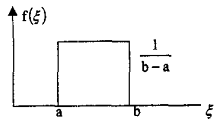

Fig. 1. Uniform pdf.

PROOF The authors [1] assumed that both the target relative position and relative speed are uniformly distributed between 0 to 60 km and −40 to 40 m/s, respectively. Let us derive the variance of uniform distributed variable. The uniform probability density function (pdf) is defined by [2]

$$f(\xi) = \frac{1}{b-a} \qquad a \le \xi \le b$$
$$= 0 \qquad \text{otherwise}$$

for real constants $-\infty < a < \infty$ and $b > a$. Fig. 1 illustrates the behavior of the above function.

$$\sigma_\xi^2 = E[\xi^2] - \bar{\xi}^2$$
$$= \int_a^b \xi^2 \left(\frac{1}{b-a}\right) d\xi - \bar{\xi}^2$$
$$\sigma_\xi^2 = \frac{1}{b-a} \frac{\xi^3}{3}\Big|_a^b - \bar{\xi}^2$$
$$= \frac{1}{(b-a)} \frac{b^3-a^3}{3} - \left(\frac{b+a}{2}\right)^2$$
$$= \frac{(b-a)^2}{12}.$$

In case of relative range, x component $b = 60\sin\beta_0$, $a = 0$ and y component $b = 60\cos\beta_0$, $a = 0$. Hence

$$\sigma_{R_x}^2 = \frac{(60\sin\beta_0)^2}{12}, \qquad \sigma_{R_y}^2 = \frac{(60\cos\beta_0)^2}{12}.$$

In case of relative speed, $a = -40$ $b = 40$

$$\sigma_{S_x}^2 = \sigma_{S_y}^2 = \frac{(40+40)^2}{12} = \frac{80^2}{12}$$

$$P(0 \mid 0) = \text{diag}\left[\frac{(60\sin\beta_0)^2}{12} \quad \frac{80^2}{12} \quad \frac{(60\cos\beta_0)^2}{12} \quad \frac{80^2}{12}\right]$$
$$= \text{diag}\left[\frac{(60\sin\beta_0)^2}{12} \quad \frac{40^2}{3} \quad \frac{(60\cos\beta_0)^2}{12} \quad \frac{40^2}{3}\right].$$

S. KOTESWARA RAO
Scientist "F"
N.S.T.L.
Visakhapatnam-27, A.P.
India

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Doppler Ambiguity Resolution Using Optimal Multiple Pulse Repetition Frequencies

Ferrari, Bérenguer, and Alengrin recently proposed an algorithm for velocity ambiguity resolution in coherent pulsed Doppler radar using multiple pulse repetition frequencies (PRFs). In this algorithm, two step estimations (folded frequency and ambiguity order) for the Doppler frequency is used by choosing particular PRF values. The folded frequency is the fractional part of the Doppler frequency and is estimated by averaging the folded frequency estimates for each PRF. The ambiguity order is the integer part of the Doppler frequency and is estimated by using the quasi-maximum-likelihood criterion. The PRF are grouped into pairs and each pair PRF values are symmetric about 1. The folded frequency estimate for each pair is the circular mean of the two folded frequency estimates of the pair due to the symmetry property.

We propose a new algorithm based on the optimal choice of the PRF values, where the PRF values are also grouped into pairs. In each pair PRF values, one is given and the other is optimally chosen. The optimality is built upon the minimal sidelobes of the maximum likelihood criterion. Numerical simulations are presented to illustrate the improved performance.

I. INTRODUCTION

Multiple pulse repetition frequency (PRF) is commonly used in modern-day radars for the velocity ambiguity resolution in coherent pulsed Doppler radars, see for example [1–4]. In this approach, the conventional method for achieving the ambiguity resolution is to search for the coincidence between unfolded Doppler frequency estimates for each PRF, see for example [2–4]. Since the Doppler frequency may take all possible real values in a range and infinite many trials are needed for all

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0018-9251/99/$10.00

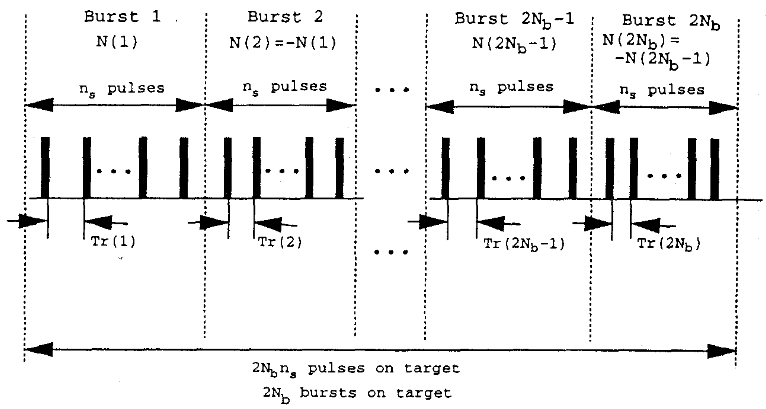

Fig. 1. Multiple PRF waveform.

the possibilities of the Doppler frequency, it maybe impossible to have an exact match. Thus, estimation errors usually occur. Based on this observation, a two step estimation algorithm has been proposed in [1] by Ferrari, Bérenguer, and Alengrin. The basic idea for the two step estimation is the following. The Doppler frequency is decomposed into two parts: the folded part, i.e., the fractional part modulo 1, and the ambiguity order part, i.e., the integer part. By grouping the PRFs into pairs where each pair is symmetric about 1, the folded part is the "circular mean" [5] of the folded estimates of the pair PRFs. This circular averaging is the first step of the algorithm in [1]. After the folded part is estimated, the second step is to find the match of the ambiguity order. By noticing that the ambiguity order takes integer values, there are only finite many possible trials needed ranging from the minimal and the maximal possible ambiguity orders. Therefore, the exact estimation of the Doppler frequency becomes possible with the two step estimation. Note that the *key* of this method is to convert the infinite many trials to the finite many trials, by converting a general real number matching to an integer matching.

The *motivation* for this paper is as follows. Since the specific PRF pairs, which are symmetric about 1, are needed in the Ferrari–Bérenguer–Alengrin approach, it may reduce the detectability of using the maximum likelihood criterion to detect the peak or the match. It is because the sidelobes of the maximum likelihood function with the specific PRFs may not be as low as the one with other PRFs. The motivation of this work is to relax the above PRF condition in the following way: one of each pair PRFs is fixed and the other of the pair is optimally determined based on the lowest sidelobes of the maximum likelihood function. With this relaxization, the "circular mean" estimation of the folded frequency may not be as good as the one in [1]. We propose an alternative approach to achieve the folded frequency estimation as follows. We first take the conventional mean of the folded frequency estimates in each pair. The true folded frequency falls in a finite number of possibilities from the conventional mean. These finite possibilities of the folded frequency can be obtained when the PRF pairs are known. Since the ambiguity order has also finite possibilities, the overall folded frequency and the ambiguity order have *finite* possibilities. This suggests us to estimate both the folded frequency and the ambiguity order simultaneously based on the maximum likelihood criterion. What is gained here is the detectability improvement of the Doppler frequency while the penalty is the increase of the computational complexity with a multiple of the one in [1] due to more possibilities to search for the folded frequency. .

This paper is organized as follows. In Section II, we briefly review the Ferrari–Bérenguer–Alengrin approach proposed in [1]. In Section III, we study the optimal PRF method. In Section IV, we present numerical examples which outperform the Ferrari–Bérenguer–Alengrin method.

II. THE FERRARI–BÉRENGUER–ALENGRIN TWO STEP ESTIMATION METHOD

First of all, we briefly describe the problem. Let radar transmit $2N_b$ bursts of n_s pulses, where the PRF of the kth burst is assumed $Fr(k)$, $1 \leq k \leq 2N_b$. The time difference between two pulses in the kth burst is $Tr(k) = 1/Fr(k)$. It is assumed that the elapsed time between the last pulse of the kth burst and the first one of the $(k+1)$th burst is $Tr(k)$. The time delays $Tr(k)$ are assumed as

$$Tr(k) = \left(1 + \frac{1}{N(k)}\right) Tr \tag{1}$$

where $N(1), N(2), \ldots, N(2N_b)$ are integers and Tr is usually assumed as 1 for simplicity. The multiple PRF waveform is shown in Fig. 1.

After coherent demodulation, the received data at the nth sample, $0 \leq n \leq n_s - 1$, in the kth burst,

$1 \leq k \leq 2N_b$, becomes

$$y_k(n) = x_k(n) + b_k(n) = a_k(f_D)\exp(j2\pi f_D nTr(k)) + b_k(n) \quad (2)$$

where f_D is the unknown Doppler frequency, $b_k(n)$ is white noise from the contribution of both thermal noise and clutter whitened residue, and $a_k(f_D)$ contains the initial phase of the target signal on the kth burst. If $a_1(f_D) = A$, then we have

$$a_k(f_D) = A\exp\left(j2\pi n_s f_D \sum_{q=1}^{k-1} Tr(q)\right), \qquad k \geq 2. \quad (3)$$

Then the ambiguity resolution problem is to estimate the Doppler frequency f_D from the noisy data $y_k(n)$ in (2). It is usually assumed that f_D is in a certain range, i.e., $|f| \leq f_{\max}$. The conventional detection method is the following maximum likelihood estimation. Find $\hat{f}_D$ that maximizes the following maximum likelihood function

$$L(f) = \left|\sum_{k=1}^{2N_b}\sum_{m=0}^{n_s-1} y_k(m)a_k^*(f)\exp(-j2\pi fmTr(k))\right|^2 \quad (4)$$

i.e.,

$$L(\hat{f}_D) = \max_{|f|\leq f_{\max}} L(f)$$

where $a_k(f)$ takes the form (3) with f_D replaced by f and $|\hat{f}_D| \leq f_{\max}$. This is a matching process and f needs to run all real numbers from $-f_{\max}$ to $f_{\max}$. Clearly, it has infinite many trials and therefore is impossible to have an exact match.

In [1], Ferrari–Bérenguer–Alengrin proposed an alternative two step approach for the above problem without implementing infinite many trials, where particular $N(k)$ in (1) were used. We next want to briefly describe this two step approach.

Let $N(2p+1)$ be a positive integer and set

$$N(2p+2) = -N(2p+1), \qquad \text{for} \quad p = 0, 1, \ldots, N_b - 1. \quad (5)$$

The Doppler frequency f_D is decomposed into its integer part (the ambiguity order) n_r and fractional part (the folded or reduced frequency) f_r as

$$f_D = f_r + n_r \qquad \text{with} \quad 0 \leq f_r < 1. \quad (6)$$

Then (2) becomes

$$y_k(n) = a_k(f_D)\exp\left(j2\pi\left(f_r + \frac{f_D}{N(k)}\right)n\right) + b_k(n), \qquad 0 \leq n \leq n_s - 1. \quad (7)$$

Let

$$f_k \triangleq f_r + \frac{f_D}{N(k)}, \qquad 1 \leq k \leq 2N_b. \quad (8)$$

If f_k could be obtained from $y_k(n)$, $0 \leq n \leq n_s - 1$, in (7), by using $N(2p+2) = -N(2p+1)$ in (5), the reduced frequency f_r would be

$$f_r = \frac{f_{2p+1} + f_{2p+2}}{2}, \qquad 0 \leq p \leq N_b - 1. \quad (9)$$

From $y_k(n)$ in (7) what we can get for f_k is, however, its folded version $\tilde{f}_k$, i.e.,

$$\tilde{f}_k = f_k + l, \qquad l \text{ is an unknown integer and } 0 \leq \tilde{f}_k < 1. \quad (10)$$

In this case, the reduced frequency f_r cannot be obtained from $\tilde{f}_k$ by simply taking their mean as $(\tilde{f}_{2p+1} + \tilde{f}_{2p+2})/2$. However, when

$$|f_{2p+1} - f_{2p+2}| < 0.5 \quad (11)$$

the reduced frequency f_r can be recovered from $\tilde{f}_k$ by taking the "circular mean" [5] as

$$\hat{f}_r(p) = \frac{1}{2\pi}\text{angle}[\exp(j2\pi\tilde{f}_{2p+1}) + \exp(j2\pi\tilde{f}_{2p+2})] \quad (12)$$

where angle(z) is the phase angle in radians in $[0, 2\pi)$ of the complex number z. With total N_b pairs of f_k, the overall estimate of the reduced frequency f_r is

$$\hat{f}_r = \frac{1}{2\pi}\text{angle}\sum_{p=0}^{N_b-1}\exp(j2\pi\hat{f}_r(p)). \quad (13)$$

When the Doppler frequency f in (4) is split into its reduced frequency part f (without confusion in understanding we also use f to denote the reduced frequency) and its ambiguity order part n, the maximum likelihood function in (4) can be written as

$$L(f,n) \triangleq \left|\sum_{k=1}^{2N_b}\sum_{m=0}^{n_s-1} y_k(m)a_k^*(f,n) \cdot \exp(-j2\pi(f+n)mTr(k))\right|^2$$

$$= \left|\sum_{k=1}^{2N_b} a_k^*(f,n)\sum_{m=0}^{n_s-1} y_k(m)\exp(-j2\pi fmTr(k)) \cdot \exp\left(-j2\pi\frac{n}{N(k)}m\right)\right|^2 \quad (14)$$

where $a_k(f,n)$ corresponds to the term $a_k(f)$ in (4) and can be expressed as

$$a_{2p+1}(f,n) = \exp(j2\pi f n_s 2p) \quad (15)$$

$$a_{2p+2}(f,n) = \exp\left(j2\pi f n_s\left(2p+1+\frac{1}{N(2p+1)}\right)\right) \cdot \exp\left(j2\pi n_s \frac{n}{N(2p+1)}\right). \quad (16)$$

After the reduced frequency f_r is estimated as in (13), the maximum likelihood function $L(f,n)$ for both f and n is reduced to the one for the ambiguity order n only:

$$L(n) \stackrel{\Delta}{=} L(\hat{f}_r, n) = \left|\sum_{k=1}^{2N_b} a_k^*(\hat{f}_r,n)\sum_{m=0}^{n_s-1} y_k(m)\exp(-j2\pi\hat{f}_r mTr(k)) \cdot \exp\left(-j2\pi\frac{n}{N(k)}m\right)\right|^2 \quad (17)$$

where n ranges all integers from $-n_{\max}$ to $n_{\max}$ and $n_{\max}$ is the maximum ambiguity order corresponding to the maximum Doppler frequency $f_{\max}$. Thus, the searching of the Doppler frequency f_D from all the real numbers $|f| \le f_{\max}$ to maximize $L(f)$ in (4) becomes the searching of the ambiguity order n from all integers $|n| \le n_{\max}$ to maximize $L(n)$ in (17). Note that there are only finite many possibilities of n, which makes the exact coincidence of the true ambiguity order possible. Let $\hat{n}_r$ denote the optimal ambiguity order estimate from $L(n)$ in (17). Then the final Doppler frequency estimate is

$$\hat{f}_D = \hat{f}_r + \hat{n}_r. \quad (18)$$

The reason for choosing $N(k)$ as integers in the whole approach is to use the discrete Fourier transform (DFT) calculations in (17) for the maximum likelihood function evaluations. For more details on the implementation issue, see [1].

The above is the main idea for the Ferrari–Bérenguer–Alengrin two step estimation method. We call it *FBA method.* It is built upon the assumption (5) and the condition (11). Condition (11) guarantees the accurate reduced frequency estimation and leads to the following condition on $N(k)$:

$$|N(k)| > 4(1+n_{\max}), \qquad 1 \le k \le 2N_b \quad (19)$$

where $n_{\max}$ is the maximum ambiguity order. Clearly, when $n_{\max}$ is large, $|N(k)|$ needs to be large. Large $|N(k)|$ may increase ambiguity order errors as mentioned in [1]. One way to relax the condition (11) or (19) is as follows, which also serves as a foundation for the optimal multiple PRF discussed latter.

Assume

$$\left|\frac{f_D}{N(k)}\right| < 1, \qquad \text{i.e.,} \quad |N(k)| > 1 + n_{\max}. \quad (20)$$

In this case, although the circular mean (12) may not be equal to the reduced frequency f_r in (8), f_r takes one of the following five values:

$$\overline{f}_r(p), \quad \overline{f}_r(p) - 0.5, \quad \overline{f}_r(p) + 0.5, \quad \overline{f}_r(p) - 1, \quad \overline{f}_r(p) + 1 \quad (21)$$

where $\overline{f}_r(p)$ is the conventional mean, $\overline{f}_r(p) = (\tilde{f}_{2p+1} + \tilde{f}_{2p+2})/2$, and $\tilde{f}_k$ are obtained from (7) and (10). It is because the unknown parameter l in (10) may only take 0, -1 or 1, when the condition (20) holds and $0 \le f_r < 1$. Thus, when $N_b = 1$, the estimation of f_r and n_r become the search of the optimal $\hat{f}_r(p)$ and $\hat{n}_r$ in the maximum likelehood function $L(f_r,n)$ in (14) among

$$f_r \in \mathcal{S}(p) \stackrel{\Delta}{=} \{\overline{f}_r(p), \overline{f}_r(p) - 0.5, \overline{f}_r(p) + 0.5, \overline{f}_r(p) - 1, \overline{f}_r(p) + 1\} \quad (22)$$

and $|n| \le n_{\max}$:

$$L(\hat{f}_r(p), \hat{n}_r) = \max_{f \in \mathcal{S}(p), |n| \le n_{\max}} L(f,n) \quad (23)$$

which also has only finite comparisons.

When $N_b > 1$, there are at least two methods to take the advantage of this multiplicity. One is to take the circular mean of all the above estimated $\hat{f}_r(p)$ as in (13). The other is to search the optimal f among all possible elements in the sets $\mathcal{S}(p)$ for $p = 0, 1, \ldots, N_b - 1$:

$$L(\hat{f}_r, \hat{n}_r) = \max_{f \in \mathcal{S}, |n| \le n_{\max}} L(f,n) \quad (24)$$

where

$$\mathcal{S} = \bigcup_{p=0}^{N_b-1} \mathcal{S}(p).$$

Note that the condition (20) can be further relaxed by allowing more possibilities for the reduced frequency f_r from the mean $\overline{f}_r$. Thus, the size of $N(k)$ can basically be arbitrary. The detection method in (20)–(24) is called *modified FBA method.* On the other hand, the condition (5) may cause high sidelobes of the maximum likelihood function $L(f_r,n)$ in (14) and therefore reduce the performance when additive white noise $b_k(n)$ in (2) is significant. The goal of the rest of this paper is to relax the condition (5) and search for the optimal linear relationship between $N(2p+1)$ and $N(2p+2)$ instead of $N(2p+2) = -N(2p+1)$.

It should be mentioned that another difference between the FBA method and the above modified FBA method is the following. In the FBA method, the angular mean is taken over the N_b bursts as shown in (13), while, in the modified FBA method, the multiplicity of the bursts gives more possibilities to search for the correct folded frequency. The angular mean may reduce the error variance of the reduced frequency, while the more possibilities of the search may provide more accurate estimate of the reduced frequency. However, the latter one clearly causes more computations.

III. OPTIMAL MULTIPLE PRF AND DOPPLER FREQUENCY DETECTION

In this section, we use the same signal model as described in Section II, where the assumption (5) is relaxed as

$$N(2p+2) = -\alpha_p N(2p+1), \quad \text{for} \quad p = 0,1,\ldots,N_b - 1 \tag{25}$$

where $N(2p+1)$ are positive integers and α_p are positive real parameters. The goal of the rest of this paper is to optimally determine the parameters α_p given $N(2p+1)$ for $p = 0,1,\ldots,N_b-1$ in terms of the lowest sidelobes of the maximum likelihood function $L(f,n)$.

With (25), an analogous formula of (9) for the reduced frequency is

$$f_r = \frac{f_{2p+1} + \alpha_p f_{2p+2}}{1+\alpha_p}, \qquad p = 0,1,\ldots,N_b - 1 \tag{26}$$

where f_k are defined in (8). One can see that the conventional mean (9) with the property (5) becomes the conventional weighted mean (26) with the property (25). The circular mean in (12), however, cannot be generalized to the general setting of the parameters α_p. In other words, the reduced frequency f_r can not be obtained as in the FBA method from the estimated individual $\tilde{f}_k$ in (8), (10), and (25) with general parameters α_p unless $\alpha_p = 1$ using the periodogram method. Fortunately, the argument in (20)–(24) can be generalized as follows.

Without loss of generality, we assume the property (20), i.e.,

$$N(2p+1) > 1 + n_{\max} \quad \text{and} \quad |N(2p+1)| > \frac{1+n_{\max}}{\alpha_p}, \qquad p = 0,1,\ldots,N_b-1. \tag{27}$$

Let

$$\overline{f}_r(p) \triangleq \frac{\tilde{f}_{2p+1} + \alpha_p \tilde{f}_{2p+2}}{1+\alpha_p}, \qquad p = 0,1,\ldots,N_b - 1 \tag{28}$$

where $\tilde{f}_k$ are obtained from (7), (8), and (10) with $N(k)$ satisfying (25) instead of (5). For $p = 0,1,\ldots,N_b - 1$, let

$$\mathcal{S}(p) \triangleq \left\{ \overline{f}_r(p), \overline{f}_r(p) \pm \frac{1}{1+\alpha_p}, \overline{f}_r(p) \pm \frac{\alpha_p}{1+\alpha_p}, \overline{f}_r(p) \pm \frac{1-\alpha_p}{1+\alpha_p}, \overline{f}_r(p) \pm 1 \right\}. \tag{29}$$

When $\alpha_p = 1$, the set $\mathcal{S}(p)$ in (29) is the same as the set $\mathcal{S}(p)$ in (22). Similar to (21), we have

$$f_r \in \mathcal{S}(p), \qquad p = 0,1,\ldots,N_b - 1. \tag{30}$$

Let

$$\mathcal{S} = \bigcup_{p=0}^{N_b-1} \mathcal{S}(p). \tag{31}$$

Then the maximum likelihood estimates for the reduced frequency f_r and the ambiguity order n_r are $\hat{f}_r$ and $\hat{n}_r$ that maximize $L(f,n)$ for $f \in \mathcal{S}$ and $|n| \le n_{\max}$, i.e.,

$$L(\hat{f}_r, \hat{n}_r) = \max_{f\in\mathcal{S}, |n|\le n_{\max}} L(f,n) \tag{32}$$

where $L(f,n)$ is similar to (14):

$$L(f,n) = \left| \sum_{k=1}^{2N_b} a_k^*(f,n) \sum_{m=0}^{n_s-1} y_k(m) \exp(-j2\pi f m Tr(k)) \cdot \exp\left(-j2\pi \frac{n}{N(k)} m\right) \right|^2 \tag{33}$$

where

$$a_k(f,n) = A \exp\left(j2\pi n_s (f+n) \sum_{q=1}^{k-1} Tr(q) \right), \qquad Tr(q) = 1 + \frac{1}{N(q)} \tag{34}$$

and $y_k(m)$ are the demodulated noisy data at the receiver:

$$y_k(m) = a_k(f_r, n_r) \exp(j2\pi f_r m Tr(k)) \cdot \exp\left(j2\pi \frac{n_r}{N(k)} m\right) + b_k(m) \tag{35}$$

where $f_D = f_r + n_r$ is the unknown Doppler frequency and $b_k(m)$ are additive white noise. The final Doppler frequency estimate is $\hat{f}_D = \hat{f}_r + \hat{n}_r$.

The performance of the above detection method depends on the property of the maximum likelihood function $L(f,n)$. The lower sidelobes of $L(f,n)$ are, the better performance of the detection is. The sidelobes depend on the choice of the parameters α_p in (35), when $N(2p+1)$ are given. We next want to discuss the optimal choice of these parameters.

By substituting (34)–(35) into (33), we have

$$L(f,n) = |A|^2 \left| \sum_{k=1}^{2N_b} \exp\left(j2\pi n_s (f_r - f + n_r - n) \sum_{q=1}^{k-1} Tr(q) \right) \cdot \sum_{m=0}^{n_s-1} \exp(j2\pi(f_r - f)mTr(k)) \exp\left(j2\pi \frac{n_r - n}{N(k)} m \right) \right|^2$$

$$= |A|^2 \left| \sum_{k=1}^{2N_b} \exp\left(j2\pi n_s (f_r - f + n_r - n) \sum_{q=1}^{k-1} Tr(q) \right) \cdot \exp\left(j\pi(n_s - 1)\left((f_r - f)Tr(k) + \frac{n_r - n}{N(k)} \right) \right) \cdot \frac{\sin\left(\pi n_s \left[(f_r - f)Tr(k) + \frac{n_r - n}{N(k)} \right] \right)}{\sin\left(\pi \left[(f_r - f)Tr(k) + \frac{n_r - n}{N(k)} \right] \right)} \right|^2. \quad (36)$$

Clearly, the mainlobe value of the above maximum likelihood function is its value when $f = f_r$ and $n = n_r$:

$$L(f_r, n_r) = |A|^2 2N_b n_s. \quad (37)$$

Since $f_r \in \mathcal{S}(p)$, the offset value $f_r - f$ in (36) may only take the values in the following set, when $f \in \mathcal{S}$ defined in (30):

$$\mathcal{S}_{\text{offset}} \triangleq \bigcup_{p=0}^{N_b-1} \left\{ \pm 1, \pm 2, \frac{\pm 1}{1+\alpha_p}, \frac{\pm 2}{1+\alpha_p}, \frac{\pm \alpha_p}{1+\alpha_p}, \frac{\pm 2\alpha_p}{1+\alpha_p}, \frac{\pm(1-\alpha_p)}{1+\alpha_p}, \frac{\pm 2(1-\alpha_p)}{1+\alpha_p}, \frac{\pm(1 \pm 2\alpha_p)}{1+\alpha_p}, \frac{\pm(2 \pm \alpha_p)}{1+\alpha_p} \right\}. \quad (38)$$

The offset value $n_r - n$ is in the set $\{\pm 1, \pm 2, \ldots, \pm 2n_{\max}\}$.

Let $E_{\text{sidelobe}}(\alpha_0, \alpha_1, \ldots, \alpha_{N_b-1})$ denote the total energy of all the sidelobe values of the maximum likelihood function $L(f,n)$ in (36). Then, by normalizing $A = 1$ it can be expressed by

$$E_{\text{sidelobe}}(\alpha_0, \alpha_1, \ldots, \alpha_{N_b-1}) = \sum_{f \in \mathcal{S}_{\text{offset}}} \sum_{0<|n|\le 2n_{\max}} \left| \sum_{k=1}^{2N_b} \exp\left(j2\pi n_s (f+n) \sum_{q=1}^{k-1} \left(1 + \frac{1}{N(q)} \right) \right) \cdot \exp\left(j\pi(n_s - 1)\left(f\left(1 + \frac{1}{N(k)}\right) + \frac{n}{N(k)} \right) \right) \cdot \frac{\sin\left(\pi n_s \left[f\left(1 + \frac{1}{N(k)}\right) + \frac{n}{N(k)} \right] \right)}{\sin\left(\pi \left[f\left(1 + \frac{1}{N(k)}\right) + \frac{n}{N(k)} \right] \right)} \right|^2 \quad (39)$$

where $N(2p+2) = -\alpha_p N(2p+1)$, $p = 0, 1, \ldots, N_b - 1$, and $\mathcal{S}_{\text{offset}}$ is defined in (38). Given $N(2p+1)$, $p = 0, 1, \ldots, N_b - 1$, and $n_{\max}$, the optimal parameters $\hat{\alpha}_p$, $p = 0, 1, \ldots, N_b - 1$, can be obtained by minimizing the cost function $E_{\text{sidelobe}}(\alpha_0, \alpha_1, \ldots, \alpha_{N_b-1})$ in (39) for $\hat{\alpha}_p > 0$, i.e.,

$$E_{\text{sidelobe}}(\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_{N_b-1}) = \min_{\alpha_0 > a_0, \alpha_1 > a_1, \ldots, \alpha_{N_b-1} > a_{N_b-1}} E_{\text{sidelobe}}(\alpha_0, \alpha_1, \ldots, \alpha_{N_b-1}) \quad (40)$$

where, by (27),

$$a_p = \frac{1 + n_{\max}}{N(2p+1)}.$$

One may see that an explicit solution for the optimal α_p is not possible. However, any existing optimization methods work for the above problem.

Let us consider the simpliest case, $N_b = 1$. In this case,

$$\mathcal{S}_{\text{offset}} \triangleq \left\{ \pm 1, \pm 2, \frac{\pm 1}{1+\alpha_0}, \frac{\pm 2}{1+\alpha_0}, \frac{\pm \alpha_0}{1+\alpha_0}, \frac{\pm 2\alpha_0}{1+\alpha_0}, \frac{\pm(1-\alpha_0)}{1+\alpha_0}, \frac{\pm 2(1-\alpha_0)}{1+\alpha_0}, \frac{\pm(1 \pm 2\alpha_0)}{1+\alpha_0}, \frac{\pm(2 \pm \alpha_0)}{1+\alpha_0} \right\} \quad (41)$$

and

$$E_{\text{sidelobe}}(\alpha_0) = \sum_{f \in \mathcal{S}_{\text{offset}}} \sum_{0<|n|\le 2n_{\max}} \left| \frac{\sin\left(\pi n_s \left[f\left(1 + \frac{1}{N(1)}\right) + \frac{n}{N(1)} \right] \right)}{\sin\left(\pi \left[f\left(1 + \frac{1}{N(1)}\right) + \frac{n}{N(1)} \right] \right)} + \exp\left(j2\pi n_s (f+n)\left(1 + \frac{1}{N(1)}\right) \right) \cdot \exp\left(j\pi(n_s - 1)\left[f\left(1 - \frac{1}{\alpha_0 N(1)}\right) - \frac{n}{\alpha_0 N(1)} \right] \right) \cdot \frac{\sin\left(\pi n_s \left[f\left(1 - \frac{1}{\alpha_0 N(1)}\right) - \frac{n}{\alpha_0 N(1)} \right] \right)}{\sin\left(\pi \left[f\left(1 - \frac{1}{\alpha_0 N(1)}\right) - \frac{n}{\alpha_0 N(1)} \right] \right)} \right|^2. \quad (42)$$

Let us see some numerical examples of $E_{\text{sidelobe}}(\alpha_0)$. Consider $N(1) = 40$ and $n_s = 12$. Figs. 2, 3, and 4 show the $E_{\text{sidelobe}}(\alpha_0)$ versus α_0 when $n_{\max}$ = 3, 5, and 12, respectively. One can see that the optimal α_0 strongly depends on the maximal ambiguity order $n_{\max}$, where the optimal α_0 are $\hat{\alpha}_0 = 0.57$, 1.85, and 2.01 for $n_{\max} = 3$, 5, and 12, respectively.

IV. NUMERICAL EXPERIMENTS

In this section, we present numerical examples to compare the performances for the modified FBA method and the method with optimized PRFs

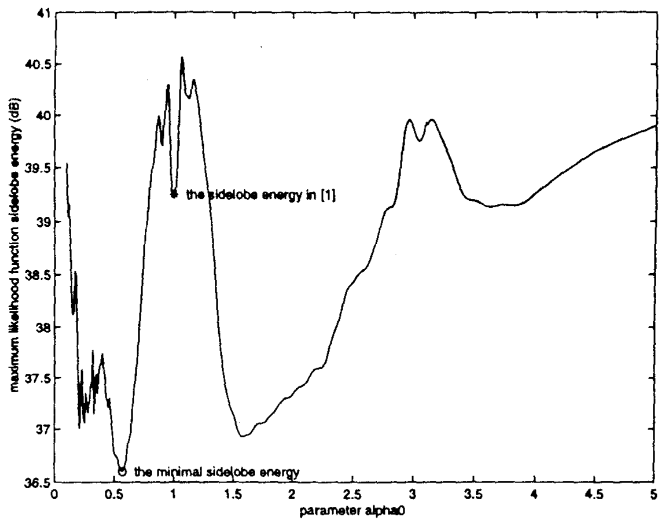

Fig. 2. $E_{\text{sidelobe}}(\alpha_0)$ when $N(1) = 40$, $n_s = 12$, and $n_{\max} = 3$. Optimal $\hat{\alpha}_0 = 0.57$.

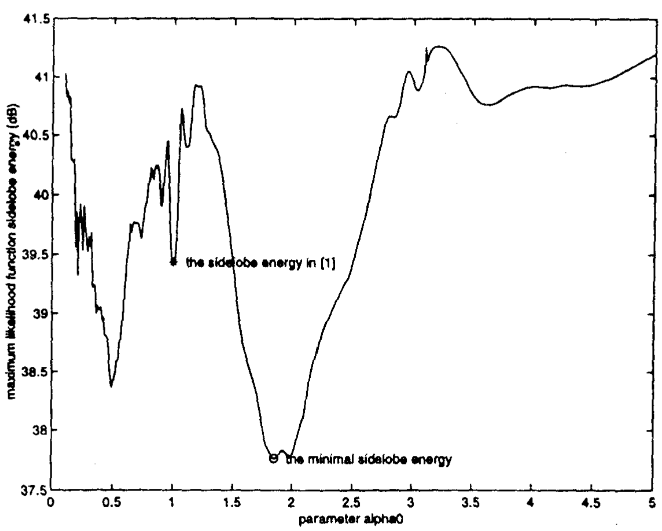

Fig. 3. $E_{\text{sidelobe}}(\alpha_0)$ when $N(1) = 40$, $n_s = 12$, and $n_{\max} = 5$. Optimal $\hat{\alpha}_0 = 1.85$.

proposed in this work. The following parameters are used in our simulations: $N(1) = 40$, $N_b = 1$, $n_s = 12$, and $N(2) = -\alpha_0 N(1)$, where $\alpha_0 = 1$ for FBA method and α_0 the optimal $\hat{\alpha}_0$ for the method proposed in this work. The additive noise $b_k(n)$ in the known noisy radar data $y_k(n)$ in (2) is assumed white Gaussian noise with mean 0 and variance σ^2. As mentioned at the end of Section III, the optimal α_0 depends on the maximal ambiguity order $n_{\max}$. Two different $n_{\max}$ are tested: $n_{\max} = 3$ and 12. Let M be the number of signal realizations. Let $f_D(k)$ be the true Doppler frequency and $\hat{f}_D(k)$ be the estimated one at the kth signal realization. Then the mean squared error (MSE) is calculated as

$$\text{MSE} = \frac{\sum_{k=1}^{M} |\hat{f}_D(k) - f_D(k)|^2}{M}. \tag{43}$$

The signal-to-noise ratio (SNR) for the additive Gaussian noise is calculated by $\text{SNR} = A^2/\sigma^2$, where A is the transmitted signal amplitude.

When $n_{\max} = 3$ and $N(1) = -N(2) = 40 > 4(1 + 3) = 16$, i.e., the condition (19) or (11) holds for the accurate circular mean formula (12). The FBA method works in this case although the parameter $\alpha = 1$ is not optimal in terms of the sidelobe values of the maximum likelihood function $E_{\text{sidelobe}}(\alpha_0)$. The optimal parameter α_0 in this case is $\hat{\alpha}_0 = 0.57$ as studied in Section III. When $\alpha_0 = 0.57$, clearly the number $N(2) = -\alpha_0 N(1) = 22.8$ is not an integer. For the DFT computation purpose, rounding $\alpha = 0.57$ to $\alpha_0 = 0.6$ may be needed for $N(2)$ to be an integer. When $\alpha_0 = 0.6$, $N(2) = -24$. As mentioned in Section III, when $\alpha_0 \neq 1$, the accurate circular mean no longer

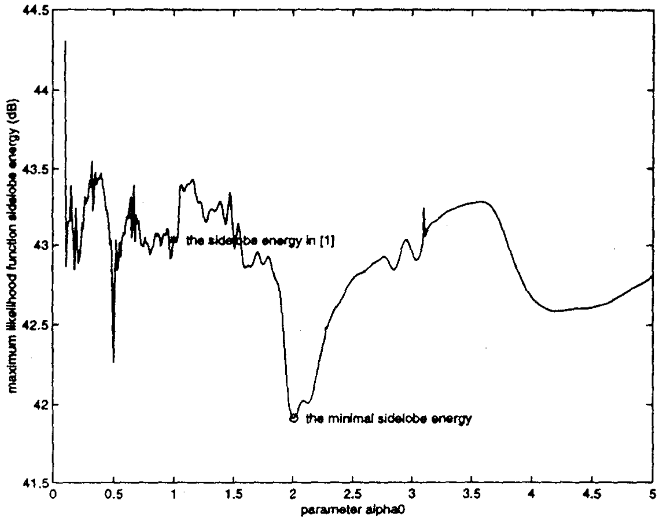

Fig. 4. $E_{\mathrm{sidelobe}}(\alpha_0)$ when $N(1) = 40$, $n_s = 12$, and $n_{\max} = 12$. Optimal $\hat{\alpha}_0 = 2.01$.

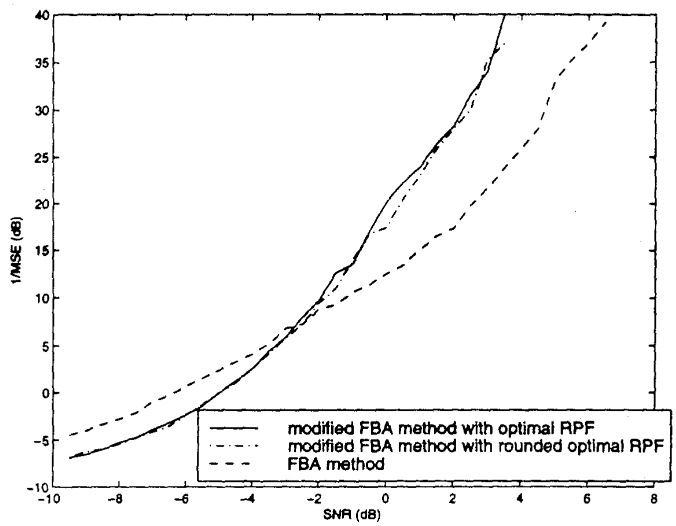

Fig. 5. Comparison of reciprocal MSE of Doppler frequency estimations: FBA method and modified FBA method with optimal PRFs (or α_0). Solid line: modified FBA method with optimal $\alpha_0 = 0.57$; dashdot line: modified FBA method with rounded optimal $\alpha_0 = 0.6$; dashed line: FBA method. $N(1) = 40$, $n_s = 12$, maximal ambiguity order $n_{\max} = 3$.

holds. In this case, we use the modified FBA method for the Doppler frequency detection. 20,000 Monte Carlo tests are implemented, i.e., $M = 20{,}000$ in (43). Three curves are plotted in Fig. 5 for the reciprocal MSE, 1/MSE, of the Doppler frequency estimations. The solid line is for the modified FBA method with the optimal $\alpha = 0.57$; the dashdot line is for the modified FBA method with the rounded α_0, 0.6; the dashed line is for the FBA method. A significant improvement of the MSE at the transition SNR band can be clearly seen.

As a remark, when $\alpha_0 = 1$, the FBA method and the modified FBA method both work. From our numerous numerical examples, these two methods have the same performance in this case.

When $n_{\max} = 12$ and $N(1) = -N(2) = 40 < 4(1 + 12) = 52$, i.e., the condition (19) or (11) for the accurate circular mean formula (12) does not hold. In this case, the FBA two step method does not work as shown in Fig. 6 and the modified FBA method should be used. The optimal parameter α_0 is $\hat{\alpha}_0 = 2.01$. 10,000 Monte Carlo tests are implemented, i.e., $M = 10{,}000$ in (43). Similar to Fig. 5, three curves are plotted in Fig. 6 for the reciprocal MSEs. The solid line is for the modified FBA method with the optimal $\alpha_0 = 2.01$. The dashdot line is for the modified FBA

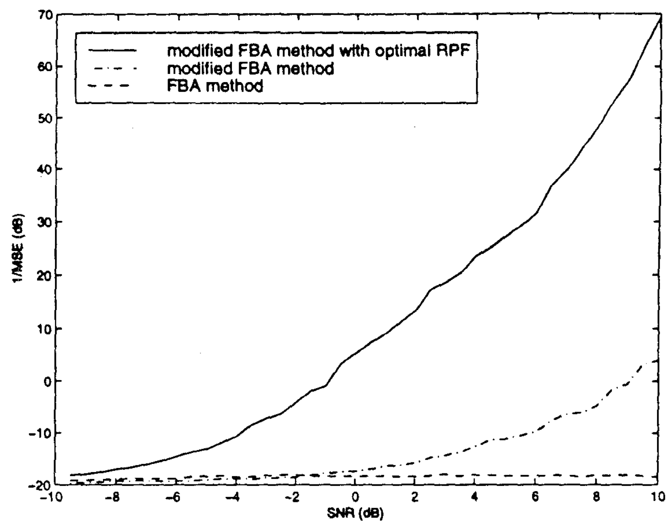

Fig. 6. Comparison of reciprocal MSE of Doppler frequency estimations: FBA method, modified FBA method, and modified FBA method with optimal PRFs (or α_0). Solid line: modified FBA method with optimal $\alpha_0 = 2.01$; dashdot line: modified FBA method; dashed line: FBA method. $N(1) = 40$, $n_s = 12$, maximal ambiguity order $n_{max} = 12$.

method with $\alpha_0 = 1$. The dashed line is for the FBA method. From Fig. 6, one can clearly see that in this case the FBA method fails, and the modified FBA method with the optimal α_0 outperforms the one with nonoptimal α_0.

V. CONCLUSION

In this paper, we studied the Ferrari–Bérenguer–Alengrin's two step Doppler frequency detection method, where the folded frequency is first estimated using the circular mean and the ambiguity order is then estimated using the quasi maximum likelihood criterion. The accuracy of the folded frequency depends on the use of the particular pairs of PRFs. When the folded frequency is not equal to the circular mean, we modified the FBA method with a finite possibilities of the folded frequency and the ambiguity order. More importantly, we studied and formulated the optimal PRFs in the modified FBA method in terms of minimizing the total sidelobe energy of the maximum likelihood function. Better performance of the modified FBA method over the FBA method was shown by numerical examples.

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XIANG-GEN XIA
Dept. of Electrical and Computer Engineering
University of Delaware
Newark, DE 19716
Email: (xxia@ee.udel.edu)

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