



Space-Time Codes, Orthogonal Designs, and Compositions of Quadratic Forms

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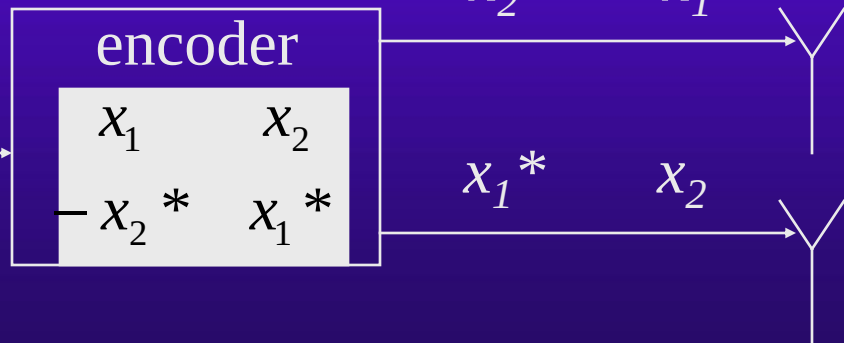
Alamouti Code for 2 Transmit Antennas (1998)

Alamouti code from 2 by 2 orthogonal design

$$\begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix}$$

It is an option in 3G

Information bits
are mapped to
complex symbols
 x_1 and x_2





Alamouti Scheme: Fast ML Decoding and Full Diversity

Signal Model:

$$Y=CA+W,$$

where

$$C \in \mathbf{C} = \left\{ \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix} : x_1, x_2 \in \mathcal{S} \right\}$$

$\mathcal{S}$ is a signal constellation, for example

$$\mathcal{S} = \{\pm 1, \pm j\}$$

Alamouti Code: Fast ML Decoding

- ML decoding is to minimize

$$\begin{aligned}\|Y - CA\|_F^2 &= \text{tr}\{(Y - CA)^H (Y - CA)\} \\ &= \text{tr}\{Y^H Y\} - \text{tr}\{Y^H CA - A^H C^H Y\} + \text{tr}\{AA^H C^H C\}\end{aligned}$$

- Orthogonality: $C^H C = (|x_1|^2 + |x_2|^2)I_2$
for any values x_1 and x_2 .

- ✓ The cross term $x_1 x_2$ can be canceled and x_1 and x_2 can be separated:

$$\|Y - CA\|_F^2 = f_1(x_1) + f_2(x_2)$$

- ✓ x_1 and x_2 can be decoded separately:

$$\min_{(x_1, x_2) \in S^2} = \min_{x_1 \in S} f_1(x_1) \quad \text{and} \quad \min_{x_2 \in S} f_2(x_2)$$

- ✓ The decoding complexity is reduced from $|S|^2$ to $2|S|$ i.e., **complex symbol-wise decoding**



Alamouti Code: Full Rank Property

- For any two different matrices

$$C = C(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix}, \tilde{C} = \tilde{C}(y_1, y_2) = \begin{pmatrix} y_1 & y_2 \\ -y_2^* & y_1^* \end{pmatrix}$$

$$C \neq \tilde{C} \Leftrightarrow (x_1, x_2) \neq (y_1, y_2)$$

- Their difference matrix is also orthogonal

$$B(C, \tilde{C}) = \begin{pmatrix} x_1 - y_1 & x_2 - y_2 \\ -(x_2 - y_2)^* & (x_1 - y_1)^* \end{pmatrix} = C(x_1 - y_1, x_2 - y_2)$$

- Because of the orthogonality, B has full rank

$$(B(C, \tilde{C}))^H B(C, \tilde{C}) = (|x_1 - y_1|^2 + |x_2 - y_2|^2) I_2$$

General Size

- For n transmit antennas and k information symbols $x_1, x_2, \dots, x_k$, the orthogonality $C^H C = (|x_1|^2 + \dots + |x_k|^2) I_n$ provides

- **Fast ML decoding:**

$$\| Y - CA \|_F^2 = \sum_{i=1}^k f_i(x_i),$$

then the minimization can be done separately

$$\min_{(x_1, x_2, \dots, x_k) \in \mathcal{A}^k} \| Y - CA \|_F^2 = \sum_{i=1}^k \min_{x_i \in \mathcal{A}} f_i(x_i).$$

In this case, the decoding complexity of the left term is $|\mathcal{A}|^k$, and that of the right term is $k|\mathcal{A}|$.

- **Full diversity:**

$$(B(C, C'))^H (B(C, C')) = (|x_1 - y_1|^2 + \dots + |x_k - y_k|^2) I_n.$$

- 
- Motivated by the Alamouti's scheme, Tarokh-Jafarkhani-Calderbank (1999) proposed space-time block codes from orthogonal designs for any number of transmit antennas.
 - Real orthogonal designs for real constellations, such as PAM.
 - Complex orthogonal designs for complex constellations, such as PSK and QAM.

Space-Time Block Codes from Real Orthogonal Designs

- A *real orthogonal design* in variables $x_1, x_2, \dots, x_k$ is a $p \times n$ matrix C such that:
 - The entries of C are $\pm x_1, \pm x_2, \dots, \pm x_k$, or their linear combinations of real coefficients.
 - $C^T C = (x_1^2 + x_2^2 + \dots + x_k^2) I_n$.
- n is the number of transmit antennas.
- p is the time delay in the en/decoding.
- k is the number of information symbols transmitted.

- The *rate* of C is defined as $R = k/p$, which means that p transmission time slots carry k information symbols.
 - For a given delay p , one wants k as large as possible to have the throughput (or transmission rate) as large as possible.
 - It is not hard to prove that $R \leq 1$, i.e., $k \leq p$.
Its proof is given in next slide.
- For a given n , a small delay p is also desired.

For $L=2$ transmit antennas:

$$k=p=2$$

$$\text{Rate } R=k/p=1$$

$$L_2 = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$$

♦ Proof of rate $R \leq 1$, i.e., $k \leq p$:

Let $C = \sum_{i=1}^k A_i x_i$, where $A_i, i = 1, 2, \dots, k$, are real $p \times n$ real constant matrices. Consider the first column of C , $(C)_1$:

$(C)_1 = \sum_{i=1}^k (A_i)_1 x_i$, where $(A_i)_1$ are the first columns of A_i . Due to the orthogonality of C ,

$$(C)_1^T (C)_1 = \sum_{i=1}^k |x_i|^2. \text{ Thus}$$

$$(C)_1^T (C)_1 = 0 \iff x_i = 0, i = 1, 2, \dots, k.$$

This proves that vectors $(A_i)_1, i = 1, 2, \dots, k$, of size $p \times 1$ are linearly independent. There are at most p linearly independent vectors in p dimensional space, so $k \leq p$.

- For 4 transmit antennas:

$$L_4 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}_{4 \times 4}.$$

- Number of transmit antennas $n = 4$
- $k = p = 4$
- Rate $R = k/p = 1$

◆ For 8 transmit antennas:

- $k=p=8$
- Rate $R=k/p=1$

$$L_8 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 & -x_8 & x_7 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 & x_8 & -x_5 & -x_6 \\ -x_4 & x_3 & -x_2 & x_1 & x_8 & -x_7 & x_6 & -x_5 \\ -x_5 & -x_6 & -x_7 & -x_8 & x_1 & x_2 & x_3 & x_4 \\ -x_6 & x_5 & -x_8 & x_7 & -x_2 & x_1 & -x_4 & x_3 \\ -x_7 & x_8 & x_5 & -x_6 & -x_3 & x_4 & x_1 & -x_2 \\ -x_8 & -x_7 & x_6 & x_5 & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}$$

Hurwitz Matrix Equations

- The real orthogonal space-time block code design is equivalent to the design of real matrices satisfying the Hurwitz matrix equations:
 - Write C as $C = A_1x_1 + A_2x_2 + \cdots + A_kx_k$, where A_i are some $p \times n$ real constant matrices and x_i are indeterminates.
 - Then, the orthogonality is equivalent to the following *Hurwitz matrix equations*:

$$\begin{aligned}A_i^T A_i &= I_n \text{ for } 1 \leq i \leq k \\ A_i^T A_j + A_j^T A_i &= 0_n \text{ for } 1 \leq i \neq j \leq k.\end{aligned}$$

Composition of Quadratic Forms

- The orthogonality is also equivalent to the following $[k,n,p]$ composition formula:

$$(x_1^2 + \cdots + x_k^2)(y_1^2 + \cdots + y_n^2) = z_1^2 + \cdots + z_p^2,$$

where $X = (x_1, \cdots, x_k)$ and $Y = (y_1, \cdots, y_n)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form in X and Y .

- View X, Y, Z as column vectors. Then $Z = CY$ and C is a $p \times n$ matrix whose entries are linear forms of X . Then

$$(x_1^2 + \cdots + x_k^2)Y^T Y = Z^T Z = Y^T C^T C Y.$$

- Since Y consists of indeterminates, the above equation is equivalent to

$$C^T C = (x_1^2 + \cdots + x_k^2)I_n.$$

Mathematical Historical Background

- n -square identities – $[n, n, n]$ composition formula, $(p = k = n)$
 - Complex numbers: $\alpha = x_1 + \mathbf{i}x_2$ and $\beta = y_1 + \mathbf{i}y_2$ then $|\alpha|^2|\beta|^2 = |\alpha\beta|^2$:

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = z_1^2 + z_2^2,$$

where $z_1 = x_1y_1 + x_2y_2$ and $z_2 = x_1y_2 - x_2y_1$.

– Quaternionic numbers (Hamilton's quaternions 1843) – Euler's formula

(1748): $x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2,$$

where

$$z_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

$$z_2 = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$$

$$z_3 = x_1y_3 - x_2y_4 - x_3y_1 + x_4y_2$$

$$z_4 = x_1y_4 + x_2y_3 - x_3y_2 - x_4y_1$$

- Legendre (1830) showed that 3-square identity is impossible.
- 8-square identity by Degen (1818) – Octonionic numbers – Cayley numbers (1845), also Graves (1843).
- Hurwitz (1898) showed that an n -square identity exists if and only if $n = 1, 2, 4, 8$, which is called “**1,2,4,8 Theorem.**”
- Radon (1922) constructively showed that, an $[p,n,p]$ formula exists over the real field if and only if $n \leq \rho(p)$, where $\rho(p)$ is the Hurwitz-Radon function defined as follows: if $p = 2^{4a+b}p_0$ for an odd p_0 and $0 \leq b \leq 3$ then $\rho(p) = 8a + 2^b$.
 - * This result implies that, for any transmit antenna number n , there exist a delay p and $k = p$ such that the $p \times n$ real orthogonal design of rate 1 exists.
 - * For n -identities, $k = n = p$.
 - * $n = \rho(p) = p$ if and only if $p = n = 1, 2, 4, 8$. This means that real $n \times n$ square orthogonal C of n variables has and only has sizes 1, 2, 4, 8.

- * For a given transmit antenna number n , the minimum time delay p is given by

$$p = \min 2^{4c+d},$$

where the minimization is taken over the set

$$\{c, d : 0 \leq d < 4, c \geq 0 \text{ and } 8c + 2^d \geq n\}.$$

For example, When $n = 2$, p is 2; When $n = 3, 4$, p is 4; When $n = 5, 6, 7, 8$, p is 8; and so on.

- $A_1, A_2, \dots, A_n$ of size $p \times k$ and $B_1, B_2, \dots, B_k$ of size $p \times n$ are two families of Hurwitz matrices **if and only if** the following two C are real orthogonal designs $C = [A_1 \mathbf{x}, \dots, A_n \mathbf{x}]$ where $\mathbf{x} = [x_1, \dots, x_k]^T$ and $C = B_1 x_1 + \dots + B_k x_k$ **Two different representations**

- There are n square Hurwitz matrices $A_1, A_2, \dots, A_n$ of size $p \times p$ by using Clifford algebra with $n \leq \rho(p) \rightarrow k=p \rightarrow \text{rate}=1$
- There are p Hurwitz matrices $B_1, B_2, \dots, B_k$ of size $p \times n$ with $n \leq \rho(p)$

The basic problem for real orthogonal designs or real space-time block codes for PAM signals is solved.

- Hurwitz (1923) independently showed the above result for $[k, n, n]$ composition formulas over the complex field, where $p = n$, i.e., square complex orthogonal designs as we shall see later.
- New proofs of the Hurwitz-Radon theory using representation theory of Clifford algebra: Eckmann (1943), Lee (1948), Albert (1942), Dubisch (1946).

- By relaxing the orthogonality of C into the linear independency, i.e., full rankness, Adams (1962) showed that the Hurwitz-Radon function bound still holds for the size of matrix C by using the Chern class over the vector fields.
- **Classic book:** A. V. Geramita and J. Seberry, *Orthogonal Designs, Quadratic Forms and Hadamard Matrices*, Lecture Notes in Pure and Appl. Math. 45, Marcel Dekker, New York, 1979.
- A recent book: D. B. Shapiro, *Compositions of Quadratic Forms*, New York, De Gruyter, 2000.
- Tarokh-Jafarkhani-Calderbank (1999) nicely linked it to the space-time coding.

Space-Time Block Codes from Complex Orthogonal Designs

- A *complex orthogonal design* (Tarokh-Jafarkhani-Calderbank 1999) in variables $x_1, x_2, \dots, x_k$ is a $p \times n$ matrix G such that:
 - The entries of G are complex linear combinations of $x_1, x_1^*, x_2, x_2^*, \dots, x_k, x_k^*$.
 - $G^H G = (|x_1|^2 + \dots + |x_k|^2) I_n$.
- A *generalized complex orthogonal design* (Tarokh-Jafarkhani-Calderbank 1999) in variables $x_1, x_2, \dots, x_k$ is a $p \times n$ matrix G such that:
 - The entries of G are complex linear combinations of $x_1, x_1^*, x_2, x_2^*, \dots, x_k, x_k^*$.
 - $G^H G = D$, where D is an $n \times n$ diagonal matrix with (i, i) th diagonal

element of the form

$$l_{i,1}|x_1|^2 + l_{i,2}|x_2|^2 + \cdots + l_{i,k}|x_k|^2,$$

where all the coefficients $l_{i,1}, l_{i,2}, \cdots, l_{i,k}$ are strictly positive numbers.

- This orthogonality also guarantees the fast ML decoding and the full diversity as in the real case.
- It is not hard to prove that $R \leq 1$, i.e., $k \leq p$.
- When the diagonal elements in matrix D are the same, its equivalent composition formula of quadratic forms becomes the $[k,n,p]$ *Hermitian composition formula*:

$$(|x_1|^2 + \cdots + |x_k|^2)(|y_1|^2 + \cdots + |y_n|^2) = |z_1|^2 + \cdots + |z_p|^2.$$

- Hurwitz (1923) constructively showed: If $p = n = 2^a b$ for an odd b , then $k \leq a + 1$ and $k = a + 1$ can be achieved.

- It was later generalized by Wolfe (1976) to the **amicable orthogonal designs**.
- It implies that the complex $n \times n$ **square** orthogonal G of n complex variables exists only when $n = 1, 2$, which is different from the real orthogonal designs even by excluding the one of size 8. This implies that the rate of a complex square orthogonal space-time code is less than 1 for more than 2 transmit antennas.
- There is not much known in the literature for *non-square* complex orthogonal designs (based on the communications with Shapiro).
- It constructively implies that the complex 4×4 square orthogonal G of 3 complex variables exists. The construction was explicitly re-given in Tarokh-Jafarkhani-Calderbank (1999)'s paper. This implies that the rates of complex orthogonal space-time codes for 3 and 4 transmit antennas can be $3/4$.

- From the amicable designs (the basic idea is similar to the Hurwitz's), the optimal rates and constructions of *square* codes ($p = n$) are obtained:
 - * $R = k/p = 1$ when $n = 1, 2$
 - * $R = k/p = 3/4$ when $n = 3, 4$
 - * $R = k/p = 1/2$ when $n = 5, 6, 7, 8$
 - * $R = k/p = 5/16$ when $9 \leq n \leq 16$
 - * $R \rightarrow 0$ as $n \rightarrow \infty$
- The rate $3/4$ codes for 3 and 4 transmit antennas can be simplified as (a few groups have presented in the literature):

$$G_4 = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ x_3^* & 0 & -x_1^* & x_2 \\ 0 & x_3^* & -x_2^* & -x_1 \end{bmatrix},$$

which is equivalent to the one given by Hurwitz in the 20's.

- Using the Hurwitz (1923) construction, the code rate is too small for more than 4 transmit antennas.
- Half rate, $R = 1/2$, complex orthogonal space-time codes from rate 1 real orthogonal designs (Tarokh-Jafarkhani-Calderbank (1999)):

$$G_n = \begin{bmatrix} C_n \\ C_n^* \end{bmatrix}, \quad \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \\ x_1^* & x_2^* \\ -x_2^* & x_1^* \end{bmatrix}$$

where C_n is a rate 1 real $k \times n$ orthogonal design. Thus, G_n is a $2k \times n$ design of rate $1/2$, i.e., $p = 2k$.

- Space-Time Codes don't have to be square!

- 
- ◆ Amicable family: family of matrices of the same size $\{A_1, \dots, A_s, B_1, \dots, B_t\}$ forms an amicable family if

$$\begin{aligned} (i) \quad & A_i^H A_i = B_j^H B_j = I, \quad 1 \leq i \leq s, 1 \leq j \leq t \\ (ii) \quad & \begin{cases} A_i^H A_j + A_j^H A_i = 0, & 1 \leq i, j \leq s \\ B_i^H B_j + B_j^H B_i = 0, & 1 \leq i, j \leq t \end{cases} \\ (iii) \quad & A_i^H B_j = B_j^H A_i, \quad 1 \leq i \leq s, 1 \leq j \leq t \end{aligned}$$

- ◆ $C(x_1, \dots, x_k) = \sum_{i=1}^k \{A_i \operatorname{Re}(x_i) + \mathbf{i} B_i \operatorname{Im}(x_i)\}$ is a complex orthogonal design iff $\{A_i, B_i, i=1, \dots, k\}$ is an Amicable design.

- 
- ◆ However the other representation

$$C(x_1, \dots, x_k) = [A_1 \mathbf{x}, \dots, A_n \mathbf{x}],$$

where $\mathbf{x} = (\text{Re}(x_1), \text{Im}(x_1), \dots, \text{Re}(x_k), \text{Im}(x_k))^T$

does not work!

- ◆ Use a different representation

$$[A_1 \mathbf{x} + B_1 \mathbf{x}^*, \dots, A_n \mathbf{x} + B_n \mathbf{x}^*]$$

where $\mathbf{x} = (x_1, \dots, x_k)^T$



Two Questions

- ◆ Can a non-square p by n complex orthogonal design have rate 1, i.e., $k=p$, when $n>2$? If not, what is the bound?
- ◆ How to construct rate over $\frac{1}{2}$ complex orthogonal designs?

Rate Upper Bounds for Complex Orthogonal Designs

- Liang-Xia'03 showed that their symbol rates, k/p , is strictly less than 1 for more than 2 transmit antennas.
- H.Wang-Xia'03 showed that their symbol rates, k/p , can not be above $\frac{3}{4}$ when $n > 2$, and conjectured that their symbol rates are upper bounded by

$$\frac{k}{p} \leq \frac{\left\lceil \frac{n}{2} \right\rceil + 1}{2 \left\lfloor \frac{n}{2} \right\rfloor}$$

For 3 and 4
transmit antennas

Rates $\frac{k}{p} = \frac{3}{4}$

- ✓ Su-Xia'03 **first** showed that $\frac{3}{4}$ holds for $n > 2$ when no linear processing is allowed.
- ✓ Liang'03 showed that this conjecture holds when no linear processing is allowed.
- H.Wang-Xia'03 showed that for a p by n generalized complex orthogonal design, the rate is upper bounded by $\frac{4}{5}$ when $n > 2$.

Two Main Lemmas

- Lemma (Liang-Xia): $\mathcal{G}$ is a generalized complex orthogonal design if and only if there exist diagonal positive definite matrices $E_i, i = 1, 2, \dots, n$, such that its associated matrices A_i and $B_i, i = 1, \dots, n$, satisfy the following conditions:

$$A_i^H A_j + B_j^t B_i^* = \delta_{ij} E_i, \quad A_i^H B_j + B_j^t A_i^* = 0, \quad B_i^H A_j + A_j^t B_i^* = 0, \quad (2)$$

or equivalently,

$$\begin{pmatrix} A_i & B_i \\ B_i^* & A_i^* \end{pmatrix}^H \begin{pmatrix} A_j & B_j \\ B_j^* & A_j^* \end{pmatrix} = \delta_{ij} \begin{pmatrix} E_i & 0 \\ 0 & E_j^* \end{pmatrix}, \quad (3)$$

for all $i, j = 1, \dots, n$, where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

In particular, $\mathcal{G}$ is a complex orthogonal design if and only if (2) or (3) holds for $E_i = I$ for $1 \leq i \leq n$.

- **Diagonalization Lemma (Wang-Xia):** Let A and B be two $p \times k$ matrices and satisfy conditions: $A^H A + B^t B^* = I$ and $A^H B + B^H A = 0$. Then, there exist a unitary matrix V of size $p \times p$ and a unitary matrix U of size $2k \times 2k$ such that the $p \times 2k$ matrix (A, B) can be diagonalized as follows

$$V(A, B)U = \Sigma \triangleq \begin{pmatrix} D_\lambda & 0 & 0 & 0 \\ 0 & I_{k-s} & 0 & 0 \\ 0 & 0 & D_\mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{p \times 2k}, \quad (4)$$

where $k - s \geq 2k - p$, $D_\lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_s)$,

$D_\mu = \text{diag}(\mu_1, \mu_2, \dots, \mu_s)$ and $\lambda_i^2 + \mu_i^2 = 1$,

$1 > \lambda_i \geq \sqrt{1/2} \geq \mu_j > 0, i, j = 1, 2, \dots, s, k + s = \kappa$, and

$\kappa = \text{rank}(A, B) \geq k$, and furthermore the $2k \times 2k$ unitary matrix U has the following form

$$U = \begin{pmatrix} W_1 & W_2 \\ W_2^* & W_1^* \end{pmatrix},$$

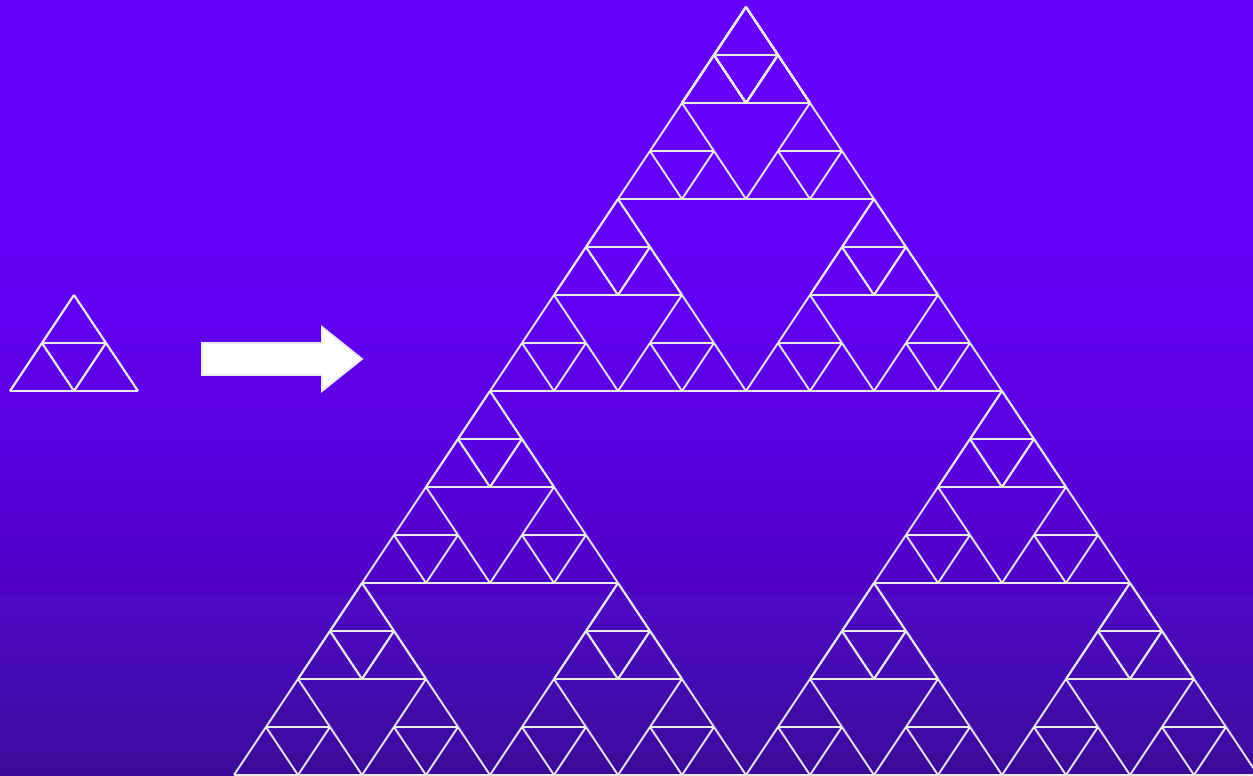
where $W_i, i = 1, 2$, are $k \times k$ matrices.



- (Wang-Xia'03) The above rate upper bounds hold for a finite QAM (excluding PSK or PAM) signal constellation.
- (Liang 2003, Su-Xia-Liu 2004, Lu-Fu-Xia 2005) constructed complex orthogonal designs with the above rates and **the constructions by Lu-Fu-Xia 2005 have closed-forms.**

Closed Form for COD

Self-Similarity



Construct COD B_{n+2} , B_{n+1} from B_n

Construction Unites for $n=2k-1$

B_n : $p_n \times n$ COD, number of nonzero complex variables is d_n

$\overline{B}_n$: $p_n \times 1$ vector, same set of complex variables as B_n

$\hat{B}_n$: $q_{1,n} \times 1$ vector, same set of complex variables as B_n

$Q_{m,n}$: $q_{m,n} \times n$ COD, $Q_{0,n} = B_n$

number of nonzero complex variables $v_{m,n} \leq d_n$

$\overline{Q}_{m,n}$: $q_{m-1,n} \times 1$ vector, $\overline{Q}_{0,n} = \overline{B}_n$

same set of complex variables as $Q_{m,n}$

$\hat{Q}_{m,n}$: $q_{m+1,n} \times 1$ COD, $\hat{Q}_{0,n} = \hat{B}_n$

same set of complex variables as $Q_{m,n}$



A Theorem (Lu-Fu-Xia'05)

Theorem 1: Let A be a complex orthogonal design and has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where (i) A_{11} and A_{22} have the same set of nonzero complex variables; (ii) A_{21} and A_{12} have the same set of nonzero complex variables; (iii) A_{11} and A_{12} do not share any common nonzero complex variable. Then, the following $\overline{A}$

$$\overline{A} = \begin{bmatrix} (-1)^k A_{11} & (-1)^l A_{12} \\ (-1)^m A_{21} & (-1)^n A_{22} \end{bmatrix}$$

is also a complex orthogonal design if $k + l + m + n$ is even.

Orthogonality among Units

$$\begin{bmatrix} B_n(i) & \bar{B}_n(j) \\ B_n(j) & (-1)^k \bar{B}_n(i) \end{bmatrix}$$

$$\begin{bmatrix} \bar{B}_n(i) & \bar{B}_n(j) \\ \hat{B}_n(j) & -\hat{B}_n(i) \end{bmatrix} \quad \text{where } n = 2k-1$$

$$\begin{bmatrix} B_n(i) & \bar{Q}_{1,n}(j) \\ Q_{1,n}(j) & -\hat{B}_n(i) \end{bmatrix}$$

$$\begin{bmatrix} \bar{B}_n(i) & \bar{Q}_{1,n}(j) \\ \bar{Q}_{1,n}(j) & -\bar{B}_n(i) \end{bmatrix}$$

$$\begin{bmatrix} \bar{Q}_{m,n}(i) & \bar{Q}_{m,n}(j) \\ \hat{Q}_{m,n}(j) & -\hat{Q}_{m,n}(i) \end{bmatrix}$$

$$\begin{bmatrix} \bar{Q}_{m,n}(i) & \bar{Q}_{m+1,n}(j) \\ Q_{m+1,n}(j) & -\hat{Q}_{m,n}(i) \end{bmatrix}$$

$$\begin{bmatrix} \hat{B}_n(i) & \bar{Q}_{2,n}(j) \\ \bar{Q}_{2,n}(j) & -\hat{B}_n(i) \end{bmatrix}$$

$$\begin{bmatrix} \hat{Q}_{m-1,n}(i) & \bar{Q}_{m+1,n}(j) \\ \bar{Q}_{m+1,n}(j) & -\hat{Q}_{m-1,n}(i) \end{bmatrix}$$

All the above matrices are COD

Orthogonal Property among Units

$B_n(i)$ has the same structure of B_n , but the indices of nonzero complex variables in $B_n(i)$ are from $(i-1)d_n+1$ to id_n , where d_n is the number of nonzero complex variables in B_n .

B_4 with $d_4=3$

$$\begin{bmatrix} x_1 & x_2^* & x_3^* & 0 \\ x_2 & -x_1^* & 0 & x_3^* \\ x_3 & 0 & -x_1^* & x_2^* \\ 0 & x_3 & -x_2 & -x_1 \end{bmatrix}$$

$B_4(i)$

$$\begin{bmatrix} x_{3(i-1)+1} & x_{3(i-1)+2}^* & x_{3(i-1)+3}^* & 0 \\ x_{3(i-1)+2} & -x_{3(i-1)+1}^* & 0 & x_{3(i-1)+3}^* \\ x_{3(i-1)+3} & 0 & -x_{3(i-1)+1}^* & x_{3(i-1)+2}^* \\ 0 & x_{3(i-1)+3} & -x_{3(i-1)+2} & -x_{3(i-1)+1} \end{bmatrix}$$

Inductive Construction for $n+1$ and $n+2$

$$B_{n+1} = \begin{bmatrix} B_n(1) & \bar{B}_n(2) \\ B_n(2) & (-1)^k \bar{B}_n(1) \end{bmatrix}$$

$$B_{n+2} = \begin{bmatrix} B_n(1) & \bar{B}_n(2) & \bar{B}_n(3) \\ B_n(2) & (-1)^k \bar{B}_n(1) & \bar{Q}_{1,n}(4) \\ B_n(3) & -\bar{Q}_{1,n}(4) & (-1)^k \bar{B}_n(1) \\ Q_{1,n}(4) & \hat{B}_n(3) & -\hat{B}_n(2) \end{bmatrix}$$

$$\bar{B}_{n+2} = \begin{bmatrix} (-1)^k \bar{Q}_{1,n}(4) \\ \bar{B}_n(3) \\ -\bar{B}_n(2) \\ \hat{B}_n(1) \end{bmatrix}$$

$$\hat{B}_{n+2} = \begin{bmatrix} (-1)^k \bar{B}_n(1) \\ \hat{B}_n(2) \\ \hat{B}_n(3) \\ -\hat{Q}_{1,n}(4) \end{bmatrix}$$

$$Q_{m,n+2} = \begin{bmatrix} Q_{m-1,n}(1) & \bar{Q}_{m,n}(2) & \bar{Q}_{m,n}(3) \\ Q_{m,n}(2) & -\hat{Q}_{m-1,n}(1) & \bar{Q}_{m+1,n}(4) \\ Q_{m,n}(3) & -\bar{Q}_{m+1,n}(4) & -\hat{Q}_{m-1,n}(1) \\ Q_{m+1,n}(4) & \hat{Q}_{m,n}(3) & -\hat{Q}_{m,n}(2) \end{bmatrix}$$

$$\bar{Q}_{m,n+2} = \begin{bmatrix} -\bar{Q}_{m-1,n}(1) \\ \bar{Q}_{m,n}(2) \\ \bar{Q}_{m,n}(3) \\ -\bar{Q}_{m+1,n}(4) \end{bmatrix}$$

$$\hat{Q}_{m,n+2} = \begin{bmatrix} \hat{Q}_{m-1,n}(1) \\ -\hat{Q}_{m,n}(2) \\ -\hat{Q}_{m,n}(3) \\ \hat{Q}_{m+1,n}(4) \end{bmatrix}$$

Orthogonality for New Units

$$\begin{bmatrix} B_n(i) & \bar{B}_n(j) \\ B_n(j) & (-1)^k \bar{B}_n(i) \end{bmatrix} \xrightarrow{\text{COD}} \begin{bmatrix} B_{n+2}(i) & \bar{B}_{n+2}(j) \\ B_{n+2}(j) & (-1)^k \bar{B}_{n+2}(i) \end{bmatrix} \quad ?$$

$$\begin{bmatrix} B_{n+2}(i) & \bar{B}_{n+2}(j) \\ B_{n+2}(j) & (-1)^k \bar{B}_{n+2}(i) \end{bmatrix} = \begin{array}{|c|c|c|c|} \hline B_n(1) & \bar{B}_n(2) & \bar{B}_n(3) & (-1)^k \bar{Q}_{1,n}(8) \\ \hline B_n(2) & (-1)^k \bar{B}_n(1) & \bar{Q}_{1,n}(4) & \bar{B}_n(7) \\ \hline B_n(3) & -\bar{Q}_{1,n}(4) & (-1)^k \bar{B}_n(1) & -\bar{B}_n(6) \\ \hline Q_{1,n}(4) & \hat{B}_n(3) & -\hat{B}_n(2) & \hat{B}_n(5) \\ \hline B_n(5) & \bar{B}_n(6) & \bar{B}_n(7) & -\bar{Q}_{1,n}(4) \\ \hline B_n(6) & (-1)^k \bar{B}_n(5) & \bar{Q}_{1,n}(8) & (-1)^{k+1} \bar{B}_n(3) \\ \hline B_n(7) & -\bar{Q}_{1,n}(8) & (-1)^k \bar{B}_n(5) & (-1)^k \bar{B}_n(2) \\ \hline Q_{1,n}(8) & \hat{B}_n(7) & -\hat{B}_n(6) & (-1)^{k+1} \hat{B}_n(1) \\ \hline \end{array}$$

COD 

Rate Formula

$$\begin{cases} q_{0,n+2} = 3q_{0,n} + q_{1,n} \\ q_{m,n+2} = q_{m-1,n} + 2q_{m,n} + q_{m+1,n}, \quad m > 0 \\ q_{0,n} = p_n \end{cases}$$



$$\begin{cases} v_{0,n+2} = 3v_{0,n} + v_{1,n} \\ v_{m,n+2} = v_{m-1,n} + 2v_{m,n} + v_{m+1,n}, \quad m > 0 \\ v_{0,n} = d_n \end{cases}$$

$$\begin{cases} q_{m,2k-1} = \frac{(2k)! \times [k + \frac{m(m+1)}{k}]}{(k+m+1)!(k-m)!} \\ v_{m,2k-1} = \frac{(2k-1)!}{(k+m)!(k-m-1)!} \end{cases}$$

$$R_{2k-1} = \frac{v_{0,2k-1}}{q_{0,2k-1}} = \frac{k+1}{2k}$$

Design Examples

$$B_1 = [x_1] \quad \bar{B}_1 = [x_1^*] \quad \hat{B}_1 = [x_1] \quad Q_{1,1} = [0] \quad \bar{Q}_{1,1} = [0] \quad \hat{Q}_{1,1} = [\phi]$$



$$B_2 = \begin{bmatrix} B_1(1) & \bar{B}_1(2) \\ B_1(2) & (-1)^k \bar{B}_1(1) \end{bmatrix} = \begin{bmatrix} x_1 & x_2^* \\ x_2 & -x_1^* \end{bmatrix}$$

$$B_3 = \begin{bmatrix} B_1(1) & \bar{B}_1(2) & \bar{B}_1(3) \\ B_1(2) & (-1)^k \bar{B}_1(1) & \bar{Q}_{1,1}(4) \\ B_1(3) & -\bar{Q}_{1,1}(4) & (-1)^k \bar{B}_1(1) \\ Q_{1,1}(4) & \hat{B}_1(3) & -\hat{B}_1(2) \end{bmatrix} = \begin{bmatrix} x_1 & x_2^* & x_3^* \\ x_2 & -x_1^* & 0 \\ x_3 & 0 & -x_1^* \\ 0 & x_3 & -x_2 \end{bmatrix}$$



Design Examples

$$B_4 = \begin{bmatrix} B_3(1) & \overline{B}_3(2) \\ B_3(2) & (-1)^k \overline{B}_3(1) \end{bmatrix} = \begin{bmatrix} x_1 & x_2^* & x_3^* & 0 \\ x_2 & -x_1^* & 0 & x_6^* \\ x_3 & 0 & -x_1^* & -x_5^* \\ 0 & x_3 & -x_2 & x_4 \\ x_4 & x_5^* & x_6^* & 0 \\ x_5 & -x_4^* & 0 & x_3^* \\ x_6 & 0 & -x_4^* & -x_2^* \\ 0 & x_6 & -x_5 & x_1 \end{bmatrix}$$

COD for 4 antennas
p = 6, d = 8, R = 3/4



$$B_5 = \begin{bmatrix} B_3(1) & \bar{B}_3(2) & \bar{B}_3(3) \\ B_3(2) & (-1)^k \bar{B}_3(1) & \bar{Q}_{1,3}(4) \\ B_3(3) & -\bar{Q}_{1,3}(4) & (-1)^k \bar{B}_3(1) \\ Q_{1,3}(4) & \hat{B}_3(3) & -\hat{B}_3(2) \end{bmatrix}$$

$$= \left[\begin{array}{ccc|cc} x_1 & x_2^* & x_3^* & 0 & 0 \\ x_2 & -x_1^* & 0 & x_6^* & x_9^* \\ x_3 & 0 & -x_1^* & -x_5^* & -x_8^* \\ 0 & x_3 & -x_2 & x_4 & x_7 \\ \hline x_4 & x_5^* & x_6^* & 0 & -x_{10}^* \\ x_5 & -x_4^* & 0 & x_3^* & 0 \\ x_6 & 0 & -x_4^* & -x_2^* & 0 \\ 0 & x_6 & -x_5 & x_1 & 0 \\ \hline x_7 & x_8^* & x_9^* & x_{10}^* & 0 \\ x_8 & -x_7^* & 0 & 0 & x_3^* \\ x_9 & 0 & -x_7^* & 0 & -x_2^* \\ 0 & x_9 & -x_8 & 0 & x_1 \\ \hline x_{10} & 0 & 0 & -x_7^* & x_4^* \\ 0 & -x_{10} & 0 & x_8 & -x_5 \\ 0 & 0 & -x_{10} & x_9 & -x_6 \end{array} \right]$$

COD for 5 antennas

p = 15, d = 10, R = 2/3



Smaller Size COD for $n=4l$

$n = 2k - 1$, if k is odd, there exists one smaller (half) size COD B'_{n+3} than B_{n+3}

$$B'_{n+3} = \begin{bmatrix} B_n(1) & \bar{B}_n(2) & \bar{B}_n(3) & -\bar{Q}_{1,n}(4) \\ B_n(2) & -\bar{B}_n(1) & \bar{Q}_{1,n}(4) & \bar{B}_n(3) \\ B_n(3) & -\bar{Q}_{1,n}(4) & -\bar{B}_n(1) & -\bar{B}_n(2) \\ Q_{1,n}(4) & \hat{B}_n(3) & -\hat{B}_n(2) & \hat{B}_n(1) \end{bmatrix}$$

$$p'_{n+3} = \frac{1}{2} p_{n+3}$$

Smaller Size COD for $n=4l$

$$B'_4 = \begin{bmatrix} B_1(1) & \bar{B}_1(2) & \bar{B}_1(3) & -\bar{Q}_{1,1}(4) \\ B_1(2) & -\bar{B}_1(1) & \bar{Q}_{1,1}(4) & \bar{B}_1(3) \\ B_1(3) & -\bar{Q}_{1,1}(4) & -\bar{B}_1(1) & -\bar{B}_1(2) \\ Q_{1,1}(4) & \hat{B}_1(3) & -\hat{B}_1(2) & \hat{B}_1(1) \end{bmatrix} = \begin{bmatrix} x_1 & x_2^* & x_3^* & 0 \\ x_2 & -x_1^* & 0 & x_3^* \\ x_3 & 0 & -x_1^* & -x_2^* \\ 0 & x_3 & -x_2 & x_1 \end{bmatrix}$$

COD from our design for 4 antennas: $d = 3, p = 4, R = 3/4$

----- coincides with the existing one

Liang's and Su-Xia-Liu's:

$d = 6, p = 8, R = 3/4$



$B'_8 =$

x_1	x_2	x_3	0	0	$-x_{20}$	$-x_{30}$	0
x_2	$-x_1^*$	0	x_6^*	x_9^*	0	0	x_{33}^*
x_3	0	$-x_1^*$	$-x_5^*$	$-x_8^*$	0	0	$-x_{32}^*$
0	x_3	$-x_2$	x_4	x_7	0	0	x_{31}
x_4	x_3^*	x_6^*	0	$-x_{10}^*$	0	0	x_{34}
x_5	$-x_4^*$	0	x_3^*	0	x_{19}^*	x_{29}^*	0
x_6	0	$-x_4^*$	$-x_2^*$	0	$-x_{18}^*$	$-x_{28}^*$	0
0	x_6	$-x_5$	x_1	0	x_{17}	x_{27}	0
x_7	x_8^*	x_9^*	x_{10}^*	0	0	0	x_{35}^*
x_8	$-x_7^*$	0	0	x_3^*	$-x_{16}^*$	$-x_{26}^*$	0
x_9	0	$-x_7^*$	0	$-x_2^*$	x_{15}^*	x_{25}^*	0
0	x_9	$-x_8$	0	x_1	$-x_{14}^*$	$-x_{24}^*$	0
x_{10}	0	0	$-x_7^*$	x_4^*	$-x_{11}^*$	$-x_{21}^*$	0
0	$-x_{10}$	0	x_8	$-x_5$	x_{12}	x_{22}	0
0	0	$-x_{10}$	x_9	$-x_6$	x_{13}	x_{23}	0
x_{11}	x_{12}^*	x_{13}^*	0	0	x_{10}^*	0	$-x_{30}^*$
x_{12}	$-x_{11}^*$	0	x_{16}^*	x_{19}^*	0	$-x_{33}^*$	0
x_{13}	0	$-x_{11}^*$	$-x_{15}^*$	$-x_{18}^*$	0	x_{32}	0
0	x_{13}	$-x_{12}$	x_{14}	x_{17}	0	$-x_{31}$	0
x_{14}	x_{15}^*	x_{16}^*	0	$-x_{20}^*$	0	$-x_{34}^*$	0
x_{15}	$-x_{14}^*$	0	x_{13}^*	0	$-x_9^*$	0	x_{29}^*
x_{16}	0	$-x_{14}^*$	$-x_{12}^*$	0	x_8^*	0	$-x_{28}^*$
0	x_{16}	$-x_{15}$	x_{11}	0	$-x_7$	0	x_{27}
x_{17}	x_{18}^*	x_{19}^*	x_{20}^*	0	0	$-x_{35}^*$	0
x_{18}	$-x_{17}^*$	0	0	x_{13}	x_6^*	0	$-x_{26}^*$
x_{19}	0	$-x_{17}^*$	0	$-x_{12}^*$	$-x_5^*$	0	x_{25}^*
0	x_{19}	$-x_{18}$	0	x_{11}	x_4	0	$-x_{24}^*$
x_{20}	0	0	$-x_{17}^*$	x_{14}^*	x_1^*	0	$-x_{21}^*$
0	$-x_{20}$	0	x_{18}	$-x_{15}$	$-x_2$	0	x_{22}
0	0	$-x_{20}$	x_{19}	$-x_{16}$	$-x_3$	0	x_{23}
x_{21}	x_{22}^*	x_{23}^*	0	0	0	x_{10}^*	x_{20}^*
x_{22}	$-x_{21}^*$	0	x_{26}^*	x_{29}^*	x_{33}^*	0	0
x_{23}	0	$-x_{21}^*$	$-x_{25}^*$	$-x_{28}^*$	$-x_{32}^*$	0	0
0	x_{23}	$-x_{22}$	x_{24}	x_{27}	x_{31}	0	0
x_{24}	x_{25}^*	x_{26}^*	0	$-x_{30}^*$	x_{34}^*	0	0
x_{25}	$-x_{24}^*$	0	x_{23}^*	0	0	$-x_9^*$	$-x_{19}^*$
x_{26}	0	$-x_{24}^*$	$-x_{22}^*$	0	0	x_8^*	x_{18}^*
0	x_{26}	$-x_{25}$	x_{21}	0	0	$-x_7$	$-x_{17}$
x_{27}	x_{28}^*	x_{29}^*	x_{30}^*	0	x_{35}^*	0	0
x_{28}	$-x_{27}^*$	0	0	x_{23}	0	x_6^*	x_{16}^*
x_{29}	0	$-x_{27}^*$	0	$-x_{22}^*$	0	$-x_5^*$	$-x_{15}^*$
0	x_{29}	$-x_{28}$	0	x_{21}	0	x_4	x_{14}
x_{30}	0	0	$-x_{27}^*$	x_{24}^*	0	x_1^*	x_{11}^*
0	$-x_{30}$	0	x_{28}	$-x_{25}$	0	$-x_2$	$-x_{12}$
0	0	$-x_{30}$	x_{29}	$-x_{26}$	0	$-x_3$	$-x_{13}$
x_{31}	x_{32}^*	x_{33}^*	$-x_{34}^*$	$-x_{35}^*$	0	0	0
x_{32}	$-x_{31}^*$	0	0	0	x_{23}	$-x_{13}^*$	x_3^*
x_{33}	0	$-x_{31}^*$	0	0	$-x_{22}^*$	x_{12}^*	$-x_2^*$
0	x_{33}	$-x_{32}$	0	0	x_{21}	$-x_{11}$	x_1
x_{34}	0	0	x_{31}^*	0	$-x_{24}^*$	x_{14}^*	$-x_4^*$
0	$-x_{34}$	0	$-x_{32}$	0	x_{25}	$-x_{15}$	x_5
0	0	$-x_{34}$	$-x_{33}$	0	x_{26}	$-x_{16}$	x_6
x_{35}	0	0	0	x_{31}^*	$-x_{27}^*$	x_{17}^*	$-x_7^*$
0	$-x_{35}$	0	0	$-x_{32}$	x_{28}	$-x_{18}$	x_8
0	0	$-x_{35}$	0	$-x_{33}$	x_{29}	$-x_{19}$	x_9
0	0	0	x_{35}	$-x_{34}$	$-x_{30}$	x_{20}	$-x_{10}$

- A design example for $n=8$ transmit antennas.
- In this case, $d=35$, $p=56$.
- Rate $=d/p=5/8$
- This construction is inductive for all n with closed-forms
- Liang's and Su-Xia-Liu's:
 $d = 70$, $p = 112$, $R = 5/8$
 - *These constructions do not have closed-forms and computer-aid or manual help is needed*

COD Construction Comparison

n	Liang & Su-Xia-Liu		Lu-Fu-Xia		Rate=d/p
	d	p	d	p	
1	1	1	1	1	1
2	2	2	2	2	1
3	3	4	3	4	3 / 4
4	6	8	3	4	3 / 4
5	10	15	10	15	4 / 6
6	20	30	20	30	4 / 6
7	35	56	35	56	5 / 8
8	70	112	35	56	5 / 8
9	126	210	126	210	6 / 10
10	252	420	252	420	6 / 10
11	462	792	462	792	7 / 12
12	924	1584	462	792	7 / 12
13	1716	3003	1716	3003	8 / 14
14	3432	6006	3432	6006	8 / 14



Conclusion

- ◆ Orthogonal designs have been applied in wireless communications
- ◆ Rate upper bounds have been presented for COD (with linear processing)
- ◆ A closed-form construction for COD of rate reaching the conjectured optimal rate has been presented
- ◆ The rate upper bound is still open
- ◆ What is a smallest p , i.e., the smallest number of rows in a p by n COD, is still open



Some Papers to Read

- ◆ S. M. Alamouti, A simple transmit diversity technique for wireless communications, *IEEE J. Select. Areas Commun.*, Oct. 1998.
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- ◆ W. Su and X.-G. Xia, [On space-time block codes from complex orthogonal designs](#), *Wireless Personal Communications*, April, 2003.
- ◆ X.-B. Liang and X.-G. Xia, [On the Nonexistence of Rate-One Generalized Complex Orthogonal Designs](#), *IEEE Trans. on Information Theory*, Nov. 2003.
- ◆ H. Wang and X.-G. Xia, [Upper Bounds of Rates of Complex Orthogonal Space-Time Block Codes](#), *IEEE Trans. on Information Theory*, Oct. 2003. [longer version](#).
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- ◆ W. Su, X.-G. Xia, and K. J. R. Liu, A Systematic Design of High-Rate Complex Orthogonal Space-Time Block Codes, *IEEE Communications Letters*, June 2004.
- ◆ K. Lu, S. Fu, and X.-G. Xia, [Closed Form Designs of Complex Orthogonal Space-Time Block Codes of Rates \$\(k+1\)/\(2k\)\$ for \$2k-1\$ or \$2k\$ Transmit Antennas](#), *IEEE Trans. on Information Theory*, Dec. 2005.



Thank You!