On Bandlimited Signals with Fractional Fourier Transform

Xiang-Gen Xia

Abstract— **In this letter, we study bandlimited signals with fractional Fourier transform (FRFT). We show that if a nonzero signal f is bandlimited with FRFT $\mathbf{F}_\alpha$ for a certain real α, then it is not bandlimited with FRFT $\mathbf{F}_\beta$ for any β with $\beta \neq \pm\alpha + n\pi$ for any integer n. This is a generalization of the fact that a nonzero signal can not be both timelimited and bandlimited. We also provide sampling theorems for bandlimited signals with FRFT that are similar to the Shannon sampling theorem.**

I. Introduction

The FRACTIONAL Fourier transform (FRFT) has been recently studied and has found some applications in solving differential equations [1], [2], physics [1], [8]–[13], and signal processing [3]–[7]. As a rotation of the traditional Fourier transform, it has been used to show that a rotation of Wigner (or Radon-Wigner) distribution is still a Wigner (or Radon-Wigner) distribution. For more details, see [3]–[8] and [14]. We now briefly review its definition and some basic properties.

We borrow the notations used in [4]. For any real α, let

$$K_\alpha(t,u) = \begin{cases} \sqrt{\frac{1-j\cot\alpha}{2\pi}} e^{j\frac{t^2+u^2}{2}\cot\alpha - jut\csc\alpha}, & \text{if } \alpha \text{ is not a multiple of } \pi, \\ \delta(t-u), & \text{if } \alpha \text{ is a multiple of } 2\pi, \\ \delta(t+u), & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi. \end{cases} \quad (1)$$

With this transformation kernel, the FRFT with an angle α of a signal f is defined as

$$(\mathbf{F}_\alpha f)(u) = \int_{-\infty}^{\infty} f(t) K_\alpha(t,u)dt. \quad (2)$$

It is not hard to see that we have (3), which appears at the top of the next page. This says that $\mathbf{F}_{2n\pi}$ is the identity transformation $(\mathbf{F}_{(2n+1)\pi} f)\ (t) = f(-t)$, and $\mathbf{F}_{\frac{\pi}{2}}$ is the traditional Fourier transform. Moreover, the following rotation property (see [2] and [4]) holds:

$$\mathbf{F}_{\alpha+\beta} = \mathbf{F}_\alpha \mathbf{F}_\beta. \quad (4)$$

The inverse FRFT is the following:

$$\mathbf{F}_\alpha^{-1}(\mathbf{F}_\alpha f)(t) = \int_{-\infty}^{\infty} (\mathbf{F}_\alpha f)(u) K_{-\alpha}(u,t)du. \quad (5)$$

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The author is with Hughes Research Laboratories, Malibu, CA 90265 USA (e-mail: xxia@maxwell.hrl.hac.com).

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The conventional bandlimited signals play an important role in communications and signal processing. In this letter, we study bandlimited signals with FRFT and their sampling theorems.

II. Bandlimited Signals

Let $f \in L^2(\mathbf{R})$ and α be a real number. If there exists a positive Ω_α such that $(\mathbf{F}_\alpha f)(u) = 0$ for $|u| > \Omega_\alpha$, then f is said to be Ω_α bandlimited with FRFT $\mathbf{F}_\alpha$ or Ω_α *bandlimited with angle* α. With this definition, the following facts are clear. When $\alpha = n\pi$, a signal f is bandlimited with angle α is equivalent to that f is bandlimited in the conventional sense. When $\alpha = \frac{\pi}{2} + n\pi$, f is bandlimited with angle α and is equivalent to that f, which is bandlimited in the conventional sense. It is known that a signal cannot be timelimited and bandlimited simultaneously, i.e., a signal cannot be bandlimited with both angles 0 and $\frac{\pi}{2}$ simultaneously. A natural question arises: Can a signal be bandlimited with other two different angles α and β simultaneously? We will answer this question negatively. Before going to the result, we briefly review some properties for entire functions.

We first define the order and the type of an entire function. Let $f(z)$ be an entire function on the complex plane. Let

$$M(r) = \max_{|z|\le r>0} |f(z)|.$$

The order of the entire function f is defined by

$$\rho = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r)}{\log r}.$$

For example, the order of the entire function $f(z) = 2e^{z^2}$ is 2. We say that an entire function f with a positive order ρ is of type τ if

$$\overline{\lim_{r\to+\infty}} r^{-\rho}\log M(r) = \tau.$$

An entire function is said to be an exponential type if its order $\rho \le 1$ and its type $\tau < \infty$. It is well known that any conventional bandlimited signal is an entire function of exponential type, and the bandwidth is its type.

Lemma 1: Let f and g be two nonzero entire functions with orders ρ_f and ρ_g and types τ_f and τ_g, respectively. Assume that $\rho_f \le \rho_g$. Then, the order of the product fg is ρ_g. If $\rho_f < \rho_g$, then the type of fg is τ_g. If $\rho_f = \rho_g$, then the type of fg is $\max\{\tau_f, \tau_g\}$.

For a proof of this lemma and more details about entire functions, see [15] and [16].

$$(\mathbf{F}_\alpha f)(u) = \begin{cases} \sqrt{\frac{1-j\cot\alpha}{2\pi}} e^{j\frac{u^2}{2}\cot\alpha} \int_{-\infty}^{\infty} f(t) e^{j\frac{t^2}{2}\cot\alpha} e^{-jut\csc\alpha} dt, & \text{if } \alpha \text{ is not a multiple of } \pi, \\ f(t), & \text{if } \alpha \text{ is a multiple of } 2\pi, \\ f(-t), & \text{if } \alpha + \pi \text{ is a multiple of } 2\pi. \end{cases} \tag{3}$$

Assume a function f is Ω_α bandlimited with angle α and $\alpha \neq n\pi$ for any integer n. Then, by the inverse FRFT in (5)

$$f(t) = \sqrt{\frac{1+j\cot\alpha}{2\pi}} e^{-\frac{jt^2}{2}\cot\alpha} \times \int_{-\Omega_\alpha}^{\Omega_\alpha} (\mathbf{F}_\alpha f)(u) e^{-j\frac{u^2}{2}\cot\alpha} e^{jut\csc\alpha} du. \tag{6}$$

Let

$$g(t) = \int_{-\Omega_\alpha}^{\Omega_\alpha} (\mathbf{F}_\alpha f)(u) e^{-j\frac{u^2}{2}\cot\alpha} e^{jut\csc\alpha} du. \tag{7}$$

Then, g is conventionally bandlimited, and its order is 1 when f is almost surely not zero, and the time variable t is extended to the complex plane. Meanwhile, the function $e^{-j\frac{t^2}{2}\cot\alpha}$ has order 2 and type $\frac{|\cot\alpha|}{2}$ when $\alpha \neq n\frac{\pi}{2}$ for any integer n. Therefore, by Lemma 1, the signal f has order 2 and type $\frac{|\cot\alpha|}{2}$ when $\alpha \neq n\frac{\pi}{2}$ for any integer n. $|\cot\alpha| \neq |\cot\beta|$ if and only if $\beta \neq \pm\alpha + n\pi$ for any integer n. This implies that f cannot be bandlimited with angle β, where $\beta \neq \pm\alpha + n\pi$ for any integer n. When f is almost surely not zero, then f is an entire function from (6), and therefore, it cannot be timelimited, i.e., it cannot be bandlimited with angle $\alpha = n\pi$ for any integer n. Overall, we have proved the following theorem.

Theorem 1: If a nonzero signal f is bandlimited with angle α, then f cannot be bandlimited with another angle β, where $\beta \neq \pm\alpha + n\pi$ for any integer n.

When $\alpha = \frac{\pi}{2}$, $\beta = 0$, Theorem 1 states that if f is bandlimited, then f is not timelimited. When $\alpha = 0$, $\beta = \frac{\pi}{2}$, Theorem 1 states that if f is timelimited, then f is not bandlimited. Therefore, Theorem 1 is a generalization of the fact that a nonzero signal cannot be both timelimited and bandlimited.

We next study the sampling theorem for an Ω_α bandlimited signal f with angle $\alpha \neq n\pi$ for any integer n. In this case, f has the representation (6) or

$$f(t) = g(t)\sqrt{\frac{1+j\cot\alpha}{2\pi}} e^{-j\frac{t^2}{2}\cot\alpha}.$$

Since $g(t)$ is $\Omega_\alpha \csc\alpha$ bandlimited in the conventional sense, we may apply the Shannon sampling theorem to g, that is

$$g(t) = \sum_n g(n\Delta_\alpha) \frac{\sin[\Omega_\alpha(\csc\alpha)(t - n\Delta_\alpha)]}{\Omega_\alpha(\csc\alpha)(t - n\Delta_\alpha)}$$

where $\Delta_\alpha = \pi\sin\alpha/\Omega_\alpha$. Therefore

$$f(t) = \sqrt{\frac{1+j\cot\alpha}{2\pi}} e^{-j\frac{t^2}{2}\cot\alpha} \sum_n f(n\Delta_\alpha) e^{j\frac{n^2\Delta_\alpha^2}{2}\cot\alpha} \times \frac{\sin[\Omega_\alpha(\csc\alpha)(t - n\Delta_\alpha)]}{\Omega_\alpha(\csc\alpha)(t - n\Delta_\alpha)}. \tag{8}$$

Let $\alpha_1 \neq n\pi$ for any integer n, and α_2 be two real numbers such that $\alpha_1 + \alpha_2 = \alpha$. By the rotation property (4), $\mathbf{F}_{\alpha_2} f$ is Ω_α bandlimited with angle α_1 when f is Ω_α bandlimited with angle α, where α can be any real number. Therefore, the above sampling theorem (7) also applies to $\mathbf{F}_{\alpha_2} f$ with α replaced by α_1

$$(\mathbf{F}_{\alpha_2} f)(t) = \sqrt{\frac{1+j\cot\alpha_1}{2\pi}} e^{-j\frac{t^2}{2}\cot\alpha_1} \times \sum_n (\mathbf{F}_{\alpha_2} f)(n\Delta_{\alpha_1}) e^{j\frac{n^2\Delta_\alpha^2}{2}\cot\alpha_1} \times \frac{\sin[\Omega_\alpha(\csc\alpha_1)(t - n\Delta_\alpha)]}{\Omega_\alpha(\csc\alpha_1)(t - n\Delta_\alpha)} \tag{9}$$

where $\Delta_{\alpha_1} = \pi\sin\alpha_1/\Omega_\alpha$.

We can also directly apply the FRFT $\mathbf{F}_{\alpha_2}$ to the both sides in (8) when $\alpha \neq n\pi$ for any integer n

$$(\mathbf{F}_{\alpha_2} f)(t) = \sqrt{\frac{1+j\cot\alpha}{2\pi}} \sum_n f(n\Delta_\alpha) e^{j\frac{n^2\Delta_\alpha^2}{2}\cot\alpha} \times \mathbf{F}_{\alpha_2}\left[e^{-j\frac{t^2}{2}\cot\alpha} \frac{\sin[\Omega_\alpha(\csc\alpha)(t - n\Delta_\alpha)]}{\Omega_\alpha(\csc\alpha)(t - n\Delta_\alpha)}\right]. \tag{10}$$

This leads to the following theorem.

Theorem 2: Let f be Ω_α bandlimited with angle α. When $\alpha \neq n\pi$ for any integer n, the sampling theorem (8) is true for f. Let $\alpha_1 + \alpha_2 = \alpha$. The sampling theorem (9) is true for $\mathbf{F}_{\alpha_2} f$ when $\alpha_1 \neq n\pi$ for any integer n, where α may be any real number. The sampling theorem (10) is true for $\mathbf{F}_{\alpha_2} f$ when $\alpha \neq n\pi$ for any integer n, where α_1 may be any real number.

As special cases of Theorem 2, when $\alpha = \frac{\pi}{2}$, the identity (8) is the Shannon sampling theorem for bandlimited signals; when $\alpha = 0$ and $\alpha_1 = \frac{\pi}{2}$, the identity (9) is the Shannon sampling theorem for timelimited signals in the frequency domain.

III. Conclusion

In this letter, we have studied bandlimited signals with fractional Fourier transforms. We have shown that a nonzero signal cannot be bandlimited with two different angles α and β simultaneously when $\beta \neq \pm\alpha + n\pi$ for any integer n. This is a generalization of the fact that a nonzero signal cannot be both timelimited and bandlimited. We have also provided several sampling theorems for bandlimited signals with FRFT.

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