Consider the real line $\Gamma = \mathbb{R}$.

* A Borel set is a set obtained from countable open sets unions, intersections, and complimentary (or closed sets) denoted by $\mathcal{B}(\mathbb{R})$

Borel algebra or Borel sigma σ-algebra.

Any measure defined on Borel sets is called Borel measure.

A non-Borel set was obtained by Lusin 1927.

All subsets of $\mathbb{R}$ is denoted by $\mathcal{P}(\mathbb{R})$

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$$

* Lebesgue measure

For any interval $I = [a, b] \subset \mathbb{R}$, or (a, b), let $l(I) = b - a$, denote its length. For any subset $E \subset \mathbb{R}$, $\in \mathcal{P}(\mathbb{R})$, the Lebesgue outer measure $\lambda^*(E)$ is defined as an infimum

$$\lambda^*(E) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) : I_k,\ k = 1, 2, \cdots, \text{ is a sequence of open intervals with } E \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

$E \in \mathcal{P}(\mathbb{R})$ is said Lebesgue measureable if for any $A \in \mathcal{P}(\mathbb{R})$ the following
$\subset \mathbb{R}$,
<u>Carathéodory criterion</u> holds:

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c),$$

where E^c is the complementary of E.

All Lebesgue measurable sets denoted by $\mathcal{L}(\mathbb{R})$ is a σ-algebra

$\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra of $\mathbb{R}$

and $\mathcal{L}(\mathbb{R}) \supset \mathcal{B}(\mathbb{R})$

A non-Lebesgue measured set is given by Vital, called Vital set.

* Countable

$\mathbb{N} = \{0, 1, 2, \cdots\}$ called countable.

For any set E, if there is a one-to-one and onto mapping between E and $\mathbb{N}$, E is called countable. Its cardinality is denoted as $\aleph_0$. Its power set, i.e., its all subsets,

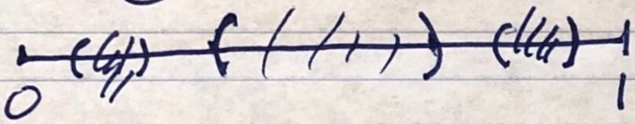
$\mathcal{P}(\mathbb{N}) \underset{\text{and onto}}{\overset{\text{one to one}}{\longleftrightarrow}} \mathbb{R}$. $\mathbb{R}$ is not countable.

The cardinality of $\mathbb{R}$ is $2^{\aleph_0}$

The cardinality of Borel σ-algebra is also the same as that of $\mathbb{R}$, i.e., also $2^{\aleph_0}$.

* What is the cardinality of $\mathcal{L}(\mathbb{R})$?

Cantor set $\mathcal{C} \subset [0, 1]$



delete or cut all the middle one third open intervals of all the left intervals. What is left is called the Cantor set.

Since $\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = 1$, that is the length of all the middle intervals deleted, the measure of the Cantor set is $1-1=0$.

so $\lambda(C)=0$.

Thus, any subset of C, i.e., any set in $\mathcal{P}(C)$ is Lebesgue measurable since it has measure 0.

$\Rightarrow \mathcal{P}(C) \subset \mathcal{L}(\mathbb{R})$

$\Rightarrow$ The cardinality of $\mathcal{L}(\mathbb{R})$ is at least that of $\mathcal{P}(C)$

For any element ~~in $\mathcal{P}(C)$~~ in C, it can be represented by $.t_1 t_2 \cdots\cdots$, where $t_1, t_2 \cdots \in \{0, 2\}$, ~~ter~~ ternary representation, vice versa

Change $2 \longrightarrow 1$,

it corresponds a binary representation.

Thus $C \underset{\text{onto}}{\overset{\text{one-to-one}}{\Longleftrightarrow}} [0, 1]$

$\Rightarrow$ The cardinality of C is also $2^{\aleph_0}$

and the cardinality of $\underline{\mathcal{P}(C) = (2)^{2^{\aleph_0}}}$

Since $\mathcal{L}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$ that has its cardinality $(2)^{2^{\aleph_0}}$

The cardinality of $\mathcal{P}(C) \leq$ that of $\mathcal{L}(\mathbb{R})$

$\leq$ that of $\mathcal{P}(\mathbb{R})$

$\Rightarrow$ The cardinality of $\mathcal{L}(\mathbb{R})$ is

also $(2)^{2^{\aleph_0}}$

while the cardinality of $\mathcal{B}(\mathbb{R})$

is only $2^{\aleph_0}$.

There are much more Lebesgue measurable sets that Borel measureable sets.

* Continuum hypothesis

$\aleph_0 < \aleph_1 < \cdots$

$2^{\aleph_0} = \aleph_1$

Gödel

$2^{\aleph_i} = \aleph_{i+1}$ Paul Cohen

Lebesgue Measureable functions.

$E \in \mathcal{L}(\mathbb{R})$ if $f^{-1}(E) \in \mathcal{L}(\mathbb{R})$,

where $f : \mathbb{R} \to \mathbb{R}$.

Lebesgue Integrable functions

Let μ be a Lebesgue measure. for a function $f(t)$, Let

$$f_+^*(t) = \mu(\{x \mid f^+(x) > t\}), \qquad f^+ : f \geq 0$$

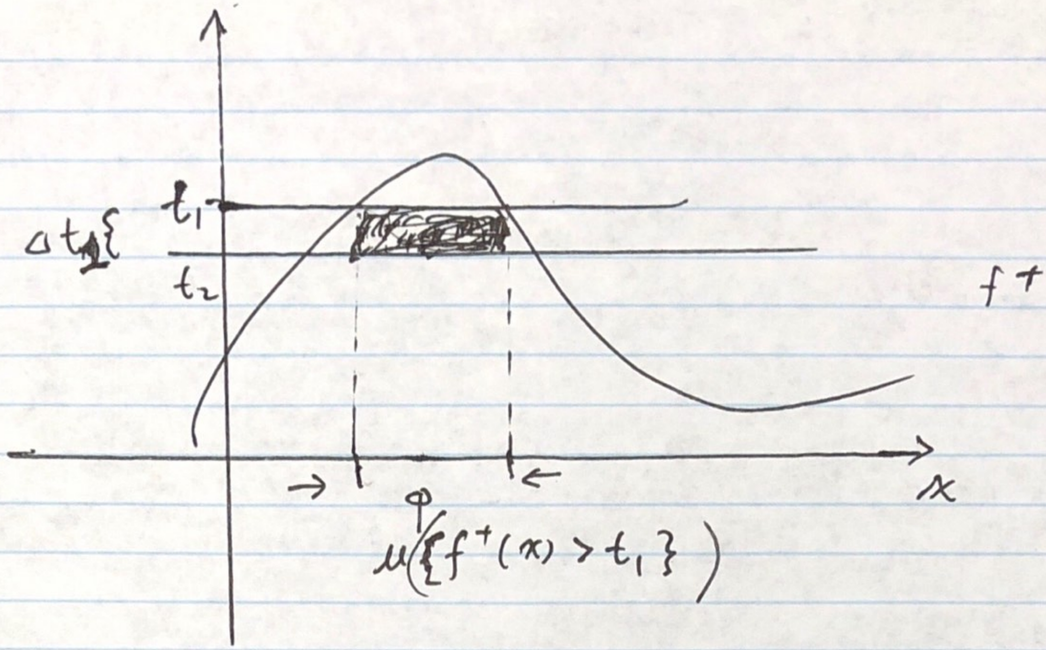
$$f_-^*(t) = \mu(\{x \mid f^-(x) > t\}) \qquad f^- : -f \geq 0$$

where f^+ and f^- are the positive and negative parts

$$\int f \, d\mu \triangleq \int_0^\infty f^+ \, dt - \int_0^\infty f^- \, dt$$

where $\int_0^\infty f^+ \, dt$ and $\int_0^\infty f^- \, dt$ are the Riemann integrals.

$$\lim_{\Delta t_i \to 0} \sum_i f(t_i) \Delta t_i \quad : \text{Riemann integral.}$$



$$f_+^*(t_1)\ \Delta t_1 = \mu(\{f^+(x)>t_1\})\cdot \Delta t_1$$

$\sum_i f_+^*(t_i)\Delta t_i \approx$ the area of $f^+(t)$ and the axis $\cancel{x}$

$\Rightarrow \int_0^\infty f^+ dt$ is the Riemann integration

Since $f_+^*(t)$ and $f_-^*(t)$ are both decreasing funtions, their Riemann integrations always exist.

Lebesgue measurable function

$\int |f| dx < \infty$

* Lebesgue measurable $\rightarrow$ Lebesgue integrable.

Why do we need Lebesgue integrability?

Why do we need Lebesgue measure?

This is related to the limit of integration of a sequence of functions.

Lebesgue's dominated convergence

Theorem: $\boxed{\text{the most important theorem in real Analysis}}$

Suppose g is Lebesgue integrable on E $\int_E |g| d < \infty$

The sequence f_n of Lebesgue measurable functions satisfies:

(i) $f_n \rightarrow f$ almost everywhere (a.e.) on E

(ii) $|f_n| \leq g$. a.e. on E for all n.

Then f is Lebesgue integrable and

$$\lim_{n\to\infty} \int_E f_n \, dx = \int_E \lim_{n\to\infty} f_n \, dx = \int_E f dx$$

This helps to calculate integrations.

* $f: \mathbb{R} \rightarrow \mathbb{R}$ if $f^{-1}(E) \in \mathcal{L}(\mathbb{R})$ for every $E \in \mathcal{L}(\mathbb{R})$

f is called Lebesgue measurable.

without Lebesgue measure and Lebesgue integration, not all bounded functions on a bounded interval are integrable!

✗ A bounded function $f: [a, b] \to \mathbb{R}$ is Riemann integrable iff it is continuous almost everywhere with respect to the Lebesgue measure on $[a, b]$.

there exist functions that are not continuous anywhere:
1 on irrationals and 0 on rationals

Free Probability Theory Voiculescu 80's

for non commutative variables

$$x_1 x_2 \neq x_2 x_1$$

such as matrices

$\underbrace{E(x)}_{\varphi}$ is a functional, as expectation, φ is a value

free random variables

free central limit theorem

"Undesired Cross Terms,"
https://www.eecis.udel.edu/~xxia/CrossTerms.pdf

"A Simple Introduction to Free Probability Theory and Its Application to Random Matrices,"
https://www.eecis.udel.edu/~xxia/Free_Probab.pdf