Lecture 8: Equality Constrained Minimization

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Outline

- Introduction
- Eliminating Equality Constraints
- Newton's Method With Equality Constraints
- Infeasible Start Newton Method

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Equality Constrained Minimization

$$\text{minimize } f(x)$$
$$\text{subject to } Ax = b$$

- f convex, twice continuously differentiable
- $p^* = \inf_x f(x)$ is finite and attained
- $A \in \mathbf{R}^{p \times n}$ with rank p

Optimality condition: from KKT condition, x^* is optimal if and only if there exists ν^* such that

$$Ax^* = b, \;\; \nabla f(x^*) + A^T \nu^* = 0$$

- a set of $n+p$ equations in $n+p$ variables x^*, ν^*

Equality Constrained Quadratic Minimization

$$\begin{aligned} \text{minimize } & (1/2)x^T P x + q^T x + r \\ \text{subject to } & Ax = b \end{aligned}$$

Optimality condition:

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ v^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

- coefficient matrix is called KKT matrix
- KKT matrix is nonsingular iff

$$Ax = 0, x \neq 0 \Rightarrow x^T P x > 0$$

- equivalent condition for nonsingulariry: $P + A^T A \succ 0$

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Eliminating Equality Constraint

represent solution of $\{x|Ax = b\}$ as

$$\{x|Ax = b\} = \{Fz + \hat{x}|z \in \mathbf{R}^{n-p}\}$$

where $\hat{x}$ is any particular solution, and range of $F \in \mathbf{R}^{n\times(n-p)}$ is null space of A (F has rank $n - p$ and $AF = 0$)

reduced or eliminated problem

$$\text{minimize } f(Fz + \hat{x})$$

- an unconstrained problem with variable $z \in \mathbf{R}^{n-p}$
- once have solution z^*, can obtain x^* and ν^* as

$$x^* = Fz^* + \hat{x}, \;\; \nu^* = -(AA^T)^{-1}A\nabla f(x^*)$$

Example

optimal allocation with resource constraint

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) \\ \text{subject to} & x_1 + x_2 + \cdots + x_n = b \end{array}$$

eliminate $x_n = b - x_1 - \cdots - x_{n-1}$, i.e. choose $x = Fz + \hat{x}$ with

$$\hat{x} = be_n, \quad F = \begin{bmatrix} I \\ -\mathbf{1}^T \end{bmatrix}$$

reduced problem:

$$\text{minimize} \quad f_1(x_1) + f_2(x_2) + \cdots + f_n(b - x_1 - \cdots - x_{n-1})$$

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Extension of Newton's Method

Newton's method can be extended to include equality constraints. The extended method is almost the same as Newton's method without constraints, except for two differences:

- The initial point must be feasible (i.e., satisfy $x \in \mathrm{dom}\ f$ and $Ax = b$)

- The definition of Newton step is modified to take the equality constraints into account. In particular, we make sure that the Newton step Δx_{nt} is a feasible direction, i.e., $A\Delta x_{\mathrm{nt}} = 0$

Newton Step

Newton step Δx_{nt} of f at feasible x is given by solution v of

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

Interpretations:
- Δx_{nt} minimizes second order approximation (with variable v)

$$\begin{array}{ll} \text{minimize} & \hat{f}(x+v) = f(x) + \nabla f(x)^T v + (1/2) v^T \nabla^2 f(x) v \\ \text{subject to} & A(x+v) = b \end{array}$$

- Δx_{nt} solves linearized optimality condition:

$$0 = \nabla f(x+v) + A^T w \approx \nabla f(x) + \nabla^2 f(x) v + A^T w, \;\; A(x+v) = b$$

Newton Decrement

$$\begin{aligned}\lambda(x) &= (\Delta x_{\mathrm{nt}}^T \nabla^2 f(x) \Delta x_{\mathrm{nt}})^{1/2} \\ &= \|\Delta x_{\mathrm{nt}}\|_{\nabla^2 f(x)}\end{aligned}$$

- gives an estimate of $f(x) - p^*$, using quadratic approximation $\hat{f}$:

$$f(x) - \inf_{Ay=b} \hat{f}(y) = \lambda(x)^2/2$$

- as before,

$$\nabla f(x)^T \Delta x_{\mathrm{nt}} = -\|\Delta x_{\mathrm{nt}}\|^2_{\nabla^2 f(x)} = -\lambda(x)^2$$

therefore it comes up in backtracking line search

- in general, $\lambda(x) \neq (\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x))^{1/2}$

Newton's Method with Equality Constraints

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given starting point $x \in \mathbf{dom}\, f$ with $Ax = b$, tolerance $\epsilon > 0$.

repeat

1. Compute the Newton step and decrement Δx_{nt}, $\lambda(x)$.
2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$.

- a feasible descent method: $x^{(k)}$ feasible and $f(x^{k+1}) < f(x^k)$
- affine invariant

Newton's Method and Elimination

Newton's method for reduced problem

$$\text{minimize } \bar{f}(z) = f(Fz + \hat{x})$$

Newton's method for $\bar{f}$, started at $z^{(0)}$, generates iterates $z^{(k)}$

Newton's method with equality constraints when started at $x^{(0)} = Fz^{(0)} + \hat{x}$, iterates are

$$x^{(k+1)} = Fz^{(k)} + \hat{x}$$

hence, don't need separate convergence analysis

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Newton Step at Infeasible Points

Linearizing optimality conditions at infeasible x gives

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\mathrm{nt}} \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax - b \end{bmatrix}$$

Primal-dual Interpretations:
- write optimality condition as $r(y) = 0$, where

$$y = (x, \nu), \; r(y) = (\nabla f(x) + A^T \nu, Ax - b)$$

- linearizing $r(y) = 0$ gives $r(y + \Delta y) \approx r(y) + Dr(y)\Delta y = 0$:

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\mathrm{nt}} \\ \Delta \nu_{\mathrm{nt}} \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + A^T \nu \\ Ax - b \end{bmatrix}$$

Infeasible Start Newton Method

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given starting point $x \in \mathbf{dom}\, f$, ν, tolerance $\epsilon > 0$, $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$.

repeat

1. Compute primal and dual Newton steps Δx_{nt}, $\Delta \nu_{\mathrm{nt}}$.
2. *Backtracking line search on* $\|r\|_2$.
   $t := 1$.
   while $\|r(x + t\Delta x_{\mathrm{nt}}, \nu + t\Delta \nu_{\mathrm{nt}})\|_2 > (1 - \alpha t)\|r(x, \nu)\|_2, \quad t := \beta t$.
3. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$, $\nu := \nu + t\Delta \nu_{\mathrm{nt}}$.

until $Ax = b$ and $\|r(x, \nu)\|_2 \leq \epsilon$.

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not a descent method: $f(x^+) > f(x)$ is possible

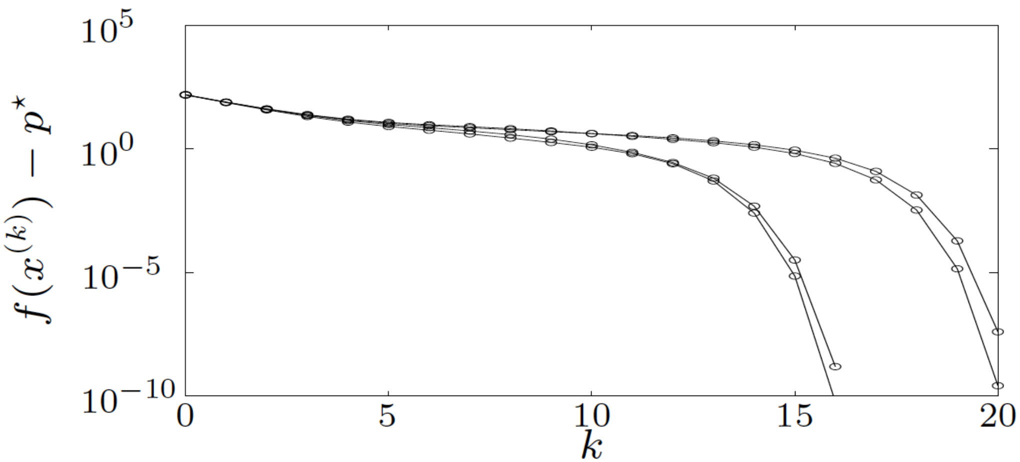
Equality Constrained Analytic Centering

$$\text{primal problem: minimize } -\sum_{i=1}^{n} \log x_i \text{ subject to } Ax = b$$

$$\text{dual problem: maximize } -b^T\nu + \sum_{i=1}^{n} \log(A^T\nu)_i + n$$

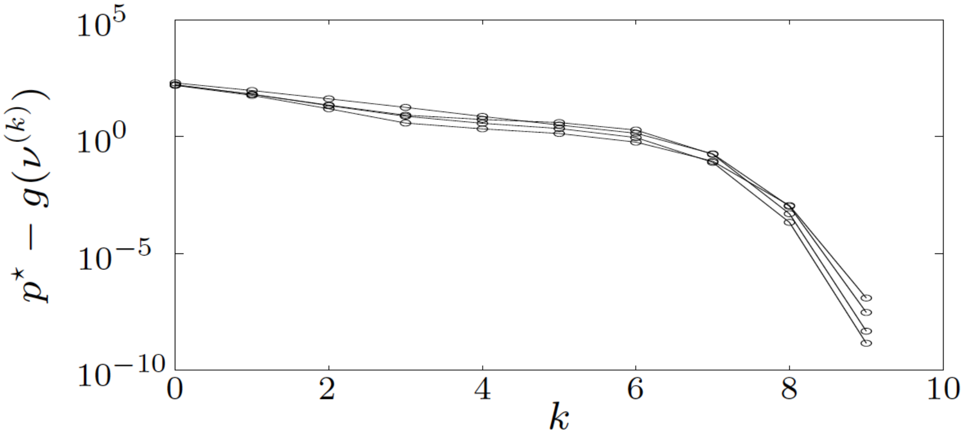
Three methods for an example with $A \in \mathbf{R}^{100\times 500}$, different starting points:

1. Newton method with equality constraints ($x^{(0)} \succ 0$, $Ax^{(0)} = b$)



Equality Constrained Analytic Centering

2. Newton method applied to dual problem $(A^T\nu^{(0)} \succ 0)$



3. infeasible start Newton method $(x^{(0)} \succ 0)$