\omega \in [A_1, \min\{A_2, B_1\}]$. Since all g, g_1, g_2 are bandlimited and therefore, entire functions, we must have $\hat{f}(\omega) = \alpha_1 \hat{f}_1(\omega) = g(\omega) = \alpha_1 g_1(\omega)$ for all ω. In other words, $\hat{f}(\omega) = \alpha_1 \hat{f}_1(\omega) + \alpha_2 \hat{f}_2(\omega) = \alpha_1 \hat{f}_1(\omega)$ for all ω. This implies $\hat{f}_2(\omega) = 0$ for all ω, i.e., f_2 is the 0 signal, which contradicts with the non-zero signal assumption. This proves the necessity. **q.e.d.**

In general, for p BT-limited signals $f_i \in \mathcal{BL}_{A_i,B_i,a_i,b_i}$, with $-\infty < A_i < B_i < \infty$ and $-\infty < a_i < b_i < \infty$, $i = 1, 2, ..., p$, their non-zero linear combination $f = \sum_{i=1}^{p} \alpha_i f_i$ for non-zero complex coefficients α_i is not BT-limited, unless $A_i = B_i$ for all $i = 1, 2, ..., p$. It is easy to see that if $A_i = B_i$ for all $i = 1, 2, ..., p$, then the above linear combination f is indeed BT-limited and $f \in \mathcal{BL}_{A,B,a,b}$, where $A = A_i$, $B = B_i$, $a = \min\{a_1, a_2, ..., a_p\}$ and $b = \max\{b_1, b_2, ..., b_p\}$.

Although the above linear combination f of p BT-limited signals is generally not BT-limited, from (14), we have

$$f(t) = \sum_{i=1}^{p} \alpha_i e^{j(A_i+\Omega_i)t} \int_{-T_i}^{T_i} \frac{\sin \Omega_i(t + a_i + T_i - s)}{\pi(t + a_i + T_i - s)} q_i(s) ds \qquad (15)$$

for any $t \in (-\infty, \infty)$, where $\Omega_i = (B_i - A_i)/2$, $T_i = (b_i - a_i)/2$, and $q_i \in L^2[-T_i, T_i]$ for $i = 1, 2, ..., p$.

7. Conclusion

In this paper, BT-limited signal space was introduced and characterized. It is a subspace of bandlimited signals where the non-zero parts of their Fourier transforms are also pieces of bandlimited signals. Some new properties about and applying BT-limited signals were also presented. For BT-limited signals, an extrapolation from inaccurate data with an analytic error estimate in the whole time domain exists. Some simulations were presented to verify the theoretical extrapolation results for BT-limited signals.

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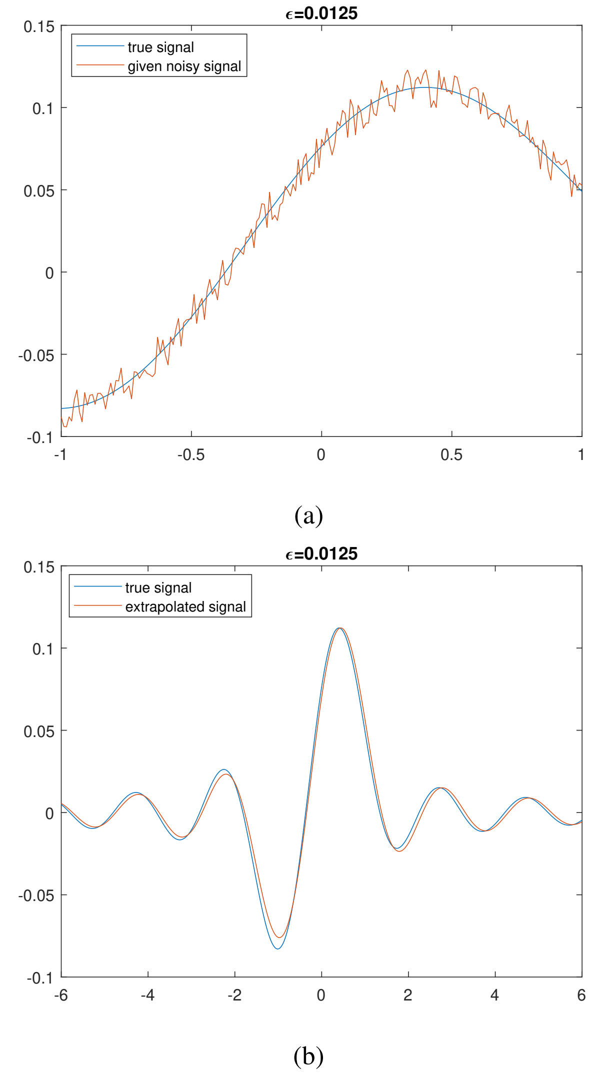
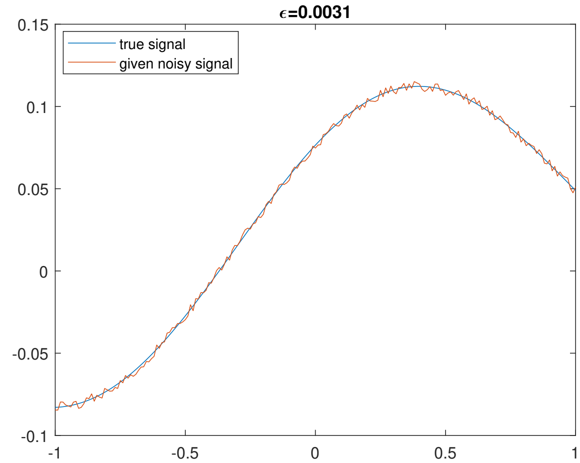
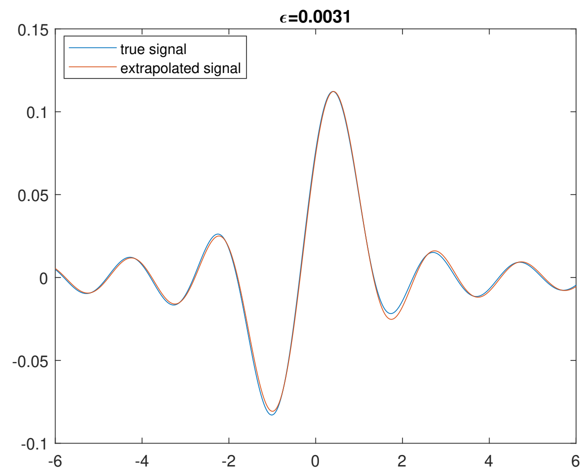


Fig. 1. BT-limited signal extrapolation from noisy data with the maximum error magnitude $\epsilon = 0.0125$: (a) given noisy data on $[-1, 1]$ and (b) extrapolated signal using the MNS method.



(a)



(b)

Fig. 2. BT-limited signal extrapolation from noisy data with the maximum error magnitude $\epsilon = 0.0031$: (a) given noisy data on $[-1, 1]$ and (b) extrapolated signal using the MNS method.

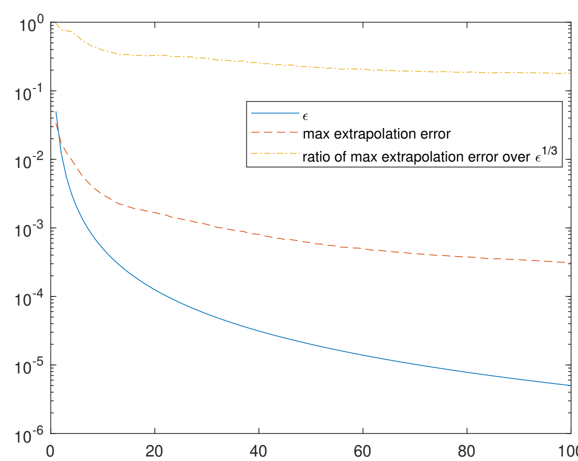


Fig. 3. The maximum error between extrapolated and true signals vs. the maximum magnitude ϵ of the errors in the given data over the time interval $[-1, 1]$, and its ratio, $R(\epsilon)$, over $\epsilon^{1/3}$.