



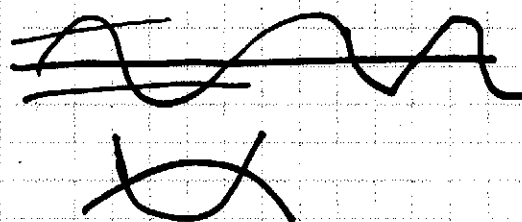
Chapter 7: Intersymbol Interference (ISI) Cancellation

§ 1. Introduction

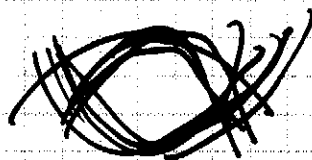
* Eye diagrams / eye patterns

In digital communication transmission waveforms, it is usually up/down

up means 1
down means -1



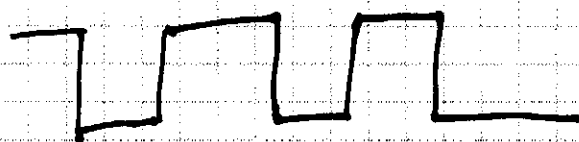
At the receiver, When you fold all segments of all symbols, the accumulated curves form a diagram, which is called eye diagram. If there is a space in the center, it is called eye open otherwise it is called eye closed.



eye open means 1 and -1 can be detected

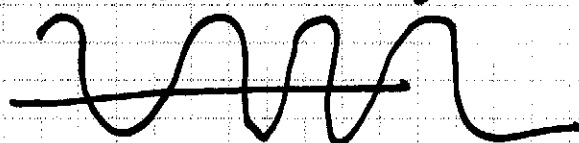
closed - - - - -

not be detected
(sometimes)



Infinite bandwidth Channel

band limited
channel



bandwidth larger →
bandwidth smaller →

eye is more open
close.



eye diagram is an evaluation of a channel.

* ISI (Intersymbol Interference)

What is ISI?

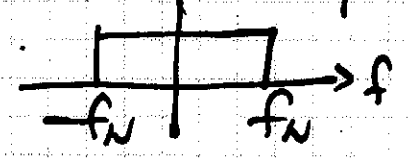
$x(n)$
current symbol

$x(n-1)$
past symbol

received : $a_0 x(n) + a_1 x(n-1) \triangleq y(n)$

↑ This is an intersymbol interference.

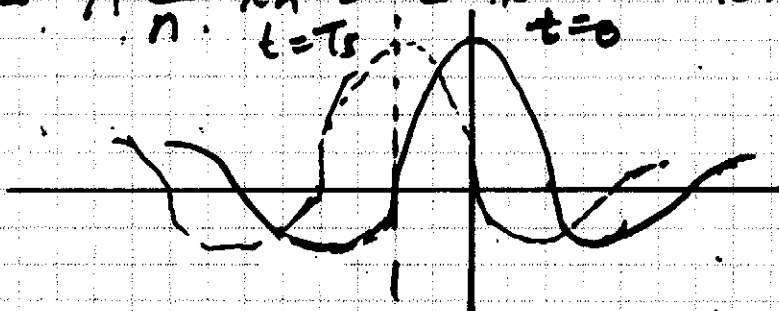
Why ISI?

Suppose the pulse shape function has ideal lowpass property with bandwidth $f_N = \frac{1}{2T_s}$: 
Nyquist rate 

$h_a(t) = \text{sinc } f_N(t - nT_s)$ and the channel is also ideally band-limited.

Then the transmit and receive waveform is,

$$r(t) = A \sum_n x_n \text{sinc } f_N(t - nT_s)$$

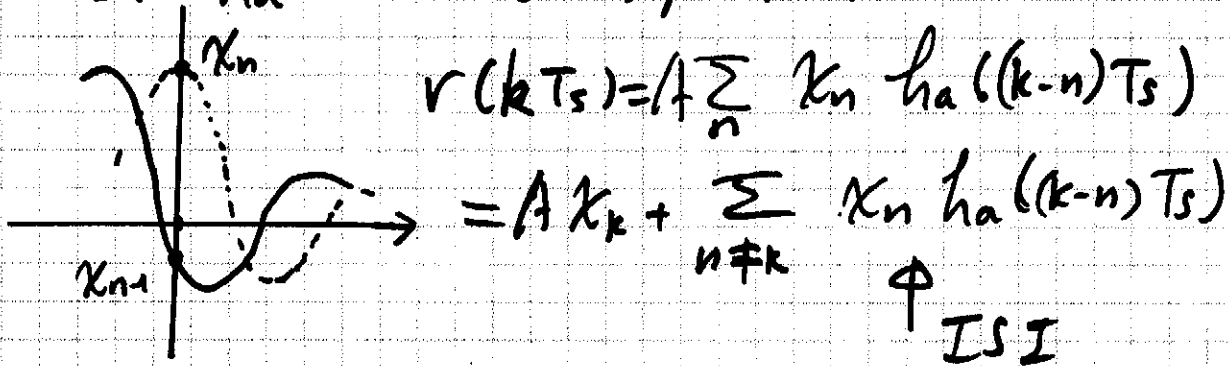


The received signal is a superposition of many waves.



Since $h_a(nT_s) = \delta(n)$, $r(kT_s) = X_k$
 $\Rightarrow$ There is no ISI

If $h_a(nT_s) \neq \delta(n)$, then there is ISI.



$$r(kT_s) = A \sum_n X_n h_a((k-n)T_s)$$

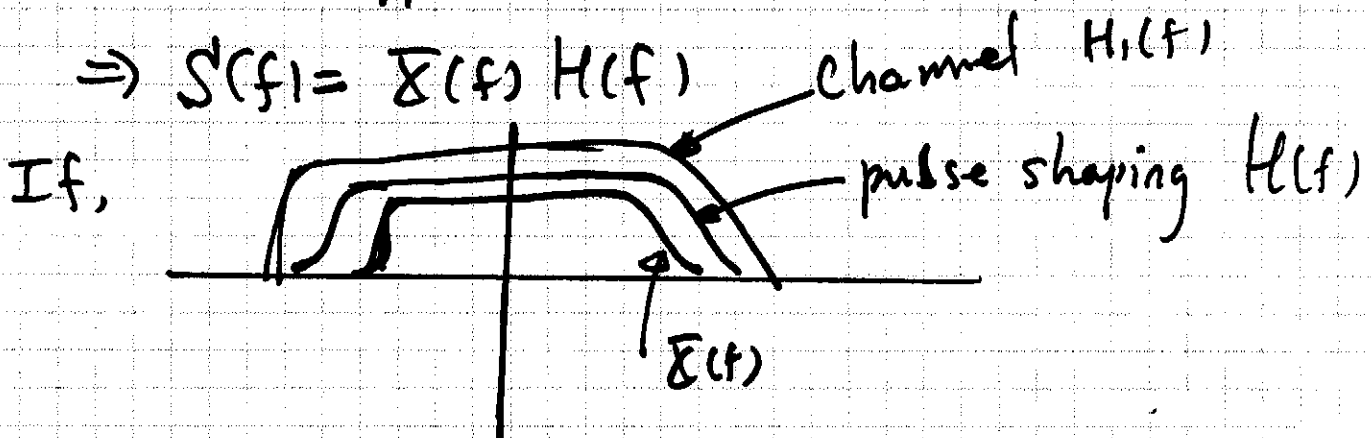
$$= A X_k + \sum_{n \neq k} X_n h_a((k-n)T_s)$$

↑
ISI

Now let us draw the Fourier spectrum of a pulse shaping function $h(t)$. Suppose the transmitted waveform is

$$S(t) = \sum_n X_n h(t - nT_s)$$

$$\Rightarrow S(f) = X(f) H(f)$$



then, the received signal is

$$r(t) = A \sum_n X_n h(t - nT_s)$$

Otherwise

$$r(t) = A \sum_n X_n \hat{h}(t - nT_s)$$

where $\hat{h}(t) = h(t) * h_c(t)$

↑
channel impulse response function



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The bottom line for ISI free condition is

$$\hat{h}(nT_s) = \delta(n).$$

When channel is ideal, i.e., ~~all~~ transmitted signal is all passed,

the pulse shaping function $h(t)$ for ISI-free needs to be

$$h(nT_s) = \delta(n) \quad (*)$$

↑
called Nyquist condition
(or Nyquist pulse).

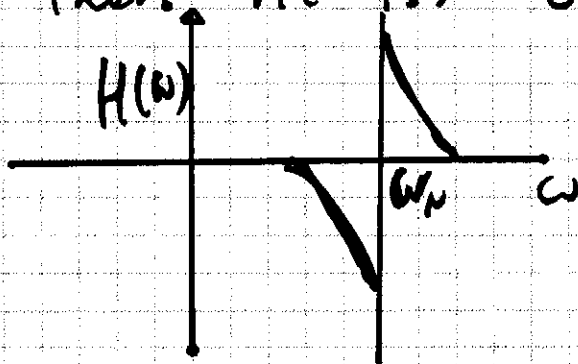
* Nyquist's Vestigial Symmetry Theorem

Raised-Cosine Filters

$$\text{Let } \omega_N = 2\pi f_N = 2\pi \frac{1}{2T_s} = \frac{\pi}{T_s}$$

If $H(\omega_N - x) = -H(\omega_N + x)$, $0 < x < \omega_N$,

then $h(nT_s) = \delta(n)$, i.e., ISI-free.



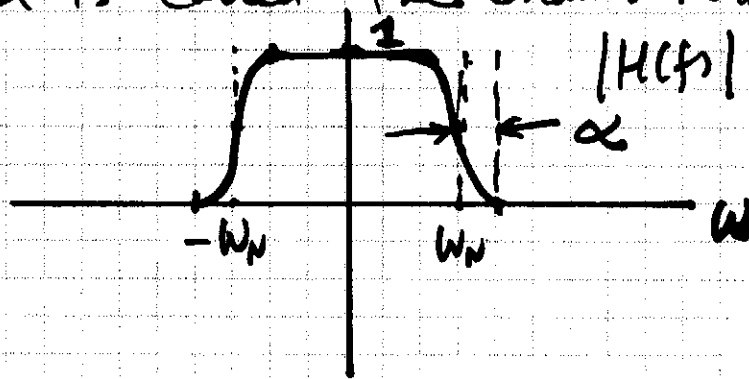


Raised-Cosine Filters

$$H(f) = \begin{cases} 1, & 0 < f < f_N - f_x \\ \frac{1}{2} \left\{ 1 - \sin \frac{\pi}{2\alpha} \left[\frac{f}{f_N} - 1 \right] \right\}, & f_N - f_x < f < f_N + f_x \\ 0, & f > f_N + f_x \end{cases}$$

$$|H(\omega)| = \begin{cases} 1, & 0 \leq \omega \leq \frac{\pi}{T_s} (1 - \alpha) \\ \cos^2 \left\{ \frac{\pi}{4\alpha} \left[\omega - \frac{\pi(1-\alpha)}{T_s} \right] \right\}, & \frac{\pi}{T_s} (1 - \alpha) \leq \omega \leq \frac{\pi}{T_s} (1 + \alpha) \\ 0, & \omega > \frac{\pi}{T_s} (1 + \alpha) \end{cases}$$

where α is called the channel roll-off factor.



* Square-Root Raised Cosine Filter

Since at the receiver, usually the matched filter is implemented first, to reduce noise,

$G(\omega) \triangleq \sqrt{|H(\omega)|}$ is usually as the pulse shaping filter. At the receiver, $|H(\omega)|$ appears and ISI is then free, i.e., $|G(\omega)|^2$ is ISI-free.



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The Question is whether there exists $H(\omega)$ such that both $H(\omega)$ and $G(\omega)$ are ISI-free?

Answer is Yes!

Xiang-Gen Xia, "A family of pulse shaping filters with ISI-free matched and unmatched filter properties," IEEE Transactions on Communications, Oct. 1997.

§2 Modeling of ISI Channels

Modulated signal $V(t) = A \sum_K x_k h_a(t - kT)$.

Suppose $C(t)$ is the channel (time-invariant) impulse response. Then the received signal is

$$W(t) = \sum_{k=-\infty}^{\infty} x_k h(t - kT) + Z(t)$$

where $h(t) = \int_{-\infty}^{\infty} h_a(\tau) C(t - \tau) d\tau$ is the convolution of the pulse shaping function $h_a(t)$ and the channel impulse response $C(t)$. $Z(t)$ is a zero-mean complex AWGN with a power spectral density of N_0 Watts/Hz.

Since $C(t)$ is causal, $h(t) = \int_0^{\infty} h_a(\tau) C(t - \tau) d\tau, t \geq 0$.



* Suppose the overall pulse $h(t)$ has a finite duration.

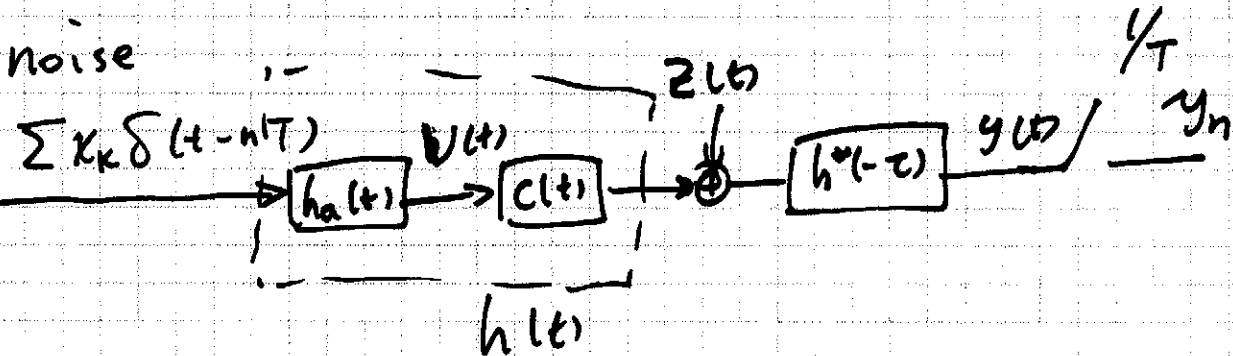
$h(t) = 0$ for $t \leq 0$ and $t \geq LT$, where L is some positive integer.

* Assuming the matched filter $h^*(-t)$ is used at the receiver,

$$y(t) = \sum_{k=-\infty}^{\infty} x_k f(t-kT) + v(t)$$

where $f(t) = \int_{-\infty}^{\infty} h^*(\tau) h(\tau+t) d\tau$

and $v(t) = \int_{-\infty}^{\infty} h^*(\tau) z(t-\tau) d\tau$ is the filtered



* Sampling the matched filter output every T seconds:

$$y_n = y(nT) = \sum_{k=-\infty}^{\infty} x_k f(nT - kT) + v(nT)$$

$$= \sum_{k=-\infty}^{\infty} x_n f_{n-k} + v_n$$

$$= x_n f_0 + \underbrace{\sum_{k \neq n} x_k f_{n-k}}_{\text{ISI}} + v_n$$

↑ noise may not be i.i.d.



The noise after the matched filter is

$$v_n = \int_{-\infty}^{\infty} h^*(-\tau) z(nT + \tau) d\tau$$

and the autocorrelation is

$$\phi_{vv}(n) = \frac{1}{2} E[v_{kn} v_k^*] = N_0 f_n$$

Since the additive noise v_n is not white now, it is sometimes important to whiten v_n into white noise.

From the above autocorrelation formula, $\phi_{vv}(n)$, we

consider the filter $F(z) = \sum_{n=-L}^L f_n z^{-n}$,

where $f_n = h_n * h_n^*$
 $\uparrow$ length L , i.e., $0 \leq n \leq L$.

Since $f_n = h_n * h_n^*$, we may write (factorise)

$$F(z) = \underset{\substack{\uparrow \\ \text{length } L}}{G(z)} \underset{\substack{\uparrow \\ \text{length } L}}{G^*(1/z^*)}$$

The whole filter may not be invertible.

where $G(z)$ and $G^*(1/z^*)$ are polynomials of degree L , and have conjugate reciprocal roots. Choose $G^*(1/z^*)$ to be minimum phase, i.e., its all zeros are inside the unit circle.

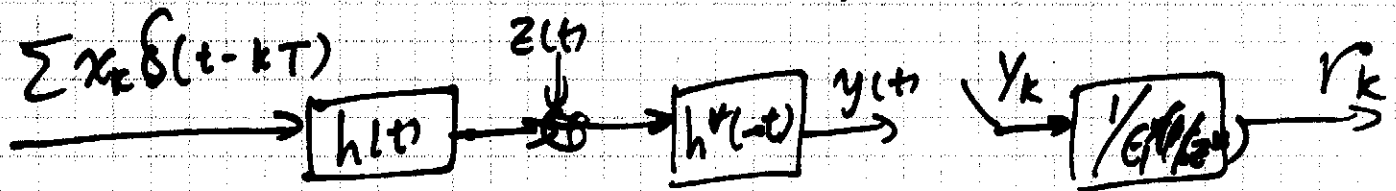
Then apply the filter $1/G^*(1/z^*)$ to the matched



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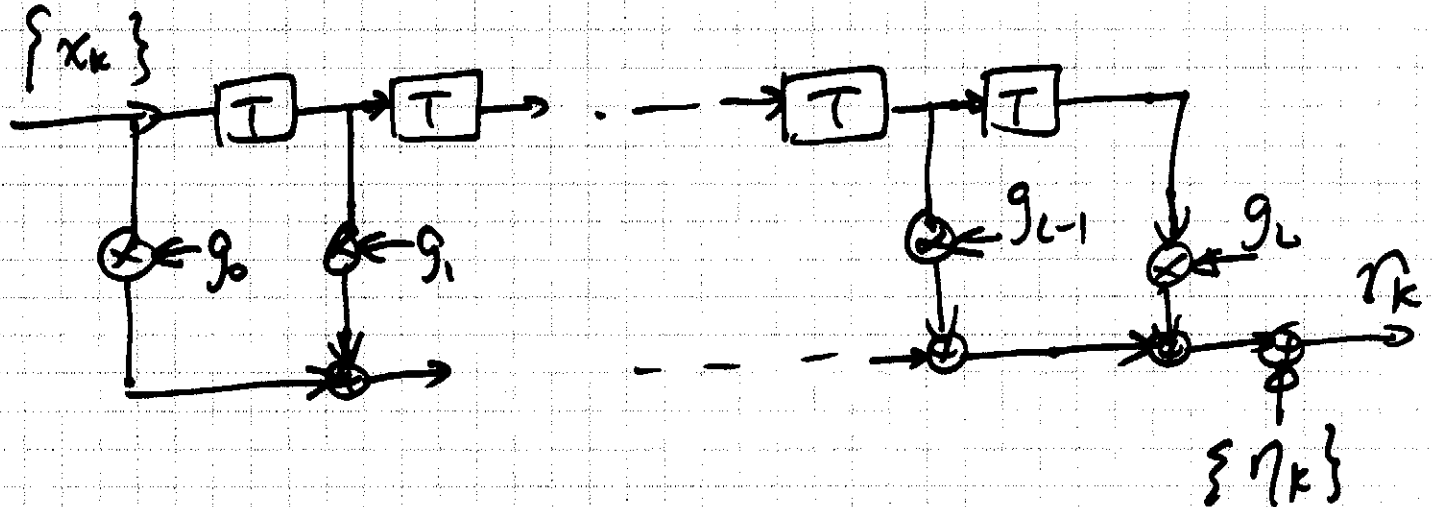
filtered signal, we have

$$R(z) = \underbrace{X(z)G(z) + U(z)}_{\text{White}} \frac{1}{G^*(1/z^*)}$$

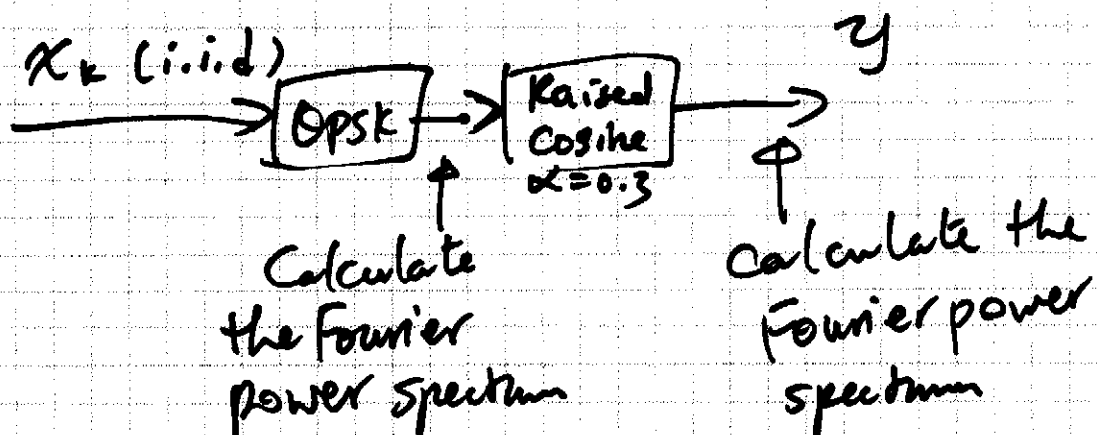


Equivalent discrete-time white noise channel model:

$$r_k = \sum_{n=0}^L g_n x_{k-n} + \eta_k$$



HWK :



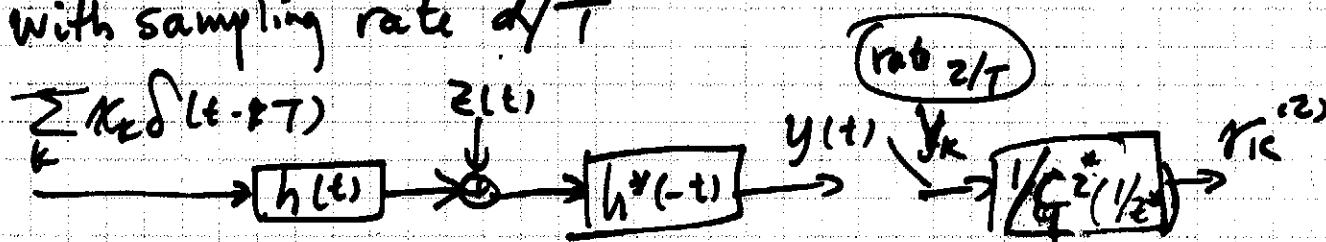


§3. Fractionally-spaced Receiver

Often used in equalizations.

In practice, the matched filter outputs are often oversampled for the purpose of extracting timing information and to mitigate the effects of timing errors.

It is usual to double the band sampling rate, i.e., with sampling rate $2/T$



In this case, the matched filtered system becomes

$$\mathbf{f}^{(2)} = (f_{-2L}^{(2)}, f_{-2L+1}^{(2)}, \dots, f_1^{(2)}, f_0^{(2)}, \dots, f_{2L-1}^{(2)}, f_{2L}^{(2)})$$

$$\rightarrow \text{where } f_n^{(2)} = f(nT/2)$$

Do the same thing for the whitening the noise,

$$\mathbf{F}^{(2)}(z) = \mathbf{G}^{(2)}(z) (\mathbf{F}^{(2)}(1/z^*))^*$$

After the whitening filter $1/(\mathbf{G}^{(2)}(1/z^*))^*$

$$r_k^{(2)} = \sum_{n=-\infty}^{\infty} g_k^{(2)} x_{n-k}^{(2)} + \eta_k^{(2)} \quad *$$

where $\{\eta_k^{(2)}\}$ is a $T/2$ -spaced white Gaussian noise sequence,



$\{g_k^{(2)}\}$ are the coefficients of a $T/2$ -spaced discrete-time transversal filter having a transfer function $G^{(2)}(z)$. The sequence $\{x_n^{(2)}\}$ is the corresponding $T/2$ -spaced input symbol sequence:

$$x_n^{(2)} = \begin{cases} x_{nh}, & n=0, 2, 4, \dots \\ 0, & n=1, 3, 5, \dots \end{cases} \quad (**)$$

Because of (**), the system (*) in terms of x_n is not time-invariant. The system (*) can be split into two LTI systems:

$$r_{2n}^{(2)} = \sum_{k=0}^L g_{2k}^{(2)} x_{n-k} + \eta_{2n}^{(2)}$$

$$r_{2n+1}^{(2)} = \sum_{k=0}^{L-1} g_{2k+1}^{(2)} x_{n-k} + \eta_{2n+1}^{(2)}$$

One of them is redundant but helpful in equalization.

$$R_1(z) = G_1(z) \tilde{x}(z)$$

$$R_2(z) = G_2(z) \tilde{x}(z)$$

$R_1(z)$ and $R_2(z)$ are known. How to blindly find $\tilde{x}(z)$ or $\{x_n\}$? Called blind equalization

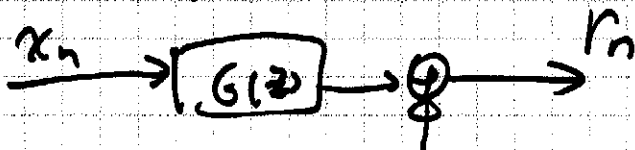
(Tong, Xu, Kailath, 80's)



§4. Symbol-by-Symbol Equalizations

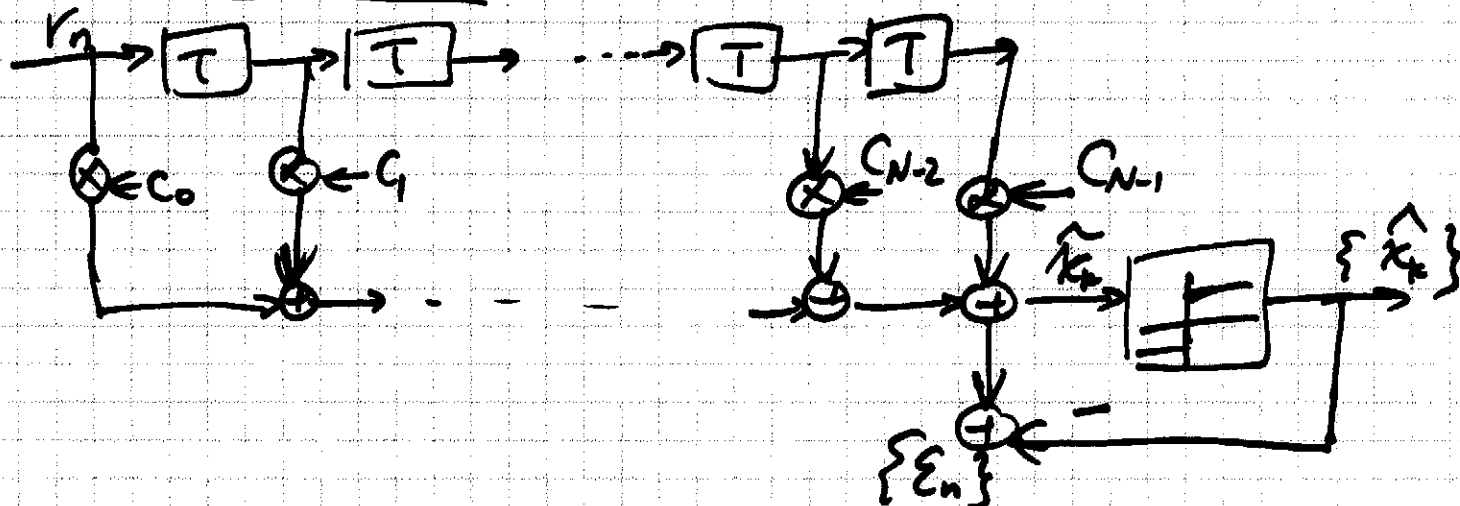
linear forward equalizers
nonlinear decision feedback equalizers
blind equalization

zero-forcing
mean-square-error



Equalizer is to determine (estimate) x_n from r_n .

* Linear equalization



$\underline{c} = (c_0, c_1, \dots, c_{N-1})^T$ to approximate $\frac{1}{G(z)}$

$$\hat{x}_k = \sum_{j=0}^{N-1} c_j r_{k-j}$$

$$\underline{g}_n = \sum_{j=0}^{N-1} c_j g_{n-j} = \underline{c}^T \cdot \underline{g}_n$$

where $\underline{g} = (g_0, g_1, \dots, g_{N-1})^T$

$$\underline{\hat{g}} = (0, 0, \dots, 0, 1, 0, \dots, 0)^T = \underline{e}_d$$

$$\underline{g}_n = (g_n, g_{n-1}, \dots, g_{n-N+1})^T, \quad d \text{ is a delay.}$$

$$\underline{g} = \underline{c} * \underline{g} \approx \underline{\delta}$$

goal



$$D_p = \frac{1}{|g_d|} \sum_{\substack{n=0 \\ n \neq d}}^{N+L-1} |g_n - \hat{g}_n|^2$$

↑ desired equalized channel.

ZF equalizer:

Let $N = 2d_2 + 1$ and $d = d_1 + d_2$

Lucky showed that if the initial distortion without equalization is less than unity, i.e.,

$$D = \frac{1}{|g_{d_1}|} \sum_{\substack{n=0 \\ n \neq d_1}}^L |g_n| < 1,$$

then D_p is minimised by those N tap values which simultaneously cause $g_j = \hat{g}_j$ for $d - d_2 \leq j \leq d + d_2$.

However, if $D > 1$, the ZF criterion is not guaranteed to minimise the peak distortion

ZF solution:

Let $\underline{G} = [g_{d_1}, \dots, g_d, \dots, g_{N+d_1-1}]$

and $\underline{\hat{G}} = (\hat{g}_{d_1}, \dots, \hat{g}_d, \dots, \hat{g}_{N+d_1-1})^T$.

Then, for the case when $\underline{\hat{G}} = \underline{G}$, the equalized channel is given by

$$\underline{g} = (g_0, \dots, g_{d_1-1}, 0, 0, \dots, 0, \dots, 0, g_{d_1+N}, \dots, g_{N+L-1})^T$$



Then the vector of optimal tap, $\underline{c}_{op}$, is

$$\underline{c}_{op} = (G^{-1})^T \underline{\hat{g}}$$

Ex: Let $\underline{g} = (0.9, -1.5, 0.2, 0.1, -0.05)^T$.

Then $D = \frac{1}{19.1} \sum_{n=1}^4 |g_n| = 0.5$

Thus, the minimum distortion is achieved with the ZF solution. Assume we use/design 3-tap ZF equaliser.

Since g_0 is the component of $\underline{g}$ having the largest magnitude, $d_1 = 0$ and the optimal equaliser delay is $d = 1$. ($d = d_1 + d_2$ and $d_1 = 0, d_2 = 1$). Let $\underline{\hat{g}}_0 = \underline{e}_0$, then $\underline{\hat{g}} = (1, 0, 0)$.

Then $G = \begin{bmatrix} 0.9 & -0.15 & 0.2 \\ 0 & 0.9 & -0.05 \\ 0 & 0 & 0.9 \end{bmatrix} = [\underline{g}_0, \underline{g}_1, \underline{g}_2]$

The optimal solution (ZF equaliser) is

$$\underline{c}_{op} = (G^{-1})^T \underline{\hat{g}} = (1.111, 0.1832, -0.2160)^T$$

The overall response of the channel & equaliser is

$$\underline{q} = (1.0, 0, 0, 0.1806, -0.08, -0.036, -0.018, 0, \dots)^T$$

The minimum distortion with this equaliser is

$$D_{min} = \frac{1}{180} \sum_{n=1}^6 |q_n - \hat{q}_n|^2 = 0.30247$$



Adaptive solution:

In practice, the channel coefficients $\{g_n\}$ are unknown, and a known finite length sequence x is used to train the channel/equalizer.

During the training, the equalizer taps c can be obtained by using the following steepest-descent recursive algorithm

$$c_j^{n+1} = c_j^n + \alpha \varepsilon_n x_{n-j-d_1}^*, \quad j=0, 1, \dots, N-1, \quad (*)$$

$$\text{Where } \varepsilon_n = x_{n-d} - \hat{x}_n \\ = x_{n-d} - \sum_{i=0}^{N-1} c_i x_{n-i}$$

is the error sequence, $\{c_j^n\}$ is the set of equalizer tap values at stage n , and α is an adaptation step-size that can be optimized to provide rapid convergence.

The adaptation rule in (*) attempts to force the crosscorrelations $\varepsilon_n x_{n-j-d_1}^*$, $j=0, 1, \dots, N-1$, to zero.

Note that $\{x_n\}$ and η_n are uncorrelated.

$$\begin{aligned} E(\varepsilon_n x_{n-j-d_1}^*) &= E(x_{n-d} x_{n-j-d_1}^*) - \sum_{i=0}^{N-1} c_i E(x_{n-i} x_{n-j-d_1}^*) \\ &= \delta_{d-j} - \sum_{i=0}^{N-1} c_i g_{j+d_1-i} \\ &= \delta_{d-j} - g_{j+d_1}, \quad j=0, 1, \dots, N-1, \end{aligned}$$



$$\begin{aligned} &= 0 \quad \text{if} \quad \begin{cases} p_d = 1 \\ p_i = 0 \text{ for } d_1 \leq i < d \\ d < i \leq d + d_2 \end{cases} \end{aligned}$$

After training the equaliser, a decision-feedback mechanism is typically employed such that the newly estimated symbol $\hat{x}_n$ is used to update the tap coefficients $\underline{c}$. (Data mode) transmitting data.

Error is

$$e_n = \hat{x}_{n-1} - \sum_{i=0}^{N-1} c_i r_{n-i}$$

$$\hat{x}_n = \text{decision}(\tilde{x}_n)$$

Performance Analysis

Assume the ZF equaliser has an infinite number of taps so that $D_p = 0$, i.e., $\hat{\mathbf{z}} = \mathbf{z}$. Then $Q(z) = 1 = C(z)G(z)$ and $C(z) = \frac{1}{G(z)}$

The cascade of the noise-whitening filter with transfer function $\frac{1}{G^H(z)}$ is $C'(z) = \frac{1}{G^H(z)G(z)} = \frac{1}{F(z)}$

Note that the noise sequence at the input to the equivalent equaliser $C'(z)$ has the discrete autocorrelation



$$\phi_{vv}(k) = N_0 f_k$$

$$\Rightarrow S_{vv}(f) = N_0 F(e^{j2\pi fT}) : \text{power spectral density (psd)} \\ \text{for } |f| \leq \frac{1}{2T}.$$

$\Rightarrow$ The psd of the noise sequence $\{z_n\}$ at the output of the equalizer is

$$S_{zz}(f) = \frac{N_0}{F(e^{j2\pi fT})}, \quad |f| \leq \frac{1}{2T}$$

and the noise samples have variance

$$\begin{aligned} \sigma_z^2 &= T \int_{-1/2T}^{1/2T} S_{zz}(f) df \\ &= T \int_{-1/2T}^{1/2T} \frac{N_0}{F(e^{j2\pi fT})} df \end{aligned}$$

If $E(|x_k|^2) = 1$ and $\hat{p}_n = \delta_{n0}$, then the SNR at the output of the infinite tap equalizer is

$$\gamma_{\infty} = \frac{E(|x_k|^2)}{\sigma_z^2} = \frac{1}{\sigma_z^2}$$