A Construction of Pairwise Co-Prime Integer Matrices of Any Dimension and Their Least Common Right Multiple

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***Abstract*—Compared with co-prime integers, co-prime integer matrices are more challenging due to the non-commutativity. In this paper, we present a new family of pairwise co-prime integer matrices of any dimension and large size. These matrices are non-commutative and have low spread, i.e., their ratios of peak absolute values to mean absolute values (or the smallest non-zero absolute values) of their components are low. When matrix dimension is larger than 2, this family of matrices differs from the existing families, such as circulant, Toeplitz matrices, or triangular matrices, and therefore, offers more varieties in applications. In this paper, we first prove the pairwise coprimality of the constructed matrices, then determine their determinant absolute values, and their least common right multiple (lcrm) with a closed and simple form. We also analyze their sampling rates when these matrices are used as sampling matrices for a multi-dimensional signal. The proposed family of pairwise co-prime integer matrices may have applications in multi-dimensional Chinese remainder theorem (MD-CRT) that can be used to determine integer vectors from their integer vector remainders modulo a set of integer matrix moduli, and also in multi-dimensional sparse sensing and multirate systems.**

***Index Terms*—Pairwise co-prime integer matrices, least common right multiple (lcrm), Smith form, Chinese remainder theorem (CRT), multi-dimensional CRT (MD-CRT), multi-dimensional sampling.**

I. Introduction

IT is well-known that a family of pairwise co-prime integers, i.e., every pair of integers in the family are co-prime, have important applications in, such as, Chinese remainder theorem (CRT) [1], [2] that has many applications in, for example, cryptography and coding theory [2], [3], [6], and signal processing [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]. Similarly, pairwise co-prime integer matrices have applications in multi-dimensional CRT (MD-CRT) [32], [33] that can be used to determine integer vectors from their integer vector remainders modulo a set of integer matrix moduli. Note that when the integer matrix moduli can be diagonalized simultaneously, MD-CRT had appeared in earlier literature [21], [5]. However, different from co-prime integers, due to the non-commutativity of matrices, co-prime integer matrices are much more challenging.

Co-prime integer matrices have been studied in [26], [27], [28], [29], [30], [31] with applications in multi-dimensional sparse sensing and multi-dimensional multirate systems. Most studies in [26], [27], [28], [29], [30], [31] are for 2 by 2 circulant integer matrices and their variants and commutative integer matrices, 3 by 3 circulant integer matrices, Toeplitz integer matrices, triangular integer matrices and their adjugate matrices. In particular, a necessary and sufficient condition for two 2×2 integer matrices are co-prime was obtained in [29], which is easy to check.

Similar to the conventional CRT, in MD-CRT pairwise co-prime integer matrices as matrix moduli may play an important role as well to have a large range of uniquely determinable integer vectors from their integer vector remainders modulo the matrix moduli. In this paper, we present a new family of pairwise co-prime integer matrices of any dimension and large size. They are non-commutative and have low spread, i.e., their ratios of peak absolute values to mean absolute values (or the smallest non-zero absolute values) of their components are low. We first prove the pairwise coprimality of the matrices in the family and then determine their determinant absolute values and also their least common right multiple (lcrm) with a closed and simple form.

When matrix dimension is 2, the family of co-prime integer matrices we construct in this paper happen to be a set of Toeplitz integer matrices, and satisfy the necessary and sufficient condition for two 2×2 Toeplitz integer matrices to be co-prime obtained in [29]. When matrix dimension is larger than 2, the family of co-prime integer matrices we construct in this paper are much different from those in [26], [27], [28], [29], [30], [31]. The key differences are that our construction of pairwise co-prime integer matrices in this paper is: i) for any dimension, ii) of large size, iii) not pairwise commutative, iv) not circulant, and v) not Toeplitz or triangular matrices. In addition, as mentioned earlier, we determine the lcrm of the

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family of pairwise co-prime matrices constructed in this paper, including the family of 2×2 co-prime integer matrices, which has not been addressed in any existing literature.

Note that the determinant absolute values of the matrix moduli are the sampling rates, i.e., the number of sampled points per unit spatial volume, using these matrices as sampling matrices [20], [21], [24], [25], [32], [33] for a multi-dimensional signal. Also, an lcrm $\mathbf{R}$ of the matrix moduli determines the range $\mathcal{N}(\mathbf{R})$ detailed in 1) in Section II, called the fundamental parallelpiped (FPD) of $\mathbf{R}$ [24], of the uniquely determinable integer vectors from their integer vector remainders modulo the matrix moduli [32]. The determinant absolute value $|\det(\mathbf{R})|$, called the *dynamic range*, is the number of these uniquely determinable integer vectors, which is given with the specified lcrm $\mathbf{R}$ in this paper for our newly constructed family of integer matrices as matrix moduli.

In this paper, we also show that the sampling rates of our newly proposed pairwise co-prime integer matrices as (non-separable) sampling matrices in *each dimension* are much smaller than the maximal ones of the necessary sampling rates of the conventional one dimensional samplings using diagonal (separable) integer sampling matrices, when their maximal determinant absolute values and the determinant absolute values of their lcrm matrices, i.e., their dynamic ranges, are the same. This is an advantage of non-separable sampling over separable sampling for a multi-dimensional signal.

This paper is organized as follows. In Section II, we briefly introduce some necessary notations and preliminaries on integer matrices including MD-CRT. In Section III, we present a novel family of pairwise co-prime integer matrices of any dimension. In Section IV, we prove the pairwise coprimality of the integer matrices in the constructed family. In Section V, we determine the determinants of the integer matrices in the constructed family and their lcrm, and also analyze their sampling rates. In Section VI, we conclude this paper.

II. Some Necessary Notations and Preliminaries on Integer Matrices

$\mathbb{Z}$ denotes the set of all integers and $\mathbb{R}$ denotes the set of all real numbers. All vectors, such as $\mathbf{n}$, $\mathbf{f}$ and $\mathbf{r}$, and matrices, such as $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{P}$, in this paper are D dimensional integer vectors and $D \times D$ dimensional integer matrices, respectively, i.e., $\mathbf{n}$, $\mathbf{f}$, $\mathbf{r} \in \mathbb{Z}^D$ and $\mathbf{M}$, $\mathbf{N}$, $\mathbf{P} \in \mathbb{Z}^{D\times D}$, unless otherwise specified. $\mathbf{I}$ is the $D \times D$ identity matrix, and $\mathbf{0}$ is the all 0 matrix or vector. $\det(\mathbf{M})$ denotes the determinant of matrix $\mathbf{M}$, and $^\top$ stands for the transpose. And diag stands for a $D \times D$ diagonal matrix. For a set $\mathcal{S}$, its cardinality is denoted by $|\mathcal{S}|$. For two positive integers n and m, the remainder of n modulo m is denoted by $\langle n \rangle_m$. Below we introduce some necessary concepts on integer matrices and for details, see, for example, [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33]. These definitions, when reduced to the one-dimensional case, do not affect any of the classical results related to co-prime integers.

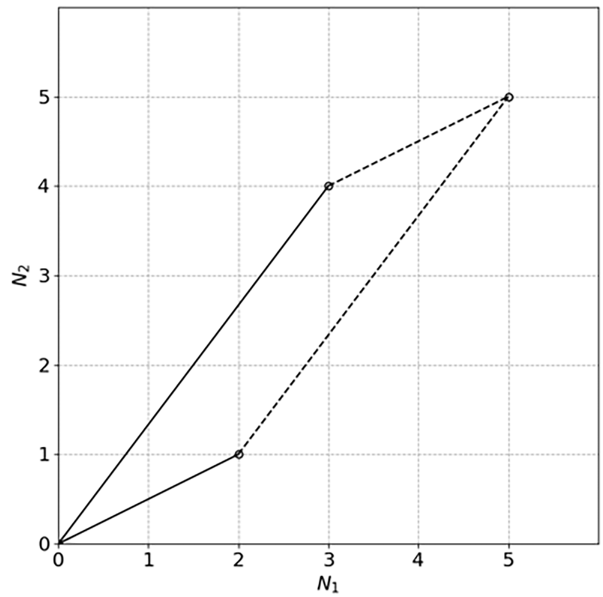

Fig. 1. FPD of $\mathbf{N}$.

1) **Set $\mathcal{N}(\mathbf{M})$ called the fundamental parallelpiped (FPD) of $\mathbf{M}$** [24]: Given a $D \times D$ nonsingular integer matrix $\mathbf{M}$, set $\mathcal{N}(\mathbf{M})$ is defined as the following set of integer vectors:
$$\mathcal{N}(\mathbf{M}) = \left\{\mathbf{k} \mid \mathbf{k} = \mathbf{M}\mathbf{x}, \mathbf{x} \in [0,1)^D \text{ and } \mathbf{k} \in \mathbb{Z}^D\right\}. \tag{1}$$
The number of elements in $\mathcal{N}(\mathbf{M})$ is equal to the absolute value of the determinant of matrix $\mathbf{M}$, i.e., $|\det(\mathbf{M})|$, [20], [24]. The FPD of
$$\mathbf{N} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$
is shown in Fig. 1, where the dashed edges and hollow vertices are not part of the FPD. We refer the reader to [24] for more details about FPD.
2) **Unimodular matrix**: A square integer matrix is called unimodular if its determinant is 1 or -1.
3) **Divisor and greatest common left divisor (gcld)**: A nonsingular integer matrix $\mathbf{A}$ is a left divisor of an integer matrix $\mathbf{M}$ if $\mathbf{A}^{-1}\mathbf{M}$ is an integer matrix. If $\mathbf{A}$ is a left divisor of each of all $L \geq 2$ integer matrices $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_L$, it is called a common left divisor (cld) of $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_L$. Moreover, if any other cld is a left divisor of $\mathbf{A}$, then $\mathbf{A}$ is a *greatest common left divisor* (gcld) of $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_L$.
4) **Co-prime matrices**: Two $D \times D$ integer matrices are left co-prime (or simply co-prime in this paper) if their gcld is a unimodular matrix. For two integer matrices $\mathbf{M}$ and $\mathbf{N}$, they are left co-prime if and only if the Smith form [18], [22] of the combined $D \times 2D$ integer matrix $(\mathbf{M}\ \mathbf{N})$ is $(\mathbf{I}\ \mathbf{0})$. In addition, an equivalent necessary and sufficient condition on all the $D \times D$ minors of matrix $(\mathbf{M}\ \mathbf{N})$ was proposed in [29]. Also, it is not hard to see that if the determinant absolute values of two integer

matrices are co-prime, these two integer matrices are coprime [27]. Note that in this paper, only left coprimality is considered.

5) **Multiple and least common right multiple (lcrm)**: A nonsingular integer matrix $\mathbf{A}$ is a right multiple of an integer matrix $\mathbf{M}$, if there exists a nonsingular integer matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{MP}$. If $\mathbf{A}$ is a right multiple of each of all $L \geq 2$ integer matrices $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_L$, $\mathbf{A}$ is called a common right multiple (crm) of $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_L$. Additionally, $\mathbf{A}$ is a *least common right multiple* (lcrm) of $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_L$, if any other crm of them, is a right multiple of $\mathbf{A}$. If $\mathbf{A}$ is an lcrm of $\mathbf{M}_1, \cdots, \mathbf{M}_L$, $\mathbf{AU}$ is also an lcrm of them when $\mathbf{U}$ is a unimodular matrix, which means that lcrm is not unique but the absolute determinant value of lcrm is unique. And this absolute determinant value is the minimum one of that of all the crms of these matrices. Although the lcrm of $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_L$ is not unique, we use lcrm$(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_L)$ to denote one fixed representative. This helps simplify later expressions. From this definition, it is not hard to see that for any groups of $D \times D$ integer matrices $\mathbf{M}_{1,1}, \cdots, \mathbf{M}_{1,L_1}, \cdots, \mathbf{M}_{k,1}, \cdots, \mathbf{M}_{k,L_k}$, we have

$$\begin{aligned} &\text{lcrm}(\mathbf{M}_{1,1}, \cdots, \mathbf{M}_{1,L_1}, \cdots, \mathbf{M}_{k,1}, \cdots, \mathbf{M}_{k,L_k}) \\ &= \text{lcrm}(\text{lcrm}(\mathbf{M}_{1,1}, \cdots, \mathbf{M}_{1,L_1}), \cdots, \\ &\qquad \text{lcrm}(\mathbf{M}_{k,1}, \cdots, \mathbf{M}_{k,L_k})). \end{aligned} \tag{2}$$

6) **Division representation for integer vectors**: Given a nonsingular integer matrix $\mathbf{M} \in \mathbb{Z}^{D\times D}$, any integer vector $\mathbf{f} \in \mathbb{Z}^D$ can be uniquely decomposed as:

$$\mathbf{f} = \mathbf{Mn} + \mathbf{r},$$

where $\mathbf{r} \in \mathcal{N}(\mathbf{M})$ and $\mathbf{n} \in \mathbb{Z}^D$. In modulo form, this is represented as:

$$\mathbf{f} \equiv \mathbf{r} \mod \mathbf{M},$$

where $\mathbf{M}$ is a modulus, and $\mathbf{n}$ and $\mathbf{r}$ are the folding integer vector and the integer vector remainder of $\mathbf{f}$ modulo $\mathbf{M}$, respectively.

7) **Multi-dimensional undersampling**: Consider the following multi-dimensional harmonic signal, [21], [24], [25], [32], [33],

$$x(\mathbf{t}) = a \exp(j2\pi \mathbf{f}^\top \mathbf{t}) + \omega(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^D, \tag{3}$$

where a is an unknown amplitude and $\mathbf{f} \in \mathbb{Z}^D$ is an unknown D dimensional frequency of the signal, and $\omega(\mathbf{t})$ is an additive noise. We want to determine the D dimensional integer frequency vector $\mathbf{f} = [N_1, N_2, \cdots, N_D]^\top$ from (possibly multiple) undersampled D dimensional signals of $x(\mathbf{t})$ with low sampling rates, where all N_i are assumed positive integers and some of them may be large, i.e., large frequencies.

We use L many $D \times D$ nonsingular integer matrices $\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_L$ to sample the multi-dimensional signal in (3) in the following sense:

$$x_i[\mathbf{n}] = a \exp(j2\pi \mathbf{f}^\top \mathbf{M}_i^{-\top} \mathbf{n}) + \omega[\mathbf{M}_i^{-\top}\mathbf{n}], \tag{4}$$

$\mathbf{n} \in \mathbb{Z}^D$, $1 \leq i \leq L$, where $\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_L$ are called *sampling matrices*. For each sampling matrix $\mathbf{M}_i$, there are $|\det(\mathbf{M}_i)|$ many sampled points per unit spatial volume of $\mathbb{R}^D$ as we can see from (1), which is called the *sampling rate* (or sampling density) of sampling matrix $\mathbf{M}_i$ for a multi-dimensional signal.

Next, by performing the multi-dimensional DFT (MD-DFT) to each $x_i[\mathbf{n}]$ with respect to $n \in \mathcal{N}(\mathbf{M}_i^\top)$, we have, for $k \in \mathcal{N}(\mathbf{M}_i)$,

$$\begin{aligned} &X_i(\mathbf{k}) \\ &= a \sum_{\mathbf{n}\in\mathcal{N}(\mathbf{M}_i^\top)} \exp(j2\pi \mathbf{f}^\top \mathbf{M}_i^{-\top}\mathbf{n}) \exp(-j2\pi \mathbf{k}^\top \mathbf{M}_i^{-\top}\mathbf{n}) \\ &\quad + \Omega_i(\mathbf{k}), \quad 1 \leq i \leq L, \end{aligned} \tag{5}$$

which arrives at

$$X_i(\mathbf{k}) = a|\det(\mathbf{M}_i)|\delta(\mathbf{k} - \mathbf{r}_i) + \Omega_i(\mathbf{k}), \quad 1 \leq i \leq L, \tag{6}$$

where $\mathbf{r}_i$ is the integer vector remainder of the integer frequency vector $\mathbf{f}$ modulo $\mathbf{M}_i$, and $\delta(\mathbf{n})$ is the discrete delta function that is 1 when $\mathbf{n} = \mathbf{0}$ and 0 otherwise. Although the sampling matrices employed in our framework are not necessarily diagonal (i.e., separable sampling), we can obtain their corresponding diagonal matrices by calculating their Smith forms. By applying appropriate input and output signal index transformations, equation (5) can be reformulated into an MD-DFT based on diagonal matrices, i.e., the separable case. Then, we can use the Fast Fourier Transform (FFT) for computational acceleration on each dimension, for more details, see [21]. From (6), one can detect the integer vector remainders $\mathbf{r}_i \equiv \mathbf{f} \mod \mathbf{M}_i$, $1 \leq i \leq L$. Now the question becomes how to determine the integer frequency vector $\mathbf{f}$ from these detected integer vector remainders. This can be solved by using MD-CRT [32].

8) **MD-CRT**: [MD-CRT for integer vectors [32]] Given L matrix moduli $\mathbf{M}_i$ for $1 \leq i \leq L$, which are arbitrary nonsingular integer matrices, let $\mathbf{R}$ be any lcrm of them. For an integer vector $\mathbf{n} \in \mathbb{Z}^D$, it can be uniquely determined from its L integer vector remainders $\mathbf{r}_i \equiv \mathbf{n} \mod \mathbf{M}_i$, $1 \leq i \leq L$, if $\mathbf{n} \in \mathcal{N}(\mathbf{R})$.

A detailed determination algorithm can be found in [32]. From this MD-CRT, the range of the uniquely determinable integer vectors $\mathbf{f}$ is $\mathcal{N}(\mathbf{R})$ for an lcrm $\mathbf{R}$ of integer matrices $\mathbf{M}_i$, $1 \leq i \leq L$, and the number of such uniquely determinable integer vectors is $|\det(\mathbf{R})|$, which is called the *dynamic range* of the sampling matrices $\mathbf{M}_i$, $1 \leq i \leq L$. Clearly one would like to have small sampling rates $|\det(\mathbf{M}_i)|$ and large dynamic range $|\det(\mathbf{R})|$.

9) **Some applications of multi-dimensional undersampling and MD-CRT**: We present three cases where multi-dimensional undersampling applies. One is the conventional multi-dimensional sampling below the Nyquist rate. This has similar applications as in the one

dimensional case, such as sensor networks, and also has applications in computational imaging [34], where the interested 2 dimensional frequencies are too high compared to the sampling rates of the sensors in a group of multiple scattered monitoring sensors with low functionalities, such as low sampling rates and low powers.

The second is in moving target parameter estimation in synthetic aperture radar (SAR) imaging, where a moving target speed and location parameters appear as frequency components in the radar return signals after some radar signal processing, such as range compression. In SAR imaging, the antenna arrays are fixed on the platform and the spatial sampling rate (corresponding to the time sampling rate) is determined by the fixed distance between adjacent antenna elements. When the target moves fast, the frequency components in the radar return signals may become too large compared to the fixed spatial sampling rate, which causes undersampling. To address this, a method using two co-prime linear arrays was proposed in [9] to accurately estimate the parameters of fast-moving targets. However, the above linear antenna arrays are not spatially efficient for platforms with limited space, such as an aircraft. A natural way is to use planar antenna arrays, turning a 1D radar return signal into a 2D radar return signal, where co-prime linear arrays turn to co-prime integer matrices. In this setting, the results developed in this paper and the MD-CRT may play a key role. Notably, 2D co-prime planar arrays have already been applied in array signal processing, for example, for direction-of-arrival (DoA) estimation in [35] where, although, the co-prime integer matrices used for co-prime arrays are diagonal (or separable).

The third application is in recent multi-channel self-reset analog-to-digital converter (SRADC) for complex-valued bandlimited signals [17]. It is a special case of 2D-CRT with 2 dimensional co-prime integer matrix moduli [36].

As we can see already from the above definitions, all matrix multiplications in this paper are in the sense of left side multiplications. Also, from the above coprimality of integer matrices, the elements of an integer matrix can be any integers including negative integers, 1, and 0. Since a scalar integer can be thought of as a special integer matrix, i.e., a 1×1 integer matrix, for the consistence with integer matrices, all integers are considered for the coprimality. Integers p and q are co-prime if and only if their gcd is 1 or -1. This relaxation does not affect any results in this paper.

III. New Construction of Non-Diagonal Pairwise Co-Prime Integer Matrices of Dimension D

We first let $1 < q_1 < q_2 < \cdots < q_L$ be L pairwise co-prime positive integers and define the following set of $D\times D$ integer matrices

$$\mathcal{S}_D = \{\mathbf{M} | \mathbf{M} \in \{0, \pm 1, \pm 2, \cdots, \pm q_L\}^{D\times D}\}. \tag{7}$$

We next present a method and an algorithm to construct a family of pairwise co-prime integer matrices in the set $\mathcal{S}_D$. To do so, we first provide a definition. Let $\mathcal{N}_D \triangleq \{1, 2, \ldots, D\}$. A permutation σ of $\mathcal{N}_D$ is a one-to-one and onto mapping from $\mathcal{N}_D$ to itself. It can be represented by a vector $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(D))$, where each $\sigma(i) \in \mathcal{N}_D$ and all $\sigma(i)$ are distinct.

Definition 1: For a given non-empty subset $\mathcal{K}$ of $\mathcal{N}_D$, a feasible permutation set $\mathcal{P}_f(\mathcal{N}_D)$ of $\mathcal{N}_D$ is defined as a subset of all permutations of $\mathcal{N}_D$:

$$\mathcal{P}_f(\mathcal{N}_D) = \{\sigma_j \text{ is a permutation of } \mathcal{N}_D \text{ and its last element } \sigma_j(D) = j, j \in \mathcal{K}\}. \tag{8}$$

From the above definition, it is not hard to see that there are a total of

$$\sum_{d=1}^{D} \binom{D}{d} ((D-1)!)^d$$

feasible permutation sets of $\mathcal{N}_D$. Any non-empty subset of d elements of $\mathcal{N}_D$ defines a feasible permutation set $\mathcal{P}_f(\mathcal{N}_D)$ of $\mathcal{N}_D$ and its size is d as well. If the whole set $\mathcal{N}_D$ is taken in defining a feasible permutation set, i.e., all the elements in $\mathcal{N}_D$ are taken as the last components of the permutations in $\mathcal{P}_f(\mathcal{N}_D)$, the feasible permutation set has the largest size D.

For a given $\mathcal{P}_f(\mathcal{N}_D)$, we can construct a family of $D\times D$ pairwise co-prime matrices as follows.

To construct a $D\times D$ integer matrix $\mathbf{M}$, we begin by selecting a permutation from $\mathcal{P}_f(\mathcal{N}_D)$ to place entries of 1 in its specific positions. The construction details are

1) For any chosen permutation
$$\sigma_j = (\sigma_j(1), \sigma_j(2), \cdots, \sigma_j(D))$$
from $\mathcal{P}_f(\mathcal{N}_D)$, we set the elements at the $(D-1)$ positions $(\sigma_j(1), \sigma_j(2))$, $(\sigma_j(2), \sigma_j(3))$, $\cdots$, $(\sigma_j(D-1), \sigma_j(D))$ of matrix $\mathbf{M}$ to 1.
2) Next, we choose the diagonal elements from the set of pairwise co-prime positive integers $\{q_1, q_2, \cdots, q_L\}$ with $q_1 > 1$. By choosing any integer q_i from this set, we set all diagonal entries of matrix $\mathbf{M}$ to q_i.
3) Set all the other elements of $\mathbf{M}$ to 0.

This completes the construction of one $D\times D$ integer matrix. For each permutation from $\mathcal{P}_f(\mathcal{N}_D)$, we can create L distinct matrices by choosing different diagonal elements from the pairwise co-prime integer set $\{q_1, q_2, \cdots, q_L\}$. Given a $\mathcal{P}_f(\mathcal{N}_D)$ with $|\mathcal{P}_f(\mathcal{N}_D)| = d$ for some positive integer d with $1 \le d \le D$, this approach allows us to construct a total of dL many $D\times D$ integer matrices. In other words, for any given feasible permutation set $\mathcal{P}_f(\mathcal{N}_D)$ of size d, $1 \le d \le D$, we can construct a family of dL many $D\times D$ integer matrices that will be shown pairwise co-prime later. The construction method is summarized in Algorithm 1.

Let $\mathcal{P}_f(\mathcal{N}_D)$ be a feasible permutation set of $\mathcal{N}_D$ of the largest size D, such as, the set of all the cyclic permutations of $\mathcal{N}_D$:

$$\mathcal{P}_f(\mathcal{N}_D) = \{(1, 2, \cdots, D-1, D), (2, 3, \cdots, D, 1), \cdots, (D, 1, \cdots, D-2, D-1)\}. \tag{9}$$

Following the above construction method, for the sake of convenience, let $\mathbf{M}_{i,j}$ denote the matrix that is constructed by

Algorithm 1 Pairwise Co-prime Matrices Construction Algorithm

Input: A set of pairwise co-prime positive integers $\{q_1, q_2, \cdots, q_L\}$ with $q_1 > 1$, dimension D and a feasible permutation set $\mathcal{P}_f(\mathcal{N}_D)$ with $|\mathcal{P}_f(\mathcal{N}_D)| = d$
Output: A set of $D \times D$ integer matrices $\{\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_{Ld}\}$
Initialize an empty list to store matrices
for each permutation $\sigma_j = (\sigma_j(1), \sigma_j(2), \cdots, \sigma_j(D))$ from $\mathcal{P}_f(\mathcal{N}_D)$ **do**
  for each $q_i \in \{q_1, q_2, \cdots, q_L\}$ **do**
    Initialize a $D \times D$ matrix $\mathbf{M}$ with zeros
    Set the element at position $(\sigma_j(1), \sigma_j(1))$ of matrix $\mathbf{M}$ to q_i
    for $k = 2$ to D **do**
      Set the element at position $(\sigma_j(k-1), \sigma_j(k))$ of matrix $\mathbf{M}$ to 1
      Set the element at position $(\sigma_j(k), \sigma_j(k))$ of matrix $\mathbf{M}$ to q_i
    end for
    Add matrix $\mathbf{M}$ to the list of matrices
  end for
end for
return the list of matrices

choosing q_i on its diagonal and choosing the permutation σ_j from $\mathcal{P}_f(\mathcal{N}_D)$ to position the 1's, for $1 \le i \le L$ and $1 \le j \le D$. We can see that in matrix $\mathbf{M}_{i,j}$, there are a total of $(D-1)$ many 1's, and at most a single 1 per row and at most a single 1 per column. Especially, the j-th row has no 1 and the $\sigma_j(1)$-th column has no 1. Furthermore, each row has at most two non-zero elements q_i and 1 and each column has at most two non-zero elements q_i and 1.

Each matrix $\mathbf{M}_{i,j}$ in the above construction can be represented as

$$\mathbf{M}_{i,j} = q_i \mathbf{I} + \mathbf{A}_j, \tag{10}$$

where $\mathbf{A}_j$ is a binary matrix constructed by following the above construction Steps 1) and 3) with the chosen permutation σ_j from $\mathcal{P}_f(\mathcal{N}_D)$. From the representation in (10), it is not hard to check that $\mathbf{M}_{i_1,j}$ and $\mathbf{M}_{i_2,j}$ are commutative, for any $1 \le i_1, i_2 \le L$ and $1 \le j \le D$. However, matrices $\mathbf{M}_{i_1,j_1}$ and $\mathbf{M}_{i_2,j_2}$ for $1 \le j_1 \ne j_2 \le D$ in the above construction are not commutative.

As an example of matrices $\mathbf{M}_{i,j}$, consider the case when $D = 4$ and the chosen feasible permutation set of $\mathcal{N}_4 = \{1, 2, 3, 4\}$ is

$$\mathcal{P}_f(\mathcal{N}_4) = \{(4,2,3,1), (1,3,4,2), (4,1,2,3), (3,1,2,4)\}.$$

Then, the matrices constructed by the method (or Algorithm 1) are: for $1 \le i \le L$,

$$\mathbf{M}_{i,1} = \begin{pmatrix} q_i & 0 & 0 & 0 \\ 0 & q_i & 1 & 0 \\ 1 & 0 & q_i & 0 \\ 0 & 1 & 0 & q_i \end{pmatrix}, \mathbf{M}_{i,2} = \begin{pmatrix} q_i & 0 & 1 & 0 \\ 0 & q_i & 0 & 0 \\ 0 & 0 & q_i & 1 \\ 0 & 1 & 0 & q_i \end{pmatrix},$$

$$\mathbf{M}_{i,3} = \begin{pmatrix} q_i & 1 & 0 & 0 \\ 0 & q_i & 1 & 0 \\ 0 & 0 & q_i & 0 \\ 1 & 0 & 0 & q_i \end{pmatrix}, \mathbf{M}_{i,4} = \begin{pmatrix} q_i & 1 & 0 & 0 \\ 0 & q_i & 0 & 1 \\ 1 & 0 & q_i & 0 \\ 0 & 0 & 0 & q_i \end{pmatrix}.$$

From this example, one can see that these matrices are not circulant, Toeplitz, triangular, or their variants as studied in [29].

When D is an even number and the chosen feasible permutation set is

$$\mathcal{P}_f(\mathcal{N}_D) = \{(1, 2, \cdots, D), (2, \langle 2+2 \rangle_{D+1}, \cdots, \langle 2D \rangle_{D+1}), \cdots, (D, \langle D+D \rangle_{D+1}, \cdots, \langle D^2 \rangle_{D+1})\}, \tag{11}$$

the family constructed by the above method (or Algorithm 1) happens to be a family of Toeplitz matrices. For example, when $D = 4$ and the chosen feasible permutation set is

$$\begin{aligned} \mathcal{P}_f(\mathcal{N}_4) &= \{(1,2,3,4), (2,4,1,3), (3,1,4,2), (4,3,2,1)\} \\ &= \{(4,3,2,1), (3,1,4,2), (2,4,1,3), (1,2,3,4)\}, \end{aligned} \tag{12}$$

the constructed matrices are: for $1 \le i \le L$,

$$\mathbf{M}_{i,1} = \begin{pmatrix} q_i & 0 & 0 & 0 \\ 1 & q_i & 0 & 0 \\ 0 & 1 & q_i & 0 \\ 0 & 0 & 1 & q_i \end{pmatrix}, \mathbf{M}_{i,2} = \begin{pmatrix} q_i & 0 & 0 & 1 \\ 0 & q_i & 0 & 0 \\ 1 & 0 & q_i & 0 \\ 0 & 1 & 0 & q_i \end{pmatrix},$$

$$\mathbf{M}_{i,3} = \begin{pmatrix} q_i & 0 & 1 & 0 \\ 0 & q_i & 0 & 1 \\ 0 & 0 & q_i & 0 \\ 1 & 0 & 0 & q_i \end{pmatrix}, \mathbf{M}_{i,4} = \begin{pmatrix} q_i & 1 & 0 & 0 \\ 0 & q_i & 1 & 0 \\ 0 & 0 & q_i & 1 \\ 0 & 0 & 0 & q_i \end{pmatrix},$$

where the index j in $\mathbf{M}_{i,j}$ corresponds to the last component in a permutation in $\mathcal{P}_f(\mathcal{N}_4)$ in (12).

When $D = 2$, i.e., the two dimension case, the feasible permutation set with the maximal size $D = 2$ has only one possibility, i.e., the one in (11) or (9), and therefore, all the constructed matrices $\mathbf{M}_{i,j}$ happen to be Toeplitz as we will study more later for their lcrm. In general, our constructed family is not a family of Toeplitz matrices and the above case with the special feasible permutation set is the only case of Toeplitz matrices. Furthermore, we do not use any property of Toeplitz matrices in the following studies.

For the above constructed family $\{\mathbf{M}_{i,j} : 1 \le i \le L, 1 \le j \le D\}$ of $D \times D$ integer matrices, we have the following results about their pairwise coprimality, determinants, lcrm, and dynamic ranges. As mentioned above for the 2 dimensional case, our contructed 2×2 integer matrices happen to be Toeplitz, every pair of which indeed satisfy the necessary and sufficient condition for them to be co-prime obtained in [29].

IV. Pairwise Coprimality

In this section, we show the pairwise coprimality of the integer matrices in the family constructed in the previous section. To do so, we first present a lemma.

Lemma 1. Let m_1 and m_2 be two non-zero integers with $gcd(m_1, m_2) = k$ for a positive integer k. For each i with

$1 \leq i \leq D$, we can obtain a new matrix $[k\mathbf{e}_i \quad \mathbf{0}]$ by performing elementary column transformations on matrix $[m_1\mathbf{e}_i \quad m_2\mathbf{e}_i]$, where $\mathbf{e}_i$ is the D-dimensional vector with the i-th component 1 and the other components 0.

Proof: When $|m_1| = |m_2|$, by multiplying 1 or -1 to each column of matrix $[m_1\mathbf{e}_i \quad m_2\mathbf{e}_i]$, we can get a new matrix $[k\mathbf{e}_i \quad k\mathbf{e}_i]$ for $k = |m_1| = |m_2|$. Then, by multiplying -1 to the first column and adding it to the second column, we get matrix $[k\mathbf{e}_i \quad \mathbf{0}]$.

When $|m_1| \neq |m_2|$, without loss of generality, we assume that $|m_1| > |m_2|$. Apply the Euclidean algorithm to m_1 and m_2 and assume that there are L equations here to calculate the gcd of m_1 and m_2:

$$\begin{aligned} m_1 &= n_1 m_2 + r_1, \quad 0 \leq r_1 < |m_2|, \\ m_2 &= n_2 r_1 + r_2, \quad 0 \leq r_2 < r_1, \\ r_1 &= n_3 r_2 + r_3, \quad 0 \leq r_3 < r_2, \\ &\vdots \\ r_{L-3} &= n_{L-1} r_{L-2} + k, \quad 0 \leq k < r_{L-2}, \\ r_{L-2} &= n_L k. \end{aligned} \tag{13}$$

Consider the first equation in (13), we can get a new matrix $[r_1\mathbf{e}_i \quad m_2\mathbf{e}_i]$ by multiplying the second column of matrix $[m_1\mathbf{e}_i \quad m_2\mathbf{e}_i]$ by $-n_1$ and adding it to the first column. Then, consider the second equation in (13), by multiplying the first column of matrix $[r_1\mathbf{e}_i \quad m_2\mathbf{e}_i]$ by $-n_2$ and adding it to the second column, we can get a new matrix $[r_1\mathbf{e}_i \quad r_2\mathbf{e}_i]$. Following all L equations in (13), we eventually get a matrix with two columns $k\mathbf{e}_i$ and $\mathbf{0}$ in the form of either $[k\mathbf{e}_i \quad \mathbf{0}]$ or $[\mathbf{0} \quad k\mathbf{e}_i]$, depending on L is odd or even. So, by switching the order of the two columns if necessary, we get matrix $[k\mathbf{e}_i \quad \mathbf{0}]$. □

We now consider the pairwise coprimality of the constructed family of integer matrices $\{\mathbf{M}_{i,j}\}$.

Theorem 1: The matrices $\mathbf{M}_{i,j}$, for $1 \leq i \leq L$ and $1 \leq j \leq D$, are pairwise co-prime.

Proof: We divide this proof into two parts. First, we prove that $\mathbf{M}_{i_1,j}$ and $\mathbf{M}_{i_2,j}$ are co-prime, for any $1 \leq i_1 \neq i_2 \leq L$ and $1 \leq j \leq D$. Second, we prove that $\mathbf{M}_{i_1,j_1}$ and $\mathbf{M}_{i_2,j_2}$ are co-prime, for any $1 \leq i_1, i_2 \leq L$ and $1 \leq j_1 \neq j_2 \leq D$.

We begin by the first part. Consider two matrices $\mathbf{M}_{i_1,j}$ and $\mathbf{M}_{i_2,j}$ for any given $1 \leq i_1 \neq i_2 \leq L$ and $1 \leq j \leq D$. Let $\sigma_j = (m_1, m_2, \cdots, m_D)$, where $m_D = j$, and $D \times 2D$ matrix $\mathbf{A} = (\mathbf{M}_{i_1,j} \quad \mathbf{M}_{i_2,j})$. We then calculate the Smith form of $\mathbf{A}$ step by step.

In the first step, we can see that the m_1-th column of $\mathbf{A}$ is $q_{i_1}\mathbf{e}_{m_1}$, and the $(D+m_1)$-th column of $\mathbf{A}$ is $q_{i_2}\mathbf{e}_{m_1}$. As q_{i_1} and q_{i_2} are co-prime for $i_1 \neq i_2$, we can get the new m_1-th column $\mathbf{e}_{m_1}$ and the new $(D+m_1)$-th column $\mathbf{0}$ by applying Lemma 1. In the second step, we can use vector $\mathbf{e}_{m_1}$ to eliminate 1's in the positions (m_1, m_2) and $(m_1, D+m_2)$ of matrix $\mathbf{A}$ and get the new m_2-th column $q_{i_1}\mathbf{e}_{m_2}$, and the new $(D+m_2)$-th column $q_{i_2}\mathbf{e}_{m_2}$. Similarly, we can get the new m_2-th column $\mathbf{e}_{m_2}$ and the new $(D+m_2)$-th column $\mathbf{0}$ by applying Lemma 1. By continuing in this manner, we can always get $q_{i_1}\mathbf{e}_{m_n}$ and $q_{i_2}\mathbf{e}_{m_n}$ in the m_n-th and the $(D+m_n)$-th columns in the n-th step, respectively, for $3 \leq n \leq D$. Therefore, we can always get two new columns $\mathbf{e}_{m_n}$ and $\mathbf{0}$ in the n-th step, for $3 \leq n \leq D$. After D steps, we get all vectors $\mathbf{e}_n$, for $1 \leq n \leq D$. Finally, by rearranging the newly obtained columns, we obtain the Smith form of $\mathbf{A}$ is $(\mathbf{I} \quad \mathbf{0})$. This proves that $\mathbf{M}_{i_1,j}$ and $\mathbf{M}_{i_2,j}$ are co-prime, for any $1 \leq i_1 \neq i_2 \leq L$ and $1 \leq j \leq D$.

We next prove the second part. Consider two matrices $\mathbf{M}_{i_1,j_1}$ and $\mathbf{M}_{i_2,j_2}$ for any given $1 \leq i_1, i_2 \leq L$ and $1 \leq j_1 \neq j_2 \leq D$. Let $\mathbf{B} = (\mathbf{M}_{i_1,j_1} \quad \mathbf{M}_{i_2,j_2})$. We then calculate the Smith form of $D \times 2D$ matrix $\mathbf{B}$.

Let $\sigma_{j_2} = (m_1, m_2, \cdots, m_D)$, where $m_D = j_2$. We first show that $\mathbf{M}_{i_2,j_2}$ can be transformed to the form of

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & q_{i_2} \\ 0 & 0 & \cdots & 1 & q_{i_2} & 0 \\ 0 & 0 & \iddots & q_{i_2} & 0 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\ 1 & q_{i_2} & \iddots & 0 & 0 & 0 \\ q_{i_2} & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

by rearranging the rows and columns.

We first rearrange the rows of $\mathbf{M}_{i_2,j_2}$. After performing some proper row permutations, we can let the m_1-th row be the first row, the m_2-th row be the second row, the m_3-th row be the third row and so on until the m_D-th row be the last row. From the construction, the positions of 1's in $\mathbf{M}_{i_2,j_2}$ are (m_1, m_2), $(m_2, m_3), \cdots, (m_{D-1}, m_D)$ and the positions of q_{i_2}'s in $\mathbf{M}_{i_2,j_2}$ are (m_1, m_1), (m_2, m_2), $\cdots$, (m_D, m_D). In the newly obtained matrix, the positions of 1's are $(1, m_2)$, $(2, m_3)$, $\cdots$, $(D-1, m_D)$ and the positions of q_{i_2}'s are $(1, m_1)$, $(2, m_2)$, $\cdots$, (D, m_D). Then, we rearrange the columns of the newly obtained matrix by implementing column permutations. Let the m_D-th column be the first column, the m_{D-1}-th column be the second column and so on until the m_1-th column be the last column. Now, the positions of 1's are $(1, D-1)$, $(2, D-2)$, $\cdots$, $(D-1, 1)$ and the positions of q_{i_2}'s are $(1, D)$, $(2, D-1)$, $\cdots$, $(D, 1)$. It is the form in (14).

After performing the above elementary transformations on $\mathbf{B}$, we get a new matrix, denoted by $\mathbf{B}'$, and the second half of $\mathbf{B}'$ has the form of (14). We claim that there must be a 1 in the last row of $\mathbf{B}'$. Otherwise, the row in $\mathbf{M}_{i_1,j_1}$ and the row in $\mathbf{M}_{i_2,j_2}$ that have no 1 have the same row number. Since in this case, from the construction Step 1), we know that the j_1-th row of $\mathbf{M}_{i_1,j_1}$ has no 1 and the j_2-th row of $\mathbf{M}_{i_2,j_2}$ has no 1, since $\sigma_{j_1}(D) = j_1$ and $\sigma_{j_2}(D) = j_2$. Because from the construction, there is only one row has no 1 in a matrix, we have $j_1 = j_2$, which leads to contradiction. Also, from the construction, each row has one q_{i_1}. Thus, there are a 1 and a q_{i_1} in the last row of $\mathbf{B}'$. Let the column containing the 1 in the last row be the k_1-th column and the column containing the q_{i_1} in the last row be the k_2-th column. Without loss of generality, we assume $k_1 < k_2$. If not, we can exchange these two columns. Therefore, the matrix $\mathbf{B}'$ now has the form of (15) at the bottom of the next page.

We then calculate the Smith form step by step. In the first step, we use the $(2D-1)$-th column to eliminate the other non-zero elements in the first row by performing elementary column transformations. Then, by using the newly obtained the first row to perform elementary row transformations, we can get the new $(2D-1)$-th column $\mathbf{e}_1$. By continuing in this manner, in each step n, for $2 \le n \le D-1$, we can always identify a 1 in the $(2D-n)$-th position of the n-th row. Then, using the $(2D-n)$-th column, we perform elementary column transformations to eliminate the other non-zero elements in the n-th row. Then, by using the newly obtained the n-th row to perform elementary row transformations, we can eliminate the other non-zero elements in this column and get the new $(2D-n)$-th column $\mathbf{e}_n$. By now, we have get all vectors e_n, for $1 \le n \le D-1$.

We now consider the $(2D)$-th column, the k_1-th column, and the k_2-th column after the above $(D-1)$ steps. Based on the previous discussion, each time 1 is used to eliminate the other non-zero elements in the same row, there exists an adjacent element q_{i_2} in the next row of the same column with the chosen 1. As a result, when we eliminate a non-zero element from the current row, a new non-zero element is introduced in the next row within the same column with this non-zero element. Therefore, in each step n, for $1 \le n \le D-1$, we need to eliminate the non-zero element $(-1)^{n-1} q_{i_2}^n$ in the n-th position of the $(2D)$-th column, so the $(2D)$-th column becomes $[0, \cdots, 0, (-1)^{D-1} q_{i_2}^D]^\top$ after $(D-1)$ steps.

From the construction, the k_1-th column of $\mathbf{B}'$ must has a single 1, a single q_{i_1}, and $(D-2)$ zeros. Assume this q_{i_1} in the k_1-th column is in the r_1-th row, $1 \le r_1 \le D-1$. All the elements in the column are zero for the first (r_1-1) rows, meaning that the first (r_1-1) steps have no effect on this column. Similar to the $(2D)$-th column, from step r_1 through step $D-1$, in each step we apply elementary column transformations to eliminate the non-zero element that appears in this column. Then, the new k_1-th column after the $(D-1)$ steps is

$$[0, \cdots, 0, 1 + (-1)^{D-r_1} q_{i_1} q_{i_2}^{D-r_1}]^\top.$$

The k_2-th column of $\mathbf{B}'$ may have or may not have a 1. Next, we will discuss these two cases separately and prove that, in both cases, we can obtain $\mathbf{e}_D$ by performing elementary column transformations on these three columns.

Case 1: the k_2-th column of $\mathbf{B}'$ has no 1.

In this case, the k_2-th column of $\mathbf{B}'$ is $[0, \cdots, 0, q_{i_1}]^\top$. Since there are no non-zero elements in the first $(D-1)$ positions, the k_2-th column remains $[0, \cdots, 0, q_{i_1}]^\top$ after the $(D-1)$ steps. Then, we can get a column of $\mathbf{e}_D$ by multiplying the k_2-th column by $(-1)^{D-r_1+1} q_{i_2}^{D-r_1}$ and adding it to the k_1-th column.

Case 2: the k_2-th column of $\mathbf{B}'$ has a 1.

In this case, assume that the 1 is in the r_2-th position of the k_2-th column, $1 \le r_2 \le D-1$. Note that $r_1 \ne r_2$, otherwise, there must be a_1 and a_2 from $\{1, 2, \cdots, D\}$ such that the elements at (a_1, a_2) position and (a_2, a_1) position of $\mathbf{M}_{i_1, j_1}$ are all 1, which is impossible from the construction. Similar to the k_1-th column, the newly obtained k_2-th column after the $(D-1)$ steps is

$$[0, \cdots, 0, q_{i_1} + (-1)^{D-r_2} q_{i_2}^{D-r_2}]^\top.$$

Since the power of q_{i_2} in the $(2D)$-th column is larger than that in the k_2-th column, we can obtain the new $(2D)$-th column

$$[0, \cdots, 0, (-1)^{r_2} q_{i_1} q_{i_2}^{r_2}]^\top$$

by multiplying $(-1)^{r_2} q_{i_2}^{r_2}$ to the k_2-th column and adding it to the $(2D)$-th column.

If $r_2 \le D - r_1$, we can get a column of $\mathbf{e}_D$ by multiplying the $(2D)$-th column by

$$(-1)^{D-r_1-r_2+1} q_{i_2}^{D-r_1-r_2}$$

and adding it to the k_1-th column.

If $r_2 > D - r_1$, we multiply the k_1-th column by

$$(-1)^{r_2-(D-r_1)+1} q_{i_2}^{r_2-(D-r_1)}$$

and add it to the $(2D)$-th column. We then get the new $(2D)$-th column

$$[0, \cdots, 0, (-1)^{r_2-(D-r_1)+1} q_{i_2}^{r_2-(D-r_1)}]^\top.$$

If $r_2 - (D - r_1) \le D - r_1$, we can obtain a column of $\mathbf{e}_D$ by multiplying

$$(-1)^{2(D-r_1)-r_2} q_{i_1} q_{i_2}^{2(D-r_1)-r_2}$$

to the $(2D)$-th column and adding it to the k_1-th column.

If $r_2 - (D - r_1) > D - r_1$, we then compare $r_2 - (D - r_1)$ and $D - r_2$. If $r_2 - (D - r_1) \le D - r_2$, we can get $q_{i_1} \mathbf{e}_D$ by multiplying

$$(-1)^{(D-r_2)-r_2+(D-r_1)} q_{i_2}^{(D-r_2)-r_2+(D-r_1)}$$

to the $(2D)$-th column and adding it to the k_2-th column. Then, we can obtain a column of $\mathbf{e}_D$ by multiplying $(-1)^{D-r_1+1} q_{i_2}^{D-r_1}$ to the k_2-th column and adding it to the k_1-th column.

From the steps above, it can be observed that when the power of q_{i_2} in the $(2D)$-th column exceeds that in the k_1-th column or

$$\begin{pmatrix} * & \cdots & * & \cdots & * & \cdots & * & 0 & 0 & \cdots & 0 & 1 & q_{i_2} \\ * & \cdots & * & \cdots & * & \cdots & * & 0 & 0 & \cdots & 1 & q_{i_2} & 0 \\ * & \cdots & * & \cdots & * & \cdots & * & 0 & 0 & \iddots & q_{i_2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots & \vdots \\ * & \cdots & * & \cdots & * & \cdots & * & 1 & q_{i_2} & \iddots & 0 & 0 & 0 \\ * & \cdots & 1 & \cdots & q_{i_1} & \cdots & * & q_{i_2} & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad (15)$$

k_2-th column, we can use elementary column transformations with the k_1-th column or the k_2-th column to reduce the power of q_{i_2} in the $(2D)$-th column by amount of $D-r_1$ or $D-r_2$, respectively. Since D is a finite integer, after a finite number of steps, we are guaranteed to obtain a new $(2D)$-th column

$$[0,\cdots,0,(-1)^{b_1}q_{i_1}q_{i_2}^{b_2}]^\top$$

with $b_2 \le D-r_1$ or a new $(2D)$-th column

$$[0,\cdots,0,(-1)^{b_3}q_{i_2}^{b_4}]^\top$$

with $b_4 \le D-r_1$ or $b_4 \le D-r_2$. Then, we can use this $(2D)$-th column to obtain a column of $\mathbf{e}_D$ by performing elementary column transformations.

Thus, combining the above two cases, we can always get a column of $\mathbf{e}_D$. Then, by eliminating the other elements in the D-th row and rearranging all the columns, the Smith form of $\mathbf{B}$ is $(\mathbf{I}\quad \mathbf{0})$. This means that $\mathbf{M}_{i_1,j_1}$ and $\mathbf{M}_{i_2,j_2}$ are co-prime, for any $1\le i_1,i_2\le L$ and $1\le j_1\ne j_2\le D$. This proves the second part.

By combining the above two parts, we have completed the proof. □

We next show that the above constructed matrices $\mathbf{M}_{i,j}$ are still pairwise co-prime when we change some elements of the matrices, which provides much more selections of pairwise co-prime integer matrices.

Corollary 1: If the signs of any (one or more) elements in any matrix within the family constructed by Algorithm 1 are changed, the modified integer matrices are still pairwise co-prime.

Proof: For the proof we only need to simply revise the proof of Theorem 1 and show that these changes don't affect their coprimality by proving that these changes don't affect the calculations of the Smith forms in the proof of Theorem 1.

First, we show that if we change the signs of any elements in the matrix $\mathbf{A}$ of the proof of Theorem 1, we can also obtain the Smith form of the newly obtained matrix, denoted by $\mathbf{A}_1$, as $(\mathbf{I}\quad \mathbf{0})$.

Compute the Smith form of $\mathbf{A}_1$ by following the same steps used in computing the Smith form of $\mathbf{A}$. In each step n, for $1\le n\le D$, we can always get $q_{i_1}\mathbf{e}_{m_n}$ and $q_{i_2}\mathbf{e}_{m_n}$ in the m_n-th and the $(D+m_n)$-th columns by multiplying 1 or -1 to the m_n-th and the $(D+m_n)$-th column. Therefore, we can always get $\mathbf{e}_{m_n}$ in the n-th step. After D steps, we get all vectors $\mathbf{e}_n$, for $1\le n\le D$. Finally, by rearranging the newly obtained columns, we obtain the Smith form of $\mathbf{A}_1$ as $(\mathbf{I}\quad \mathbf{0})$. This proves the first part.

Second, we show that if we change the signs of any elements in the matrix $\mathbf{B}$ of the proof of Theorem 1, we can also obtain the Smith form of the newly obtained matrix, denoted by $\mathbf{B}_1$, as $(\mathbf{I}\quad \mathbf{0})$.

When we apply the same transformations to matrix $\mathbf{B}_1$ as we did to matrix $\mathbf{B}$ in the proof of Theorem 1, we obtain a new matrix $\mathbf{B}_1^{'}$, which can also be obtained by changing the signs of any elements in matrix $\mathbf{B}^{'}$. Next, we calculate the Smith form of $\mathbf{B}_1^{'}$ with the same steps in Theorem 1.

In each step n, for $1\le n\le D-1$, we can always identify a 1 in the $(2D-n)$-th position of the n-th row by multiplying 1 or -1 to the $(2D-n)$-th column. Therefore, we can always get $\mathbf{e}_n$ in the n-th step.

After $(D-1)$ steps, the $(2D)$-th column becomes

$$[0,\cdots,0,q_{i_2}^D]^\top \text{ or } [0,\cdots,0,-q_{i_2}^D]^\top,$$

the k_1-th column becomes

$$[0,\cdots,0,1+q_{i_1}q_{i_2}^{D-r_1}]^\top \text{ or } [0,\cdots,0,1-q_{i_1}q_{i_2}^{D-r_1}]^\top$$

by multiplying it by 1 or -1. If the k_2-th column of $\mathbf{B}_1^{'}$ has no 1, it is

$$[0,\cdots,0,q_{i_1}]^\top \text{ or } [0,\cdots,0,-q_{i_1}]^\top$$

after $D-1$ steps. We can get $\mathbf{e}_D$ by multiplying the k_2-th column by $q_{i_2}^{D-r_1}$ or $-q_{i_2}^{D-r_1}$ and adding it to the k_1-th column. If the k_2-th column of $\mathbf{B}_1^{'}$ has a 1, the newly obtained k_2-th column after $(D-1)$ steps is

$$[0,\cdots,0,q_{i_1}+q_{i_2}^{D-r_2}]^\top \text{ or } [0,\cdots,0,q_{i_1}-q_{i_2}^{D-r_2}]^\top$$

by multiplying it by 1 or -1. Then, we can use the same steps as in the proof of Theorem 1 to reduce the power of q_{i_2} in the $(2D)$-th column until we obtain a new $(2D)$-th column

$$[0,\cdots,0,(-1)^{b_1}q_{i_1}q_{i_2}^{b_2}]^\top$$

with $b_2\le D-r_1$ or a new $(2D)$-th column

$$[0,\cdots,0,(-1)^{b_3}q_{i_2}^{b_4}]^\top$$

with $b_4\le D-r_1$ or $b_4\le D-r_2$. Then, we can use this $(2D)$-th column to obtained $\mathbf{e}_D$ by performing elementary column transformations. Finally, by eliminating the other elements in the D-th row and rearranging all the columns, we get the Smith form of $\mathbf{B}_1$ as $(\mathbf{I}\quad \mathbf{0})$. This proves the second part.

Combining these two parts, the corollary is proved. □

Since multiplying any unimodular matrices from the right does not change the coprimality, the above newly constructed matrices after multiplying any unimodular matrices are still pairwise co-prime.

V. Determinants and Least Common Right Multiples

In this section, we first determine the determinants of the integer matrices constructed in Section III, which correspond to the sampling rates as mentioned in Introduction using the sampling matrices $\mathbf{M}_{i,j}$ in the multi-dimensional sampling problem described in 7) and 8) in Section II.

Theorem 2: The determinant of the matrix $\mathbf{M}_{i,j}$ is the product of all the diagonal elements, i.e., $\det(\mathbf{M}_{i,j})=q_i^D$, for $1\le i\le L$ and $1\le j\le D$.

Proof: From the construction Steps 1) and 2), the diagonals of $\mathbf{M}_{i,j}$ are all q_i and the chosen permutation is σ_j. For convenience, let $\sigma_j=(m_1,\cdots,m_{D-1},j)$. Next, we calculate the determinant of $\mathbf{M}_{i,j}$ by applying Laplace expansion.

From the construction Step 1), we know that the m_1-th column has no 1, which means that the m_1-th column of $\mathbf{M}_{i,j}$ is $q_i\mathbf{e}_{m_1}$. So, we expand the $\det(\mathbf{M}_{i,j})$ by the m_1-th column, and we have

$$\begin{aligned}\det(\mathbf{M}_{i,j}) &= (-1)^{m_1+m_1}q_i\det(\mathbf{M}_{i,j}^{\backslash(m_1)})\\ &= q_i\det(\mathbf{M}_{i,j}^{\backslash(m_1)}).\end{aligned}$$

where $\mathbf{M}_{i,j}^{\backslash(m_1)}$ is the $(D-1)\times(D-1)$ submatrix obtained by removing the m_1-th column and the m_1-th row of $\mathbf{M}_{i,j}$.

After removing the m_1-th row and the m_1-th column of $\mathbf{M}_{i,j}$, from the construction Step 1) we have that the elements in the m_2-th column of $\mathbf{M}_{i,j}$ are all zeros expect a single q_i, since the element 1 located at the position of (m_1, m_2) in $\mathbf{M}_{i,j}$ has been removed. Then, we can identify this column in matrix $\mathbf{M}_{i,j}^{\backslash(m_1)}$. Without loss of generality, we assume that this column is the k_1-th column of $\mathbf{M}_{i,j}^{\backslash(m_1)}$, for $1\le k_1\le D-1$. Besides, all q_i's of $\mathbf{M}_{i,j}^{\backslash(m_1)}$ are also in all the diagonals of $\mathbf{M}_{i,j}^{\backslash(m_1)}$, because $\mathbf{M}_{i,j}^{\backslash(m_1)}$ is obtained by deleting the m_1-th row and the m_1-th column of $\mathbf{M}_{i,j}$. We then expand $\det(\mathbf{M}_{i,j}^{\backslash(m_1)})$ by the k_1-th column and have

$$\begin{aligned}\det(\mathbf{M}_{i,j}^{\backslash(m_1)}) &= (-1)^{k_1+k_1} q_i \det(\mathbf{M}_{i,j}^{\backslash(m_1,m_2)})\\ &= q_i \det(\mathbf{M}_{i,j}^{\backslash(m_1,m_2)}),\end{aligned}$$

where $\mathbf{M}_{i,j}^{\backslash(m_1,m_2)}$ is the $(D-2)\times(D-2)$ submatrix obtained by removing the k_1-th column and the k_1-th row of $\mathbf{M}_{i,j}^{\backslash(m_1)}$, which is the same as removing the m_1-th and the m_2-th rows and the m_1-th and the m_2-th columns of $\mathbf{M}_{i,j}$.

Similarly, for each $3\le n\le D-2$, we can always identify a column of all zeros expect a single q_i in the k_{n-1}-th column of matrix $\mathbf{M}_{i,j}^{\backslash(m_1,\cdots,m_{n-1})}$, which is obtained by removing all m_l-th rows and all m_l-th columns of $\mathbf{M}_{i,j}$ for $1\le l\le n-1$. Actually, this column is the m_n-th column of the original matrix $\mathbf{M}_{i,j}$. Since all q_i's are in the diagonals of matrix $\mathbf{M}_{i,j}$ and the deleted rows and columns have the same indices in all the steps to get the next new matrices, the remaining q_i's are always in the diagonals of matrix $\mathbf{M}_{i,j}^{\backslash(m_1,\cdots,m_{n-1})}$. So, we can expand the $\det(\mathbf{M}_{i,j}^{\backslash(m_1,\cdots,m_{n-1})})$ by this column and have

$$\begin{aligned}\det(\mathbf{M}_{i,j}^{\backslash(m_1,\cdots,m_{n-1})}) &= (-1)^{k_{n-1}+k_{n-1}} q_i \det(\mathbf{M}_{i,j}^{\backslash(m_1,\cdots,m_n)})\\ &= q_i \det(\mathbf{M}_{i,j}^{\backslash(m_1,\cdots,m_n)}),\end{aligned}$$

where $\mathbf{M}_{i,j}^{\backslash(m_1,\cdots,m_n)}$ is the $(D-n)\times(D-n)$ submatrix obtained by removing all the m_l-th columns and all the m_l-th rows of $\mathbf{M}_{i,j}$, for $1\le l\le n$.

Eventually, we can get that

$$\det(\mathbf{M}_{i,j}) = q_i^{D-2}\det\left(\begin{bmatrix} q_i & 0\\ 1 & q_i\end{bmatrix}\right)$$

or

$$\det(\mathbf{M}_{i,j}) = q_i^{D-2}\det\left(\begin{bmatrix} q_i & 1\\ 0 & q_i\end{bmatrix}\right),$$

which means that $\det(\mathbf{M}_{i,j}) = q_i^D$, i.e., the product of all the diagonal elements of $\mathbf{M}_{i,j}$. This completes the proof of Theorem 2. □

Similar to Corollary 1, from the above proof we immediately have the following corollary.

Corollary 2: If the signs of any (one or more) elements in any matrix $\mathbf{M}_{i,j}$ within the family constructed by Algorithm 1 are changed, the determinant of the modified matrix is the product of all the diagonal elements, i.e., $det(\mathbf{M}_{i,j}) = \pm q_i^D$, where $1\le i\le L$ and $1\le j\le D$.

Since $q_1, q_2, \cdots, q_L$ are pairwise co-prime, we can directly obtain the following corollary using one result in 4) of Section II, i.e., two integer matrices are co-prime if their determinants are co-prime [27].

Corollary 3: $\mathbf{M}_{i_1,j_1}$ and $\mathbf{M}_{i_2,j_2}$, for any $1\le i_1\ne i_2\le L$ and $1\le j_1, j_2\le D$, are co-prime.

This corollary directly leads to the proof of the first part in the proof of Theorem 1 in the previous section, i.e., for each fixed j, $1\le j\le D$, the group of matrices $\{\mathbf{M}_{i,j}, 1\le i\le L\}$ are pairwise co-prime, which, however, has only L integer matrices. One of our main contributions of the constructed family $\{\mathbf{M}_{i,j}\}$ in the previous section is being able to add $D-1$ many more integer matrices with the same determinant to the pairwise co-prime integer matrix family for each fixed i, i.e., for each fixed integer q_i in the known set of co-prime integers q_i, $1\le i\le L$. This leads to that the family of pairwise co-prime integer matrices in our new construction has DL many matrices. In addition, it also implies that the coprimality of determinants of two integer matrices is only a sufficient but not necessary condition for two integer matrices to be co-prime.

With Theorem 2, we immediately have the following absolute value of the determinant of the product of all the matrices in the constructed family:

$$\left|\det\left(\prod_{1\le i\le L, 1\le j\le D}\mathbf{M}_{i,j}\right)\right| = (q_1 q_2\cdots q_L)^{D^2}. \tag{16}$$

It is known that in one dimensional case, for the given pairwise co-prime integers $1<q_1<q_2<\cdots<q_L$, their least common multiple (lcm) is their product $q_1 q_2\cdots q_L$ that corresponds to the dynamic range of the conventional CRT using $q_1, q_2, \cdots, q_L$ as moduli. This means that all the nonnegative integers within the dynamic range can be uniquely determined by using CRT from their remainders modulo moduli q_i, $1\le i\le L$. However, due to the non-communtativity of the constructed integer matrices $\mathbf{M}_{i,j}$, their lcrm $\mathbf{R}$ may not be their product. Thus, it is not clear whether the value in (16) is the determinant absolute value of their lcrm, which is the dynamic range, i.e., the number of integer vectors that can be uniquely determined by using MD-CRT and their integer vector remainders modulo the matrix moduli as mentioned in 8) in Section II) similar to the conventional CRT [32], [33]. Next, we first determine an lcrm of the constructed family $\mathbf{M}_{i,j}$ for the two dimensional case, i.e., $D=2$, without the need of symbolic computations.

The 2×2 pairwise co-prime integer matrices constructed by our proposed method (or Algorithm 1) are: for $1\le i\le L$,

$$\mathbf{M}_{i,1} = \begin{pmatrix} q_i & 0\\ 1 & q_i\end{pmatrix}, \mathbf{M}_{i,2} = \begin{pmatrix} q_i & 1\\ 0 & q_i\end{pmatrix}. \tag{17}$$

As mentioned before, the above 2×2 integer matrices happen to be Toeplitz and do satisfy the necessary and sufficient condition for two 2×2 integer (Toeplitz) matrices to be co-prime obtained in [29].

We now calculate lcrm$(\mathbf{M}_{i,1}, \mathbf{M}_{i,2})$ following [22], [33], for $1\le i\le L$. First, we calculate

$$\mathbf{M}_{i,1}^{-1}\mathbf{M}_{i,2} = \begin{pmatrix} 1 & 1/q_i\\ -1/q_i & (q_i^2-1)/q_i^2\end{pmatrix}.$$

The lcm of the denominators of all the elements in $\mathbf{M}_{i,1}^{-1}\mathbf{M}_{i,2}$ is q_i^2. One can easily check that the Smith form of matrix $q_i^2\mathbf{M}_{i,1}^{-1}\mathbf{M}_{i,2}$ is

$$\begin{aligned}&\mathbf{U}(q_i^2\mathbf{M}_{i,1}^{-1}\mathbf{M}_{i,2})\mathbf{V}\\&=\begin{pmatrix}0 & 1\\ 1 & q_i^3+q_i\end{pmatrix}\begin{pmatrix}q_i^2 & q_i\\ -q_i & q_i^2-1\end{pmatrix}\begin{pmatrix}-q_i & q_i^2-1\\ -1 & q_i\end{pmatrix}\\&=\begin{pmatrix}1 & 0\\ 0 & q_i^4\end{pmatrix}.\end{aligned}$$

Let

$$\mathbf{\Lambda}=\frac{1}{q_i^2}\begin{pmatrix}1 & 0\\ 0 & q_i^4\end{pmatrix}=\begin{pmatrix}\frac{1}{q_i^2} & 0\\ 0 & q_i^2\end{pmatrix},$$

$\mathbf{\Lambda}_\alpha$ be a diagonal matrix whose diagonal elements are formed by taking the numerators of all the diagonal elements of $\mathbf{\Lambda}$ and $\mathbf{\Lambda}_\beta$ be a diagonal matrix whose diagonal elements are formed by taking the denominators of all the diagonal elements of $\mathbf{\Lambda}$.[1] Then, we can get $\mathbf{\Lambda}_\alpha=\mathrm{diag}(1,q_i^2)$ and $\mathbf{\Lambda}_\beta=\mathrm{diag}(q_i^2,1)$. Therefore, an lcrm of $\mathbf{M}_{i,1}$ and $\mathbf{M}_{i,2}$ is

$$\mathbf{M}_{i,1}\mathbf{U}^{-1}\mathbf{\Lambda}_\alpha=\mathbf{M}_{i,2}\mathbf{V}\mathbf{\Lambda}_\beta=\begin{pmatrix}-q_i^4-q_i^2 & q_i^3\\ -q_i^3 & q_i^2\end{pmatrix},$$

and $\det(\mathrm{lcrm}(\mathbf{M}_{i,1},\mathbf{M}_{i,2}))=-q_i^4$.

Let $\mathbf{R}$ be any lcrm of matrices $\mathbf{M}_{i,1},\mathbf{M}_{i,2}$ for all $1\le i\le L$. From (2), $\mathbf{R}$ is also an lcrm of matrices $\mathrm{lcrm}(\mathbf{M}_{i,1},\mathbf{M}_{i,2})$ for all $1\le i\le L$. For the sake of convenience, let $\mathbf{M}_i=\mathrm{lcrm}(\mathbf{M}_{i,1},\mathbf{M}_{i,2})$. There exists a nonsingular integer matrix $\mathbf{P}_i$ such that $\mathbf{R}=\mathbf{M}_i\mathbf{P}_i$ for each $1\le i\le L$. Then, we can get

$$|\det(\mathbf{R})|=|\det(\mathbf{M}_i)||\det(\mathbf{P}_i)|=q_i^4|\det(\mathbf{P}_i)|,$$

which means that $|\det(\mathbf{R})|$ has a divisor q_i^4, for every $1\le i\le L$. Since $q_1,q_2,\cdots,q_L$ are pairwise co-prime integers, we have that $|\det(\mathbf{R})|$ has a divisor $(q_1q_2\cdots q_L)^4$. Therefore, $|\det(\mathbf{R})|\ge(q_1q_2\cdots q_L)^4$.

We now show the following matrix

$$\mathbf{R}_2=\begin{pmatrix}(q_1q_2\cdots q_L)^2 & 0\\ 0 & (q_1q_2\cdots q_L)^2\end{pmatrix}\tag{18}$$

is an lcrm of matrices $\mathbf{M}_{i,1},\mathbf{M}_{i,2}$ for all $1\le i\le L$. For each matrix $\mathbf{M}_{i,j}$, $1\le i\le L$ and $1\le j\le 2$, matrix $\mathbf{M}_{i,j}^{-1}\mathbf{R}_2$ is a nonsingular integer matrix since

$$\mathbf{M}_{i,j}^{-1}=\mathrm{adj}(\mathbf{M}_{i,j})/\det(\mathbf{M}_{i,j})\text{ and }\det(\mathbf{M}_{i,j})=q_i^2.$$

Therefore, it is a crm of matrices $\mathbf{M}_{i,1},\mathbf{M}_{i,2}$ for all $1\le i\le L$. As $\det(\mathbf{R}_2)=(q_1q_2\cdots q_L)^4$ and for any lcrm $\mathbf{R}$, its determinant absolute value $\det(\mathbf{R})\ge(q_1q_2\cdots q_L)^4$ as proved above, $\mathbf{R}_2$ has to be an lcrm of matrices $\mathbf{M}_{i,1},\mathbf{M}_{i,2}$ for all $1\le i\le L$. This proves the following theorem.

Theorem 3: The matrix $\mathbf{R}_2$ in (18) is an lcrm of the 2×2 pairwise co-prime matrices in (17) constructed by Algorithm 1.

We next study the case when the dimension D is more than 2, i.e., $D>2$. For all $3\le D\le 75$, when the feasible permutation set is the set of all cyclic permutations (9), we have utilized Mathematica to perform symbolic calculations following the algorithm in [22], [33] and determined that for each group of matrices $\{\mathbf{M}_{i,j},1\le j\le D\}$,

$$|\det(\mathrm{lcrm}(\mathbf{M}_{i,j},1\le j\le D))|=q_i^{D^2}.$$

Similar to the case of $D=2$, the determinant absolute value of an lcrm of all the matrices $\mathbf{M}_{i,j}$, $1\le i\le L$ and $1\le j\le D$, in the constructed family is greater than or equal to $(q_1q_2\cdots q_L)^{D^2}$.

Let $\mathbf{R}_D$ be the following $D\times D$ diagonal matrix:

$$\mathbf{R}_D=(q_1q_2\cdots q_L)^D\,\mathbf{I}.\tag{19}$$

For each matrix $\mathbf{M}_{i,j}$, $1\le i\le L$ and $1\le j\le L$, matrix $\mathbf{M}_{i,j}^{-1}\mathbf{R}_D$ is a nonsingular integer matrix since

$$\mathbf{M}_{i,j}^{-1}=\mathrm{adj}(\mathbf{M}_{i,j})/\det(\mathbf{M}_{i,j})\text{ and }\det(\mathbf{M}_{i,j})=q_i^D.$$

Therefore, it is a crm of all the matrices $\mathbf{M}_{i,j}$, $1\le i\le L$ and $1\le j\le L$. Since

$$\det(\mathbf{R}_D)=(q_1q_2\cdots q_L)^{D^2},\tag{20}$$

$\mathbf{R}_D$ has to be an lcrm of all the matrices $\mathbf{M}_{i,j}$, $1\le i\le L$ and $1\le j\le L$, in the constructed family. This proves the following theorem.

Theorem 4: For $3\le D\le 75$, the matrix $\mathbf{R}_D$ in (19) is an lcrm of the $D\times D$ pairwise co-prime matrices constructed by Algorithm 1, where the feasible permutation set is chosen to be the set of all cyclic permutations (9).

For all $D>75$, due to our limited computational power, although we are not able to confirm the above result, we have the following conjecture.

Conjecture 1: The matrix $\mathbf{R}_D$ in (19) is an lcrm of the $D\times D$ pairwise co-prime matrices constructed by Algorithm 1 for $D>75$, where the feasible permutation set is chosen to be the set of all cyclic permutations (9).

If we can add any other integer matrix $\mathbf{M}$ with $|\det(\mathbf{M})|=q_i^d$ for some integers i and d, $1\le i\le L$ and $1\le d\le D$, which is not included in our constructed family, to our constructed family, $\mathbf{R}_D$ is still an lcrm of these $DL+1$ matrices, i.e., $\mathrm{lcrm}(\mathbf{R}_D,\mathbf{M})=\mathbf{R}_D$, since in this case, $\mathbf{R}_D$ is a right multiple of $\mathbf{M}$. This implies that for all $D\times D$ integer matrices with their determinant absolute values q_i^d for $1\le i\le L,1\le d\le D$, no matter they are pairwise co-prime or not, $\mathbf{R}_D$ is an lcrm of all these $D\times D$ integer matrices.

In the meantime, from the above discussions, it is not hard to see that if any member in our constructed family is removed from the family, the lcrm of the newly formed family of integer matrices has strictly less determinant absolute value than $|\det(\mathbf{R}_D)|$, since in this case q_i^D for some i, $1\le i\le L$, will not be included in $|\det(\mathbf{R}_D)|$. Thus, we have proved the following corollary.

Corollary 4: The constructed family $\mathbf{M}_{i,j}$, $1\le i\le L,1\le j\le D$, is the smallest family of integer matrices with determinant absolute values q_i^d for $1\le i\le L$ and $1\le d\le D$ such that $\mathbf{R}_D$ in (19) is their lcrm.

As mentioned in 7) and 8) of Section II, when $\mathbf{M}_{i,j}$ are used as sampling matrices, $|\det(\mathbf{M}_{i,j})|$ are their sampling rates

[1]In the algorithm of calculating an lcrm of two non-singular integer matrices, all fractions used to generate these two diagonal matrices Λ_α and Λ_β are in irreducible forms.

and $|\det(\mathbf{R_D})|$ is their dynamic range. Thus, under the same sampling rates, our constructed family is the smallest set of sampling matrices to achieve the maximal dynamic range.

From the above results, we also see that the number of integer vectors that can be uniquely determined from their integer vector remainders modulo the constructed integer matrix moduli $\mathbf{M}_{i,j}$, $1 \le i \le L, 1 \le j \le D$, using MD-CRT, i.e., the dynamic range, is $(q_1 q_2 \cdots q_L)^{D^2}$. Notice that since these matrix moduli $\mathbf{M}_{i,j}$, $1 \le i \le L, 1 \le j \le D$, do not commute, they cannot be diagonalized simultaneously in any sense. Thus, the corresponding MD-CRT cannot be equivalently converted to multiple conventional CRTs for integers. In other words, they are non-separable. On the other hand, to have the same dynamic range as that using multiple individual CRT for integers, i.e., the separable case, it is obvious to construct diagonal integer matrix moduli as, for $1 \le i \le L$,

$$\mathbf{D}_{i,1} = \text{diag}(q_i^D, 1, \cdots, 1), \cdots, \mathbf{D}_{i,D} = \text{diag}(1, \cdots, 1, q_i^D). \tag{21}$$

These diagonal integer matrices are clearly pairwise co-prime and their product is their lcrm, and thus their dynamic range is also $(q_1 q_2 \cdots q_L)^{D^2}$, the same as that of $\mathbf{M}_{i,j}$, $1 \le i \le L, 1 \le j \le D$. In the meantime, $\det(\mathbf{D}_{i,j}) = q_i^D = \det(\mathbf{M}_{i,j})$, i.e., the sampling rates of $\mathbf{D}_{i,j}$ and $\mathbf{M}_{i,j}$ are the same as well for $1 \le i \le L, 1 \le j \le D$. Note that only for the convenience in comparison, these sampling rates are counted in the sense of overall D dimensional volume-wise sampling rates for the multi-dimensional sampling matrices applied to the D dimensional real vector $(t_1, t_2, \cdots, t_D)^\top$. Below we analyze the sampling rates for each dimension t_j, i.e., how fast a sampling of each continuous real variable t_j is.

We first see that the ratios of the peak values and the average values of the components in integer matrices $\mathbf{M}_{i,j}$ and integer matrices $\mathbf{D}_{i,j}$ are, respectively,

$$\gamma_{\mathbf{M}_{i,j}} = \frac{D^2 q_i}{D q_i + D - 1} \approx D, \text{ when } q_i \text{ are large,} \tag{22}$$

and

$$\gamma_{\mathbf{D}_{i,j}} = \frac{D^2 q_i^D}{q_i^D + D - 1} \approx D^2, \text{ when } q_i \text{ are large.} \tag{23}$$

Also, the ratios of the peak values and the smallest non-zero absolute values of the components in integer matrices $\mathbf{M}_{i,j}$ and integer matrices $\mathbf{D}_{i,j}$ are, respectively, q_i and q_i^D. Clearly the above two ratios of $\mathbf{M}_{i,j}$ are smaller than those of $\mathbf{D}_{i,j}$.

We next consider the sampling rates on each dimension. For the above diagonal sampling matrices $\mathbf{D}_{i,j}$, for each i the diagonal element q_i^D means that the sampling rate in dimension j is q_i^D that could be too high in practice when q_i and (or) D are (is) large.

For the newly proposed sampling matrices $\mathbf{M}_{i,j}$ constructed in Algorithm 1, where all the elements in $\mathbf{M}_{i,j}$ are non-negative, from 1) in Section II one can see that FPD $\mathcal{N}(\mathbf{M}_{i,j})$ (or the component-wise inverses of its elements) corresponds to the unit spatial volume of $\mathbb{R}^D$ in the multi-dimensional sampling. We next show that for any dimension k, any line $\mathcal{L}_k$ included in set $\mathcal{N}(\mathbf{M}_{i,j})$, that is parallel to the k-th dimensional coordinate axis of variable t_k, has the largest value not above $q_i + 1$ and the smallest value not below 0. Since all elements in $\mathbf{M}_{i,j}$ are not negative, the smallest value on line $\mathcal{L}_k$ is not below 0. Next, we show that the largest value on line $\mathcal{L}_k$ is not above $q_i + 1$. This means that the sampling rate in dimension k for continuous variable t_k is not larger than $q_i + 1$ that is much smaller than the largest sampling rate q_i^D for the above diagonal sampling matrix $\mathbf{D}_{i,j}$.

For line $\mathcal{L}_k$ included in $\mathcal{N}(\mathbf{M}_{i,j})$ that is parallel to the k-th dimensional coordinate axis, let the two ending points of this line be $\mathbf{x}_l = [x_1, \cdots, x_{k,l}, \cdots, x_D]^\top$ for $l = 1, 2$. To show the largest value on the line $\mathcal{L}_k$ is smaller than $q_i + 1$, we only need to show that $x_{k,1}$ and $x_{k,2}$ are smaller than $q_i + 1$. From the definition of $\mathcal{N}(\mathbf{M}_{i,j})$ in 1) in Section II, there must be two vectors $\mathbf{a} = [a_1, a_2, \cdots, a_D]^\top$ and $\mathbf{b} = [b_1, b_2, \cdots, b_D]^\top$ in $[0, 1)^D$ such that

$$\mathbf{M}_{i,j}\mathbf{a} = \mathbf{x}_1 \quad \text{and} \quad \mathbf{M}_{i,j}\mathbf{b} = \mathbf{x}_2. \tag{24}$$

From the construction of $\mathbf{M}_{i,j}$, the k-th row of $\mathbf{M}_{i,j}$ may have no 1 or have a single 1.

If the k-th row of $\mathbf{M}_{i,j}$ has no 1, we can get $q_i a_k = x_{k,1}$ and $q_i b_k = x_{k,2}$ from (24). Since $0 \le a_k, b_k < 1$, we can obtain that $x_{k,1}$ and $x_{k,2}$ are smaller than q_i.

If the k-th row of $\mathbf{M}_{i,j}$ has a single 1, without loss of generality, let $\sigma_j = (m_1, \cdots, m_{D-1}, m_D)$. There must be an i, $1 \le i \le D - 1$, such that $k = m_i$. Then, from (24) we can get

$$a_{m_{i+1}} + q_i a_{m_i} = x_{k,1}, \quad \text{and} \quad b_{m_{i+1}} + q_i b_{m_i} = x_{k,2}. \tag{25}$$

Since $0 \le a_{m_{i+1}}, a_{m_i}, b_{m_{i+1}}, b_{m_i} < 1$, we conclude that $x_{k,1}$ and $x_{k,2}$ are smaller than $q_i + 1$. This completes the proof.

In summary, the above analysis tells us that for any i, $1 \le i \le L$, the sampling rate in each dimension of the sampling matrix $\mathbf{M}_{i,j}$ is no larger than $q_i + 1$ for any j, $1 \le j \le D$, while the sampling rate in one dimension of the sampling matrix $\mathbf{D}_{i,j}$ is q_i^D. This shows an advantage of non-separable sampling over separable sampling for a multi-dimensional signal. Next, for the illustration convenience, we only take a two dimensional example to show the sampling rate analysis result on each dimension.

Example 1: Let $q_i = 3$, the diagonal 2×2 sampling matrices are

$$\mathbf{D}_{i,1} = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_{i,2} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix},$$

and the sampling matrices $\mathbf{M}_{i,j}$ are

$$\mathbf{M}_{i,1} = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_{i,2} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

The FPDs of these four matrices, i.e., $\mathcal{N}(\mathbf{M}_{i,j})$ and $\mathcal{N}(\mathbf{D}_{i,j})$, are shown in Fig. 2. From the figure, we can easily see that the maximal sampling rate of $\mathbf{M}_{i,j}$ on each dimension is $4 = q_i + 1$, while the maximal sampling rate of $\mathbf{D}_{i,j}$ on each dimension is $9 = q_i^2$.

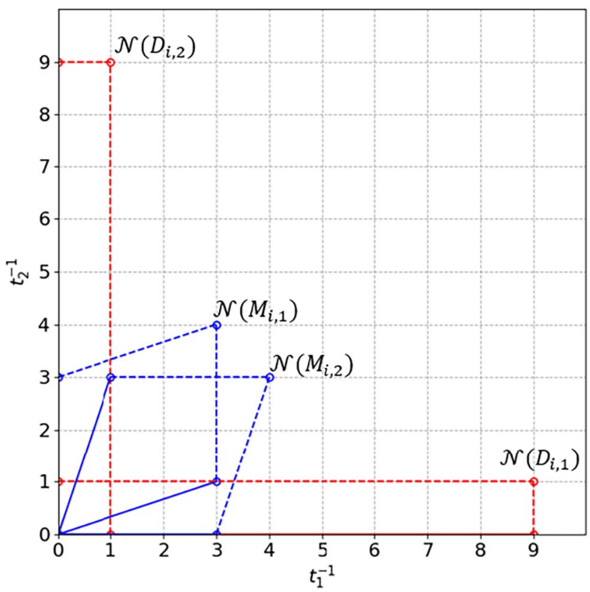

Fig. 2. FPDs of the four matrices in Example 1.

VI. Conclusion

In this paper, we have presented a new construction of pairwise co-prime integer matrices of any dimension and large size. They are non-commutative and have low ratios of peak absolute values over mean absolute values (or the smallest nonzero absolute values) of their components. We have also determined their least common right multiple (lcrm) with a closed and simple form. These integer matrices have applications in MD-CRT to determine integer vectors from their integer vector remainders, which may occur in undersamplings of multidimensional harmonic signals. Although the dynamic range of these non-diagonal integer matrices using MD-CRT can be achieved by diagonal integer matrices using separable CRT for each dimension, their sampling rates in each dimension are much smaller than the conventional ones. In other words, nonseparable sampling has a true advantage over separable sampling for a multi-dimensional signal. This means that the newly constructed pairwise co-prime integer matrices may also have applications in multi-dimensional sparse sensing and multidimensional multirate systems. Since any new construction of families of pairwise co-prime objects, such as co-prime integers, co-prime algebraic numbers, and co-prime integer matrices, is fundamental, we believe that the new families of pairwise co-prime integer matrices presented in this paper may have other applications as well.

References

[1] N. S. Szabo and R. I. Tanaka, *Residue Arithmetic and Its Application to Computer Technology*, New York, NY, USA: McGraw-Hill, 1967.

[2] C. Ding, D. Pei, and A. Salomaa, *Chinese Remainder Theorem: Applications in Computing, Coding, Cryptography.* Singapore: World Scientific, 1996.

[3] V. Guruswami, A. Sahai, and M. Sudan, "Soft-decision decoding of Chinese remainder codes," in *Proc. 41st IEEE Symp.,* Foundations Computer Sci., Redondo Beach, CA, 2000, pp. 159–168.

[4] J. H. McClellan and C. M. Rader, *Number Theory in Digital Signal Processing.* Englewood Cliffs, NJ, USA: Prentice-Hall, 1979.

[5] H. Krishna, K.-Y. Lin, and B. Krishna, "Rings, fields, the Chinese remainder theorem and an extension–Part II: Applications to digital signal processing," *IEEE Trans. Circuits Syst. II*, vol. 41, no. 10, pp. 656–668, Oct. 1994.

[6] H. Krishna, B. Krishna, K.-Y. Lin, and J.-D. Sun, *Computational Number Theory and Digital Signal Processing: Fast Algorithms and Error Control Techniques.* Boca Raton, FL, USA: CRC, 1994,

[7] M. A. Richards, *Fundamentals of Radar Signal Processing.* New York, NY, USA: McGraw-Hill, 2005.

[8] X.-G. Xia, "On estimation of multiple frequencies in undersampled complex valued waveforms," *IEEE Trans. Signal Process.*, vol. 47, no. 12, pp. 3417–3419, Dec. 1999.

[9] G. Li, J. Xu, Y.-N. Peng, and X.-G. Xia, "Location and imaging of moving targets using non-uniform linear antenna array," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 43, no. 3, pp. 1214–1220, Jun. 2007.

[10] A. Christlieb, D. Lawlor, and Y. Wang, "A multiscale sub-linear time Fourier algorithm for noisy data," *Appl. Comput. Harmon. Anal*, vol. 40, no. 3, pp. 553–574, 2016.

[11] W. C. Li, X. Z. Wang, and B. Moran, "Wireless signal travel distance estimation using non-coprime wavelengths," *IEEE Signal Process. Lett.*, vol. 24, no. 1, pp. 27–31, Jan. 2017.

[12] B. Silva and G. Fraidenraich, "Performance analysis of the classic and robust Chinese remainder theorems in pulsed Doppler radars," *IEEE Trans. Signal Process.*, vol. 66, no. 18, pp. 4898–4903, Sep. 2018.

[13] L. Xiao and X.-G. Xia, "Frequency determination from truly sub-Nyquist samplers based on robust Chinese remainder theorem," *Signal Process.*, vol. 150, pp. 248–258, Sep. 2018.

[14] C. Li, L. Gan, and C. Ling, "Coprime sensing via Chinese remaindering over quadratic fields–Part I: Array designs," *IEEE Trans. Signal Process.*, vol. 67, no. 11, pp. 2898–2910, Jun. 2019.

[15] C. Li, L. Gan, and C. Ling, "Coprime sensing via Chinese remaindering over quadratic fields–Part II: Generalizations and applications," *IEEE Trans. Signal Process*, vol. 67, no. 11, pp. 2911–2922, Jun. 2019.

[16] C. Li, L. Gan, and C. Ling, "3D coprime arrays in sparse sensing," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP),* Brighton, U.K., May 2019, pp. 4200–4204.

[17] Y. Gong, L. Gan, and H. Liu, "Multi-channel modulo samplers constructed from Gaussian integers," *IEEE Signal Process. Lett.*, vol. 28, pp. 1828–1832, 2021.

[18] H. J. S. Smith, "On systems of linear indeterminate equations and congruences," *Philos. Trans. Roy. Soc. London*, vol. 151, pp. 293–326, Dec. 1861.

[19] C. C. MacDuffee, *The Theory of Matrices*, New York, NY, USA: Chelsea, 1946.

[20] D. E. Dudgeon and R. M. Mersereau, *Multidimensional Signal Processing.* Englewood Cliffs, NJ, USA: Prentice Hall, 1984.

[21] A. Guessoum and R. Mersereau, "Fast algorithms for the multidimensional discrete Fourier transform," *IEEE Trans. Acoust. Speech Signal Process.*, vol. 34, no. 4, pp. 937–943, Aug. 1986.

[22] T. Chen and P. P. Vaidyanathan, "The role of integer matrices in multidimensional multirate systems," *IEEE Trans. Signal Process.*, vol. 41, no. 3, pp. 1035–1047, Mar. 1993.

[23] T. Chen and P. P. Vaidyanathan, "Recent developments in multidimensional multirate systems," *IEEE Trans. Circuits Syst. Video Technol.*, vol. 3, no. 2, pp. 116–137, Apr. 1993.

[24] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Englewood Cliffs, NJ, USA: Prentice Hall, 1993.

[25] Y.-P. Lin, S.-M. Phoong, and P. P. Vaidyanathan, "New results on multidimensional Chinese remainder theorem," *IEEE Signal Process. Lett.*, vol. 1, no. 11, pp. 176–178, Nov. 1994.

[26] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb. 2011.

[27] P. Pal and P. P. Vaidyanathan, "Coprimality of certain families of integer matrices," *IEEE Trans. Signal Process.*, vol. 59, no. 4, pp. 1481–1490, Apr. 2011.

[28] P. P. Vaidyanathan and P. Pal, "Generating new commuting coprime matrix pairs from known pairs," *IEEE Signal Process. Lett.*, vol. 18, no. 5, pp. 303–306, May 2011.

[29] P. P. Vaidyanathan and P. Pal, "A general approach to coprime pairs of matrices, based on minors," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3536–3548, Aug. 2011.

[30] P. P. Vaidyanathan and P. Pal, "Theory of sparse coprime sensing in multiple dimensions," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3592–3608, Aug. 2011.

[31] P. Pal and P. P. Vaidyanathan, "Nested arrays in two dimensions, Part I: Geometrical considerations," *IEEE Trans. Signal Process.*, vol. 60, no. 9, pp. 4694–4705, Sep. 2012.

[32] L. Xiao, X.-G. Xia, and Y.-P. Wang, "Exact and robust reconstructions of integer vectors based on multidimensional Chinese remainder theorem (MD-CRT)," *IEEE Trans. Signal Process.*, vol. 68, pp. 5349–5362, Sep. 2020.

[33] L. Xiao, H. Huo, and X.-G. Xia, "Robust multidimensional Chinese remainder theorem for integer vector reconstruction," *IEEE Trans. Signal Process.*, vol. 72, pp. 2364–2380, 2024.

[34] C. A. Bouman, *Foundations of Computational Imaging: A Model-Based Approach.* Philadelphia, PA, USA: SIAM, 2022.

[35] Q. Wu, F. Sun, P. Lan, G. Ding, and X. Zhang, "Two-dimensional direction-of-arrival estimation for co-prime planar arrays: A partial spectral search approach," *IEEE Sensors J.*, vol. 16, no. 14, pp. 5660–5669, Jul. 2016.

[36] X. Li, S. Sun, Q. Liao, and X.-G. Xia, "Maximum likelihood estimation based complex-valued robust Chinese remainder theorem and its fast algorithm," Mar. 2025, *arXiv:2503.18625*.

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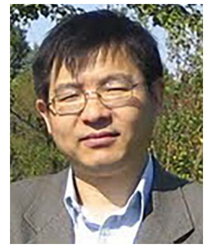

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