

Space-Time Coding: Fundamentals

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Outline

- Background
- Single Antenna Modulation
- Multi-Antenna Modulation
 - Pairwise Error Bound and Criterion
 - Some Optimal Designs
- Conclusion and Some Open Problems

Background

- The ultimate goal of our communications: for everyone to be able to communicate anything at anytime and anywhere!
- Have we achieved? **No and far away from it!**
 - ✓ Although we are able to call everywhere (almost) anytime, but not anything can be transmitted (**It is still not possible for everyone to send images/videos, visit internets like voices over cell phones**).
- What is the problem?
 - ✓ Too many people → too many things to send!
 - ✓ Limited bandwidth
- Can we improve? **Yes!**

Background

Wired phones



In the last twenty years
in the last century

Wired computer modems

Wireless cell-phones



Now

Wireless modems

Progress Review of Wired Modems (Impact of Coding): Two Ways to Improve Data Rates

< 9.6 kbs/s

equalization (Lucky 60s)

9.6 kbs/s 1984

TCM +equalization (DFE)

14.4 kbs/s

TCM

19.20 kbs/s

TCM

28.8 kbs/s

high dim TCM

33.6 kbs/s

high dim TCM

56 kbs/s

high dim TCM

Bandwidth efficient coding
Several bits/s/Hz

+ equalization

Squeeze more
bits to a symbol

better

Higher SNR

Asymmetric Digital Subscriber Line (ADSL)

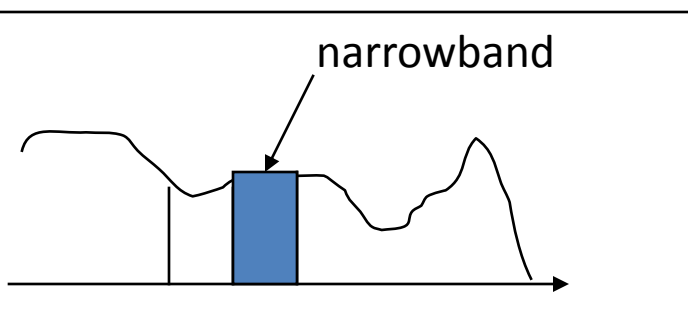
6 Mbs/s

orthogonal frequency division
multiplexing (OFDM)

or called discrete multi-tone (DMT)

More advanced DSP

Use more bandwidth



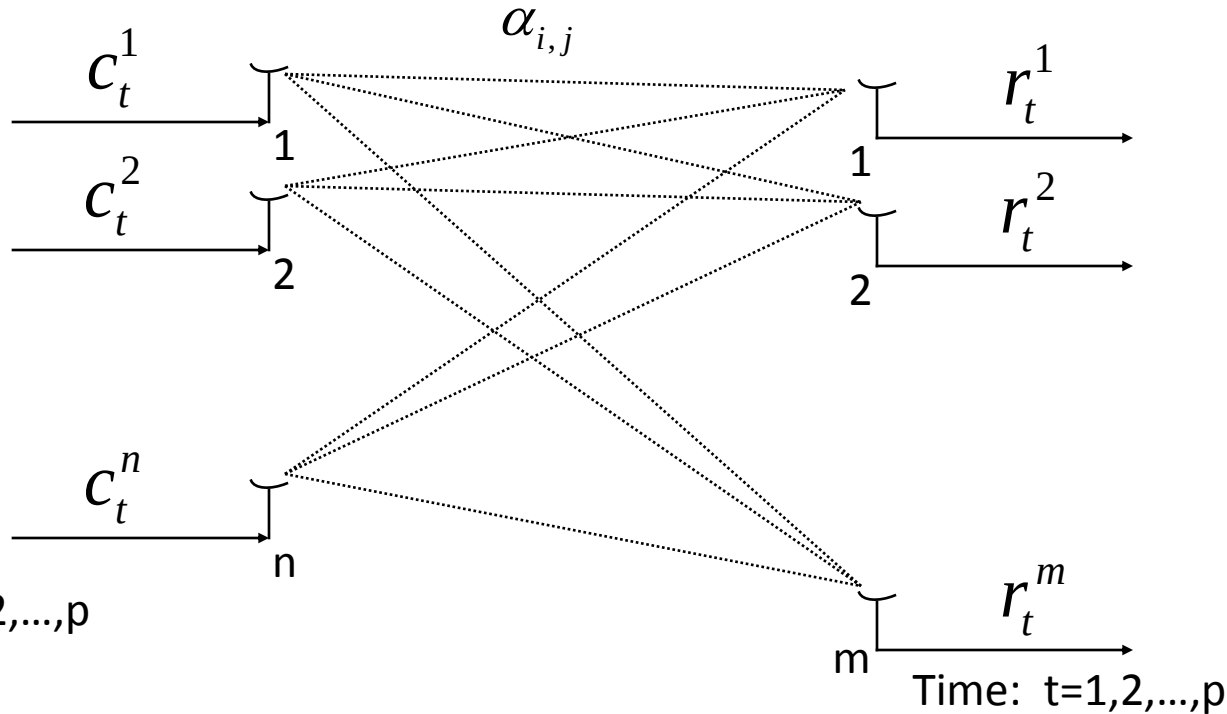
How About Wireless Systems ?

- Why trellis coded modulation (TCM) for single antenna has not been used in wireless systems?
 - ✓ TCM → high bandwidth efficient coding schemes
 - ✓ When the bandwidth efficiency is too high, the TCM distance is too small to combat wireless low SNR, interference, fading.
- Why by far wireless OFDM is only successfully used in LAN (802.11a/g), although it is proposing to go MAN/WAN (802.16d/e) with multiple antennas?
 - ✓ The same reason: Fading, interference, and low SNR
- Is there any way to combat wireless fading?
 - ✓ Yes! **To use spatial diversity!**

What About Wireless Systems ?

- Can modulation and coding for multiple antennas similar to TCM work for wireless systems to achieve high bandwidth efficiency?
 - ✓ YES! But **How?**
 - ✓ Currently, instead, more bandwidth is used in high speed wireless systems by adopting OFDM, such as 802.15 etc. **However,**
 - **Bandwidth is always limited**
 - **More bandwidth costs more, also has more interferences and fading**
- **Current IEEE standards: 802.16e (WiMax), 802.11n (WiFi), 3GPP**
- **In my opinion: not too wide bandwidth but with bandwidth efficiency coding/modulation for multiple antennas!**

Multiple Antenna System

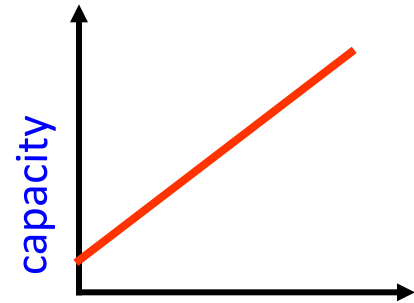


n transmit antennas

m receive antennas

$\alpha_{i,j}$: channel coefficient from i^{th} transmit
to j^{th} receive antenna
independent random variables

Capacity of Multi-Antenna System



- Teletar (1995), Foschini and Gans (1998) proved that the capacity of a multi-antenna system is proportional to $\min\{m, n\}$.
 - ✓ Theoretically, the more transmit and receive antennas, the larger the capacity !
 - ✓ Practically, how can we achieve the capacity ?
 - ✓ Shannon communication theory tells us that the capacity can be achieved by coding and modulation.
 - How to do the coding and modulation ? ? ?

Single Antenna Modulation

- For a given channel SNR and a transmission rate, we want to have the error probability as small as possible!
- Let $\mathbf{S} = \{s_0, \dots, s_{N-1}\}$ be a signal constellation
- A single antenna channel is $y=Ax+w$, x belongs to the signal constellation $\mathbf{S}$, w is the AWGN, and A is the channel coefficient
- What is the error probability $\Pr(S_i \rightarrow S_j)$?
- Consider the ML demodulation:

$$\arg \min_{l=0,1,\dots,N-1} \|Y - As_l\| = s_j$$

Single Antenna Modulation

$$\arg \min_{l=0,1,\dots,N-1} \|Y - As_l\|$$

$$= \arg \min_{l=0,1,\dots,N-1} \left\| (s_i - s_l) + \frac{w}{A} \right\|$$

$$\Pr(s_i \rightarrow s_j) \propto \exp(-c|s_i - s_j|^2)$$

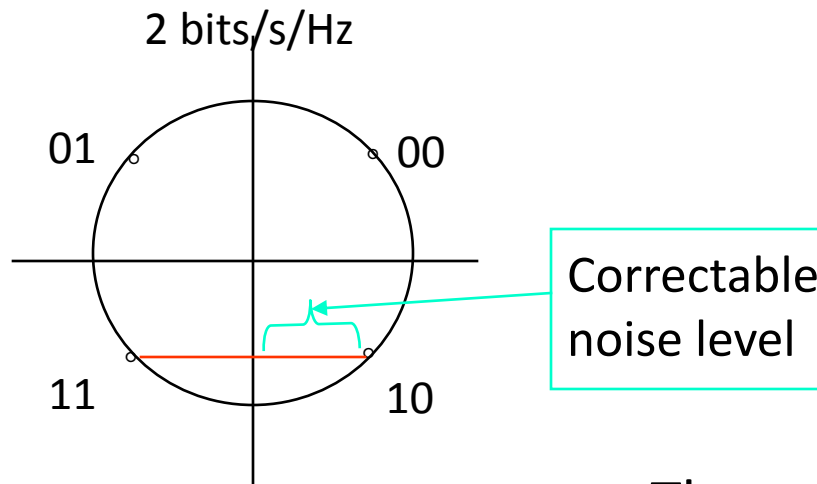
$$P_{\text{SER}} \propto \exp\{-cd_{\min}^2\}$$

$$\text{where } d_{\min} = \min_{0 \leq i \neq j \leq N-1} |s_i - s_j|$$

So, we need to have a signal constellation **S** with its minimum distance $d_{\min}$ as large as possible

Single Antenna Modulation

- Low rate transmission: 1 bit is modulated to 1 number/symbol (BPSK)
- High rate transmission: multiple bits modulated to 1 number/symbol (QAM)
- Consider 2 bits to a number: QPSK is optimal

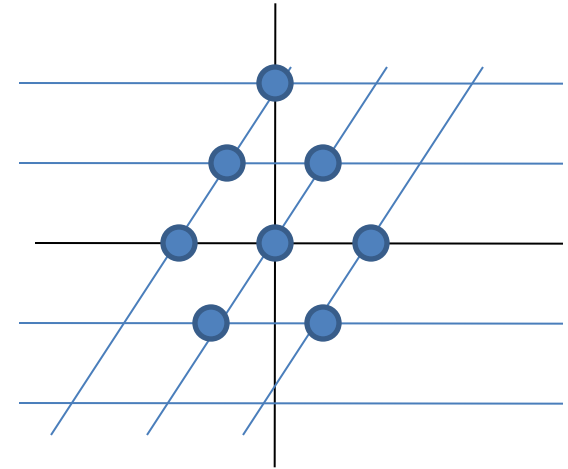


These 4 points are optimal:
The minimum distance is maximal

Single Antenna Modulation

- Consider 3 bits to a number: 8 QAM

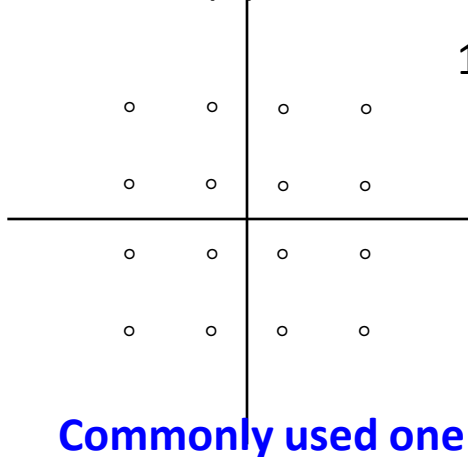
Minimum distance has been conjectured maximal



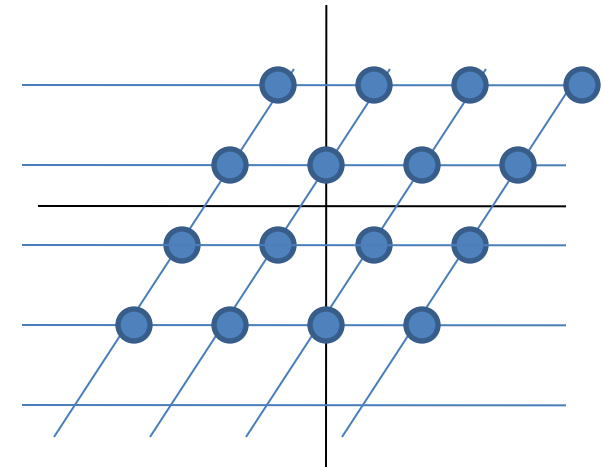
- Consider 4 bits to a symbol: 16 QAM

4 bits/s/Hz

16-QAM



These 16 points does not have the optimal minimum distance but close and they have Gray mapping



These 16 points are conjectured to have the maximum minimum distance but do not have Gray mapping

What Happens to Multiple Antenna Systems?

➤ Transmit and receive signal model:

$$Y=CA+W,$$

where

$Y = (r_t^j)_{p \times m}$	Receive signal matrix
$C = (c_t^i)_{p \times n}$	Transmit signal matrix
$A = (\alpha_{i,j})_{n \times m}$	Channel coefficient matrix
$W = (w_{t,j})_{p \times m}$	AWGN matrix

What Is Multiple Antenna Coding and Modulation?

- Multiple antenna coding/modulation: bits are modulated/mapped to $p \times n$ matrices and these matrices are taken from a pre-designed $p \times n$ matrix set $\mathbf{C}$. This matrix set $\mathbf{C}$ is called a *Space-Time Code*.
 - ✓ Information bits are mapped to matrices (in single antenna case, bits are mapped to complex numbers)
- A space-time code $\mathbf{C}$ needs to be designed such that the **error probability** at the receiver is minimized for a given SNR.
 - ✓ Depends on a receiver to be used!

The ML Receiver and Error Probability for Gaussian Noise

- The ML receiver $\min_{C \in \mathbf{C}} \|Y - CA\|_F$
where $\mathbf{C}$ is a space-time code and

$$\|B\|_F^2 = \sum_{i,j} |b_{i,j}|^2$$

- **The pairwise error probability:** (Guey-Fitz et al, Tarokh et al)

$$P(C \rightarrow \tilde{C}) \leq \left(\prod_{i=1}^v \lambda_i \right)^{-2m} \cdot (SNR)^{-vm}$$

Determinant absolute value of B
when the space-time code is squared.

Lead to two design
criteria

where $\lambda_1, \dots, \lambda_v$ are the non-zero singular values of the difference matrix

$$B(C, \tilde{C}) = \tilde{C} - C$$

Diversity Order

- The rank ν of matrix $B(C, \tilde{C})$ can not be above its number of rows, n , or its number of columns, p , i.e., $\nu \leq n, p$
 - ✓ The maximal ν is n , i.e., the difference matrix B has full rank.
- The time delay (or block size) p is free to choose but to increase time delay p does not increase the rank ν when the number n of transmit antennas is fixed, as long as p is not smaller than n
- νm is called **diversity order**. The larger the diversity is, the smaller the pairwise error probability is.
 - ✓ It is the largest, nm , when the difference matrix B has full rank.
 - ✓ The total diversity is nm that is, in fact, the total number of freedoms in the m by n channel matrix
 - MIMO
 - Cooperative systems (such as relay networks)
 - ✓ Matrix forms at both transmitter and receiver achieve the diversity.

Criteria for STC Design Based on ML Receiver

(Guey-Fitz et al and Tarokh et al)

- Rank criterion: any difference matrix of any two distinct matrices in a code $\mathbf{C}$ has **full rank**.

✓ This is for full diversity and relatively easy to satisfy.

- Diversity product criterion (or coding gain/advantage or product distance):

$$\zeta(\mathbf{C}) = \min_{\mathbf{C} \neq \tilde{\mathbf{C}} \in \mathbf{C}} \lambda_1 \cdots \lambda_v \quad \text{Diversity product of } \mathbf{C}$$

$$\max_{\mathbf{C}} \zeta(\mathbf{C}) \quad \text{The maximal diversity product}$$

- Diversity product is upper bounded by (Liang-Xia'02)

$\zeta(\mathbf{C}) \leq \frac{2L}{L-1}$
 L is the size of $\mathbf{C}$ and the mean power is normalized to 1/p and p is the number of time slots needed for the transmission

Why Determinant: An Intuitive Answer

- Consider n independent diagonal channel
$$A = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$$
- In this case, the space-time code is also diagonal
$$C = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix}$$
- The MIMO channel $Y=CA+W$ is equivalent to n independent SISO channel $y_i = \alpha_i c_i + w_i, \quad i = 1, 2, \dots, n$

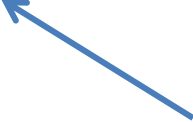
where $c_i \in \mathcal{S}_i$, signal constellation

- All the information c_i have to be **equally protected** since the transmitter does not know which channel is good
- Thus, each minimum distance $d_{\min, i}$ of c_i in $\mathcal{S}_i$ has to be large

Why Determinant: An Intuitive Answer

- One way to ensure all $d_{\min, i}$ are not small is to maximize their product

$$\prod_{i=1}^n d_{\min, i} = \prod_{i=1}^n \min_{c_i^1 \neq c_i^2 \in \mathcal{S}_i} |c_i^1 - c_i^2|$$
$$= \min_{C_1 \neq C_2 \in \mathcal{C}} |\det(C_1 - C_2)|$$



Diversity product

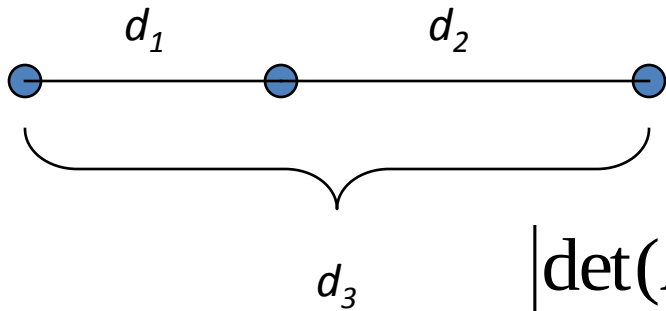
- For non-diagonal channels, they can be diagonalized.

Mathematical Challenges Compared to Single Antenna Coding and Modulation

- Matrices in multiple antenna space-time coding do not commute while scalars in the conventional single antenna coding and modulation commute.

$$AB \neq BA$$

- The diversity product (or product distance) criterion is not a distance in the mathematical sense while the Euclidean distance in single antenna coding and modulation is a distance.



$$d_3 \neq d_1 + d_2$$

$$|\det(A - C)| \not\leq |\det(A - B)| + |\det(B - C)|$$

Two Major Methods to Design STC Matrices/Modulation for Both Block and Trellis Codes

- **Direct mapping method:** p by n matrices are directly designed and mapped to information bits.
 - Advantage:** diversity product (performance) may be optimized
 - Disadvantage:** may not have fast or soft decoding
 - ✓ Unitary space-time codes

- **Symbol embedding method:** information bits are first mapped to complex symbols and these complex symbols are then put/embedded into a p by n matrix.
 - Advantage:** may have simplified and soft decoding
 - Disadvantage:** diversity product (performance) may not be optimized
 - ✓ (Quasi) orthogonal space-time codes (Alamouti code)
 - ✓ Linear dispersion codes, linear lattice based codes
 - ✓ Nonlinear algebraic codes proposed by Hammons and El Gamal

Direct Mapping Method

- Assume the transmission data rate is R bits/s/Hz, For a space-time code $\mathbf{C}$, how many $p \times n$ matrices do we need?
- Consider two transmit antennas: $n=2$ & $p=2$
 - ✓ $R=1$, i.e., 1 bits/s/Hz (BPSK corresponding to single antenna case): $\mathbf{C}$ has to have $2^{2 \cdot 1} = 4$ matrices (only two points needed for single antenna case). The best 4 matrices ([Liang-Xia'02](#))

$$\begin{pmatrix} ja_1 & a_3 - ja_2 \\ -a_3 - ja_2 & -ja_1 \end{pmatrix}, \begin{pmatrix} -ja_1 & -a_3 - ja_2 \\ a_3 - ja_2 & ja_1 \end{pmatrix}, \\ \begin{pmatrix} -ja_3 & a_1 + ja_2 \\ -a_1 + ja_2 & ja_3 \end{pmatrix}, \begin{pmatrix} ja_3 & -a_1 + ja_2 \\ a_1 + ja_2 & -ja_3 \end{pmatrix}$$

where $a_1^2 + a_2^2 + a_3^2 = 1$, $a_1^2 + a_3^2 = \frac{2}{3}$

- Its product diversity is $8/3 = (2L)/(L-1)$ that reaches the upper bound. It turns out that the above four matrices are also unitary.

Direct Mapping Method (Continued)

- ✓ When $R=2$, i.e., 2 bits/s/Hz (4-QAM corresponding to single antenna case), $\mathbf{C}$ has to have $2^{2 \cdot 2} = 16$ matrices of size 2 by 2 (only 4 points needed for single antenna case).
- ✓ When $R=3$, i.e., 3 bits/s/Hz (8-QAM corresponding to single antenna case), $\mathbf{C}$ has to have 64 matrices (only 8 points needed for single antenna case).
- ✓ In general, 2^{pR} matrices are needed.
 - With the same throughput R bits/s/Hz, a single antenna modulation only needs 2^R points/complex numbers.
 - The ML decoding complexity may be significantly increased over a single antenna modulation.

Diversity-Multiplexing Tradeoff by Zheng and Tse: A Necessary Condition for ML Receiver Based STC Designs

➤ Let r be normalized rate: $r = R / \log(\text{SNR})$

✓ Referred as multiplexing gain

➤ Diversity gain

$$p_e \approx \text{SNR}^{-d(r)}$$

$$d(r) = - \lim_{\text{SNR} \rightarrow \infty} \frac{\log(P_e)}{\log(\text{SNR})}$$

➤ The Tradeoff:

$$d(r) = (m-r)(n-r)$$

This means that the Tradeoff does not depend on an SNR

For a fixed SNR:

$$P(C \rightarrow \tilde{C}) \leq \left(\prod_{i=1}^v \lambda_i \right)^{-2m} \bullet (\text{SNR})^{-nm}$$

Diversity order

Direct Mapping Method: Some Existing Unitary Space-Time Codes

- Unitary diagonal/cyclic codes (Hughes, Hochwald-Sweldens)
- Unitary codes from orthogonal designs (Tarokh et al)
- Unitary codes from fixed-point free groups (Shokrollahi-Hassibi et al)
- Unitary codes from Caley transforms (Hassibi et al)
- Parametric codes (Liang-Xia, *IEEE Trans. IT*, Aug. 2002)
 - ✓ The 2 by 2 code of size 5 reaches the optimal diversity product.
 - ✓ The 2 by 2 codes of sizes 16, 128 and 256 have the best known diversity products.
- Unitary codes from sphere packing theory (H.Wang-G.Wang-Xia, *IEEE Trans. IT*, Dec. 2004)
 - ✓ The 2 by 2 code of size 6 has the optimal diversity product.
 - ✓ For sizes 16, 32, 48, 64 of 2 by 2 unitary codes, our codes have the best known diversity products.

Parametric Codes (2 by 2 unitary), Liang-Xia 2002

➤ Codes of size L and parameters $(k_1, k_2, k_3) \in \mathbf{Z}^3$

$$\mathbf{V}(k_1, k_2, k_3) = \{A(l k_1, l k_2, l k_3) : l = 0, 1, \dots, L-1\}$$

$$\text{where } A(l k_1, l k_2, l k_3) = \begin{pmatrix} e^{j\theta_L} & 0 \\ 0 & e^{jk_1\theta_L} \end{pmatrix}^l \bullet$$

$$\begin{pmatrix} \cos k_2\theta_L & \sin k_2\theta_L \\ -\sin k_2\theta_L & \cos k_2\theta_L \end{pmatrix}^l \bullet \begin{pmatrix} e^{jk_3\theta_L} & 0 \\ 0 & e^{-jk_3\theta_L} \end{pmatrix}^l$$

$$\text{where } \theta_L = 2\pi / L$$

Parametric Codes (2 by 2 unitary), Liang-Xia 2002

- Any 2 by 2 unitary matrix can be parameterized as

$$\begin{pmatrix} e^{j\varphi_1} & 0 \\ 0 & e^{j\varphi_2} \end{pmatrix} \begin{pmatrix} \cos \varphi_3 & \sin \varphi_3 \\ -\sin \varphi_3 & \cos \varphi_3 \end{pmatrix} \begin{pmatrix} e^{j\varphi_4} & 0 \\ 0 & e^{-j\varphi_4} \end{pmatrix}$$

- The parametric code of size 5 and parameters 4,2,0 has the optimal product diversity 5/2.
 - It also reaches the optimal minimum Euclidean distance 5/2.
- The parametric code of size 16 has the best known product diversity and is a subset of a group of size 32.
- Codes of sizes 32, 64, 128, 256 obtained from the subsets of parametric codes of sizes 37, 75, 135, 273, respectively, have the best known product diversities.

Theorem : The parametric code of size 16,
 $\mathcal{V} = \{ V_\ell(k_1, k_2, k_3) \mid \ell \in \mathbb{Z}_{16} \}$ with $(k_1, k_2, k_3) = (3, 4, 2)$, is a
subset of the finite group of order 32 given by

$$\{ j^m V_\ell(k_1, k_2, k_3) \mid \ell \in \mathbb{Z}_{16} \text{ and } m = 0, 1 \},$$

where $(k_1, k_2, k_3) = (3, 4, 2)$.

**The best known 2 by 2
Unitary code of size 16
from parametric code
family and is a subset
of group of 32
elements.**

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & e^{j\frac{15}{8}\pi} \\ e^{j\frac{13}{8}\pi} & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} e^{j\frac{7}{4}\pi} & 0 \\ 0 & e^{j\frac{5}{4}\pi} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & e^{j\frac{5}{8}\pi} \\ e^{j\frac{15}{8}\pi} & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} e^{j\frac{3}{2}\pi} & 0 \\ 0 & e^{j\frac{1}{2}\pi} \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0 & e^{j\frac{11}{8}\pi} \\ e^{j\frac{1}{8}\pi} & 0 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} e^{j\frac{5}{4}\pi} & 0 \\ 0 & e^{j\frac{7}{4}\pi} \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 0 & e^{j\frac{1}{8}\pi} \\ e^{j\frac{3}{8}\pi} & 0 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_9 = \begin{pmatrix} 0 & e^{j\frac{7}{8}\pi} \\ e^{j\frac{5}{8}\pi} & 0 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} e^{j\frac{3}{4}\pi} & 0 \\ 0 & e^{j\frac{1}{4}\pi} \end{pmatrix}$$

$$A_{11} = \begin{pmatrix} 0 & e^{j\frac{13}{8}\pi} \\ e^{j\frac{7}{8}\pi} & 0 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} e^{j\frac{1}{2}\pi} & 0 \\ 0 & e^{j\frac{3}{2}\pi} \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 0 & e^{j\frac{3}{8}\pi} \\ e^{j\frac{9}{8}\pi} & 0 \end{pmatrix}$$

$$A_{14} = \begin{pmatrix} e^{j\frac{1}{4}\pi} & 0 \\ 0 & e^{j\frac{3}{4}\pi} \end{pmatrix}$$

$$A_{15} = \begin{pmatrix} 0 & e^{j\frac{9}{8}\pi} \\ e^{j\frac{11}{8}\pi} & 0 \end{pmatrix}$$

L	$\zeta(L, \mathcal{V})$	codes and comments
2	1	parametric code of $(k_1, k_2, k_3) = (1, 0, 0)$
3	$\sqrt{3}/2$	parametric code of $(k_1, k_2, k_3) = (0, 0, 1)$
4	0.7071	cyclic code, quaternion code, orthogonal design, parametric code
4	$\sqrt{2}/3$	optimal code
5	$\sqrt{5}/8$	parametric code of $(k_1, k_2, k_3) = (4, 2, 0)$ with the optimal diversity product
8	0.5946	cyclic code
8	0.7071	quaternion code, parametric code of $(k_1, k_2, k_3) = (7, 2, 0)$
16	0.3827	cyclic code, quaternion code
16	0.5000	orthogonal design
16	$\sqrt[4]{2}/2$	parametric code of $(k_1, k_2, k_3) = (3, 4, 2)$
24	0.5000	FPF group code, parametric code of $(k_1, k_2, k_3) = (5, 6, 0)$
32	0.1951	quaternion code
32	0.2494	cyclic code
32	0.3827	parametric code of $(k_1, k_2, k_3) = (7, 8, 2)$
37	0.4461	parametric code of $(k_1, k_2, k_3) = (30, 6, 0)$
48	0.3868	FPF group code
55	0.3874	parametric code of $(k_1, k_2, k_3) = (34, 15, 0)$
64	0.0980	quaternion code
64	0.1985	cyclic code
64	0.2706	orthogonal design
64	0.3070	parametric code of $(k_1, k_2, k_3) = (7, 10, 0)$
75	0.3535	parametric code of $(k_1, k_2, k_3) = (49, 18, 0)$

32	0.3827	parametric code of $(k_1, k_2, k_3) = (7, 8, 2)$
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64	0.1985	cyclic code
64	0.2706	orthogonal design
64	0.3070	parametric code of $(k_1, k_2, k_3) = (7, 10, 0)$
75	0.3535	parametric code of $(k_1, k_2, k_3) = (49, 18, 0)$
91	0.3451	parametric code of $(k_1, k_2, k_3) = (64, 21, 0)$
105	0.3116	parametric code of $(k_1, k_2, k_3) = (34, 42, 0)$
120	0.3090	FPF group code
121	0.1992	orthogonal design
128	0.0491	quaternion code
128	0.1498	cyclic code
128	0.2606	parametric code of $(k_1, k_2, k_3) = (1, 8, 20)$
135	0.2869	parametric code of $(k_1, k_2, k_3) = (14, 34, 0)$
145	0.2841	parametric code of $(k_1, k_2, k_3) = (62, 38, 0)$
203	0.2603	parametric code of $(k_1, k_2, k_3) = (146, 35, 0)$
217	0.2511	parametric code of $(k_1, k_2, k_3) = (125, 84, 0)$
240	0.2257	FPF group code
273	0.2152	parametric code of $(k_1, k_2, k_3) = (104, 71, 0)$

2 by 2 Unitary Codes from Sphere Packing (Wang-Wang-Xia 2004)

$$\mathbf{U}(2) \triangleq \{A \mid A \text{ is a } 2 \times 2 \text{ matrix with } A^H A = I\}$$

$$\mathbf{SU}(2, \theta) \triangleq \{A \in \mathbf{U}(2) \mid \det(A) = e^{j\theta}\}$$

 All 2 by 2 unitary matrices

$$\mathbf{U}(2) = \bigcup_{\theta \in [0, 2\pi)} \mathbf{SU}(2, \theta)$$

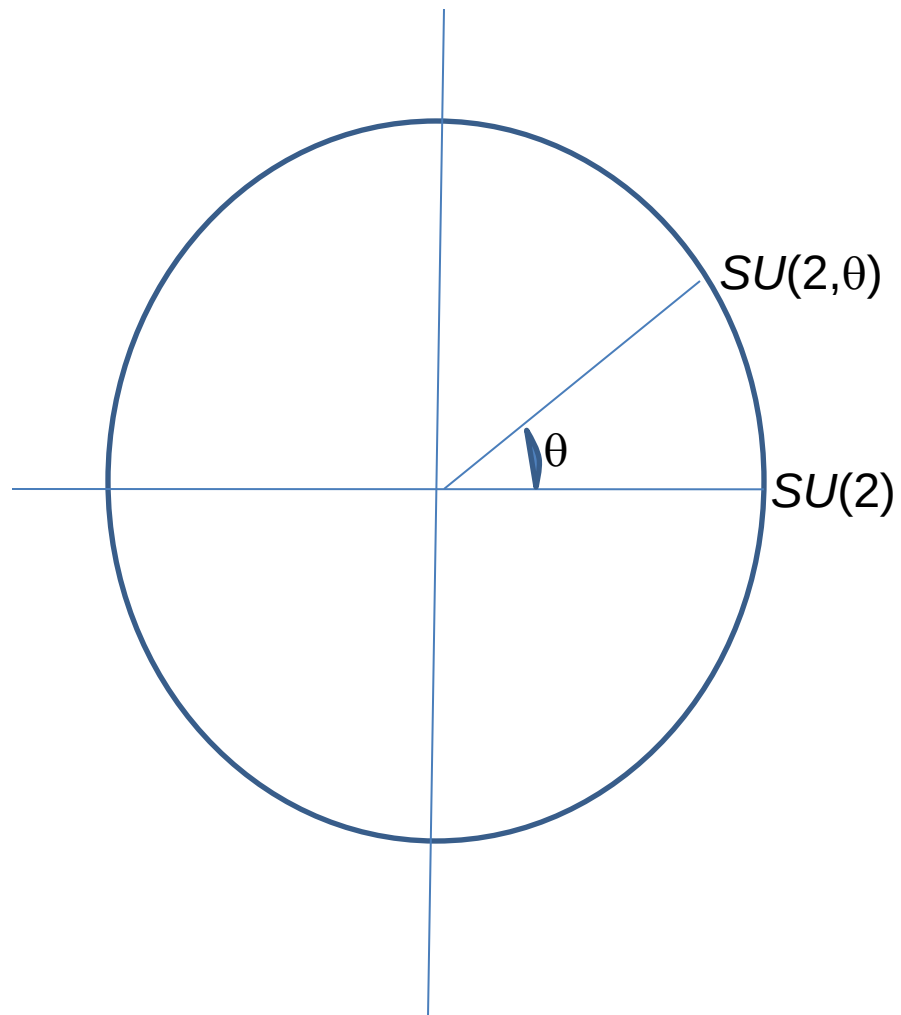
When $\theta=0$, let $\mathbf{SU}(2)=\mathbf{SU}(2,0)$. Then, every A in $\mathbf{SU}(2)$ can be represented by

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where $a = a_1 + ja_2$ and $b = b_1 + jb_2$

a_1, a_2, b_1, b_2 are real numbers

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1, \text{ i.e., } |a|^2 + |b|^2 = 1$$



$SU(2)$ can be isometrically dmbded onto the 4 dimensional Euclidean real unit sphere:

Let $\mathcal{S}^3 = \{\mathbf{x} \in \mathbb{R}^4 \mid \|\mathbf{x}\| = 1\}$

Let i be the mapping $i : A \mapsto i(A)$ from $SU(2)$ into $\mathcal{S}^3$ defined by

$$i(A) \triangleq (a_1, a_2, b_1, b_2)$$

the mapping i is one-to-one and onto.

$$\text{And } \det(A - B) = \|i(A) - i(B)\|^2 \quad (8)$$

This equation also implies that all determinants of difference matrices of two distinct 2×2 unitary matrices in $SU(2)$ are positive. From (8), one can see that the problem to find an optimal 2×2 space-time code in $SU(2)$, i.e., it is restricted to have determinant 1, becomes to find optimal packing points on the sphere $\mathcal{S}^3$, which is called *Hamiltonian constellations*

if we denote D_L as maximal minimum-distance of L -point packing on $\mathcal{S}^3$, then

$$\zeta \geq D_L^2$$

The set of $SU(2)$ is not enough to find good 2 by 2 unitary codes
 We need to consider the whole set $U(2)$: **to first have good packing points from $SU(2)$, then leverage them to $SU(2,\theta)$ using the distance property**

Let us consider a relationship between $SU(2)$ and $U(2)$ or equivalently between $SU(2)$ and $SU(2, \theta)$ for any $\theta \in [0, 2\pi)$.

For a fixed θ , we define a mapping J_θ from $SU(2, \theta)$ to $SU(2)$ as follows:

$$J_\theta(A) \triangleq e^{-j\theta/2} A, \quad \text{for } A \in SU(2, \theta). \quad (9)$$

This mapping is one to one and onto and every 2 by 2 unitary matrix A in $U(2)$ can be represented by

$$A = e^{j\theta/2} J_\theta(A), \quad \text{for some } \theta \in [0, 2\pi).$$

An important property from this mapping is that it also provides a determinant formula for a difference matrix of two matrices selected from different sets $SU(2, \theta_1)$ and $SU(2, \theta_2)$, which is stated in the following

Corollary 1: For any $A_1 \in SU(2, \theta_1)$ and $A_2 \in SU(2, \theta_2)$, we have result

$$|\det(A_1 - A_2)| = |\det(J_{\theta_1}(A_1) - J_{\theta_2}(A_2)) - 4 \sin^2((\theta_1 - \theta_2)/4)|.$$

Optimal 2 by 2 Unitary Code of 6

1) Size $L = 6$: Let $d = -5/2 + \sqrt{22}$. Select a four-point packing on $\mathcal{S}^3$ as follows:

$$\mathbf{a}_1 = (-a, -b, b, -b), \quad \mathbf{a}_2 = (-a, b, b, b)$$

$$\mathbf{a}_3 = (-a, -b, -b, b), \quad \mathbf{a}_4 = (-a, b, -b, -b)$$

where $a = \sqrt{1 - 3d/8}$ and $b = \sqrt{(1 - a^2)/3}$. By mapping these points back to $\mathbf{SU}(2)$, we have the following four unitary matrices:

$$A_1 = \begin{pmatrix} -a - bj & b - bj \\ -b - bj & -a + bj \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -a + bj & b + bj \\ -b + bj & -a - bj \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -a - bj & -b + bj \\ b + bj & -a + bj \end{pmatrix}$$

$$A_4 = \begin{pmatrix} -a + bj & -b - bj \\ b - bj & -a - bj \end{pmatrix}.$$

For other two unitary matrices, we use angle θ . Let

$$\theta_1 = 2 \arccos(d/2 - a) \quad \text{and} \quad \theta_2 = 2\pi - \theta_1$$

and

$$A_5 = e^{j\theta_1/2} I \in \mathbf{SU}(2, \theta_1), \quad A_6 = -e^{j\theta_2/2} I \in \mathbf{SU}(2, \theta_2).$$

normalized



Theorem 1: The maximal diversity product of a 2×2 unitary space–time code of size 6 is $\frac{1}{2} \sqrt{-5/2 + \sqrt{22}}$, i.e.,

$$\zeta(6) = \frac{1}{2} \sqrt{-5/2 + \sqrt{22}}.$$

Other sizes of 32, 48, 64 can be similarly constructed but we are not able to prove the optimality

TABLE I
DIVERSITY PRODUCT AND SUM COMPARISONS

	Hamiltonian Codes [9]		Parametric Codes [11]		New Codes	
Size	Diversity product	Diversity sum	Diversity product	Diversity sum	Diversity product	Diversity sum
6	0.7071	0.7071	0.7071	0.7746 (opt.)	0.7400 (opt.)	0.7400
32	0.4496	0.4496	0.4461	0.5621	0.4536	0.5217
48	0.3938	0.3938	0.3875		0.4278	0.5000
64	0.3609	0.3609	0.3535	0.4852	0.3676	0.3827

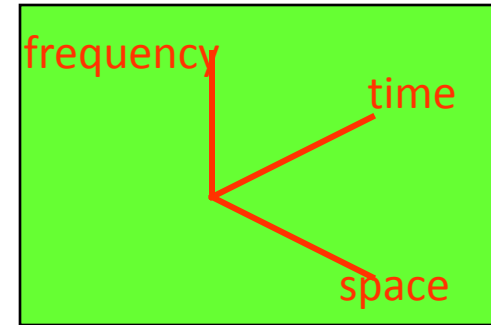
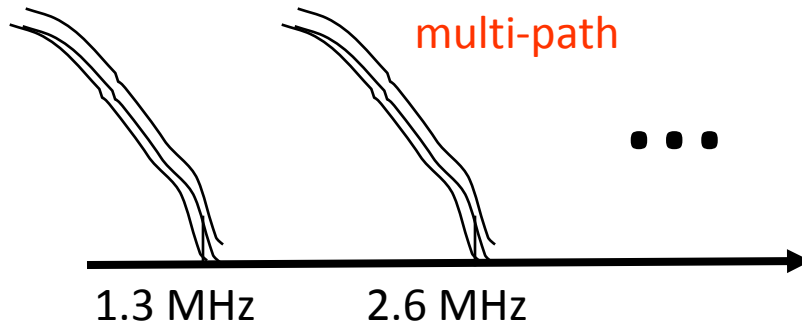
H. Wang, G. Wang, and X.-G. Xia, *IEEE Trans. Inform. Theory*, Dec. 2004.

Symbol Embedding Method

- Binary information bits are first mapped to complex symbols x_n in a signal constellation $\mathbf{S}$ and the complex symbols are embedded into a p by n matrix to transmit.
- Data rate is determined by the number of complex symbols embedded in a matrix, i.e., the symbol rate, and how many bits of a complex symbol carries, i.e., the size of a signal constellation $\mathbf{S}$.
- BLAST, OSTBC/QOSTBC, Linear lattice codes etc.

Why MIMO-OFDM?

- For broadband systems, the fading becomes frequency-selective fading



- OFDM is a good choice for frequency-selective fading channels **when the channel bandwidth is not too wide**
- MIMO is used to combat fading (low SNR)
- MIMO-OFDM is a good choice for broadband wireless systems
- In order to have a high speed wireless transmission system, both broad (**but not too broad**) bandwidth and more bandwidth efficient coding and modulation is needed
 - ✓ **Efficient space-time-frequency coding/modulation is important**

Some Open Questions

- What is the optimal space-time modulation for 2 bits/s/Hz for 3, 4, ..., transmit antennas?
- What is the optimal space-time modulation for 3 bits/s/Hz for 2,3,4,... transmit antennas?
- All these (uncoded) optimal space-time modulations are not known

Some Papers to Read

- B. M. Hochwald and W. Sweldens, Differential unitary space-time modulation," *IEEE Trans. on Information Theory*, Dec. 2000.
- B. L. Hughes, Differential space-time modulation, *IEEE Trans. on Information Theory*, Nov. 2000.
- X.-B. Liang and **X.-G. Xia**, [Unitary Signal Constellations for Differential Space-Time Modulation with Two Transmit Antennas: Parametric Codes, Optimal Designs, and Bounds](#), *IEEE Trans. on Information Theory*, August 2002.
- H. Wang, G. Wang, and **X.-G. Xia**, [Some 2 by 2 Unitary Space-Time Codes from Sphere Packing Theory with Optimal Diversity Product of Code Size 6](#), *IEEE Trans. on Information Theory* , Dec. 2004. Its [longer version](#).