Lecture 4: Convex Optimization Problems

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Outline

• Standard Form Optimization Problem
• Convex Optimization Problem
• Quasiconvex Optimization
• Linear Optimization
• Quadratic Optimization
• Geometric Optimization
• Generalized Inequality Constraints
• Semidefinite Programming
• Vector Optimization
Outline

• **Standard Form Optimization Problem**
• Convex Optimization Problem
• Quasiconvex Optimization
• Linear Optimization
• Quadratic Optimization
• Geometric Optimization
• Generalized Inequality Constraints
• Semidefinite Programming
• Vector Optimization
Standard Form Optimization

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \ i = 1, 2, \ldots, m \)
\( h_i(x) = 0, \ i = 1, 2, \ldots, p \)

Optimal value:

\( p^* = \inf \{ f_0(x) | f_i(x) \leq 0, \ i = 1, \ldots, m; h_i(x) = 0, \ i = 1, \ldots, p \} \)

- \( p^* = \infty \) if problem is infeasible (no \( x \) satisfies the constraints)
- \( p^* = -\infty \) if problem is unbounded below
- a feasible \( x \) is optimal if \( f_0(x) = p^* \); \( X_{\text{opt}} \) is the set of optimal points
- \( x \) is locally optimal if there is \( R > 0 \) such that

\( f_0(x) = \inf \{ f_0(z) | f_i(z) \leq 0, \ i = 1, \ldots, m; h_i(z) = 0, \ i = 1, \ldots, p; \|z - x\|_2 \leq R \} \)
Implicit Constraints

The standard form optimization problem has an implicit constraint:

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \text{dom } f_i \cap \bigcap_{i=0}^{p} \text{dom } h_i$$

- we call $\mathcal{D}$ the domain of the problem
- the constraints $f_i(x) \leq 0, h_i(x) = 0$ are the explicit constraints
- a problem is unconstrained if it has no explicit constraints

Example:

$$\text{minimize } f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$
Feasibility Problem

find \( x \)
subject to \( f_i(x) \leq 0, \ i = 1, 2, \ldots, m \)
\( h_i(x) = 0, \ i = 1, 2, \ldots, p \)

can be considered a special case of the general problem with \( f_0(x) = 0 \):

minimize \( 0 \)
subject to \( f_i(x) \leq 0, \ i = 1, 2, \ldots, m \)
\( h_i(x) = 0, \ i = 1, 2, \ldots, p \)

- \( p^* = 0 \) if constraints are feasible; any feasible \( x \) is optimal
- \( p^* = \infty \) if constraints are infeasible
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Convex Optimization Problem

Standard form convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \ i = 1, 2, \ldots, m \\
& \quad a_i^T x = b_i, \ i = 1, 2, \ldots, p
\end{align*}
\]

- \( f_1, f_2, \ldots, f_m \) are convex; equality constraints are affine
- Problem is quasiconvex if \( f_0 \) is quasiconvex (and \( f_1, f_2, \ldots, f_m \) are convex)

Can be more compactly written as

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \ i = 1, 2, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- Feasible set of a convex optimization problem is convex
Example

minimize \( f_0(x) = x_1^2 + x_2^2 \)

subject to \( f_1(x) = x_1/(1 + x_2^2) \leq 0 \)
\( h_1(x) = (x_1 + x_2)^2 = 0 \)

- \( f_0 \) is convex; feasible set \( \{(x_1, x_2)|x_1 = -x_2 \leq 0\} \) is convex
- not a convex problem: \( f_1 \) is not convex, \( h_1 \) is not affine
- equivalent (not identical) to the convex problem

\[
\begin{align*}
\text{minimize} & \quad x_1^2 + x_2^2 \\
\text{subject to} & \quad x_1 \leq 0 \\
& \quad x_1 + x_2 = 0
\end{align*}
\]
any locally optimal point of a convex problem is globally optimal

Proof: Since $x$ is locally optimal, there exists an $R > 0$ such that $f_0(z) \geq f_0(x)$ for any $z$ feasible and $\|z - x\|_2 \leq R$. Consider an arbitrary feasible $y$ that is not necessarily in $B(x, R)$. There must exist some $\alpha > 0$ s.t. $(1 - \alpha)x + \alpha y \in B(x, R)$ and therefore

$$f(x) \leq f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y).$$

This immediately implies that $f(x) \leq f(y)$ for any feasible $y$. 
Optimality Criterion For Differentiable Objective

$x$ is optimal iff it is feasible and

$$\nabla f_0(x)^T (y - x) \geq 0, \text{ for all feasible } y$$

if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set $X$ at $x$
Examples

- unconstrained problem: $x$ is optimal iff

$$x \in \text{dom} f_0, \nabla f_0(x) = 0$$

- equality constrained problem

minimize $f_0(x)$ subject to $Ax = b$

$x$ is optimal iff there exists a $\nu$ such that

$$x \in \text{dom} f_0, \ Ax = b, \ \nabla f_0(x) + A^T \nu = 0$$

- minimization over nonnegative orthant

minimize $f_0(x)$ subject to $x \geq 0$

$x$ is optimal iff

$$x \in \text{dom} f_0, \ x \geq 0, \ \nabla f_0(x)_i \geq 0 \text{ for } x_i = 0, \ \nabla f_0(x)_i = 0 \text{ for } x_i > 0$$
Equivalent Convex Problems

Informally, two problems are equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa.

Common transformations that preserve convexity:

- eliminating equality constraints:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, i = 1, 2, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} & \quad f_0(Fz + x_0) \\
\text{subject to} & \quad f_i(Fz + x_0) \leq 0, i = 1, 2, \ldots, m
\end{align*}
\]

where \( F \) and \( x_0 \) are such that

\[
Ax = b \iff x = Fz + x_0,
\]

i.e. \( \mathcal{R}(F) = \mathcal{N}(A) \).
Equivalent Convex Problems

introducing equality constraints:

\[
\begin{align*}
\text{minimize} & \quad f_0(A_0x + b) \\
\text{subject to} & \quad f_i(A_ix + b_i) \leq 0, \ i = 1, 2, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} & \quad f_0(y_0) \\
\text{subject to} & \quad f_i(y_i) \leq 0, \ i = 1, 2, \ldots, m \\
y_i &= A_ix + b_i, \ i = 0, 1, 2, \ldots, m
\end{align*}
\]

introducing slack variables for linear inequalities:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad a_i^Tx \leq b_i, \ i = 1, 2, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad a_i^Tx + s_i = b_i, \ i = 1, 2, \ldots, m \\
s_i & \geq 0, \ i = 1, 2, \ldots, m
\end{align*}
\]
Equivalent Convex Problems

epigraph form: standard form convex problem is equivalent to

minimize \( t \)
subject to \( f_0(x) - t \leq 0 \)
\( f_i(x) \leq 0, i = 1, 2, \ldots, m \)
\( Ax = b \)

minimizing over some variables:

minimize \( f_0(x_1, x_2) \)
subject to \( f_i(x_1) \leq 0, i = 1, 2, \ldots, m \)

is equivalent to

minimize \( \tilde{f}_0(x_1) \)
subject to \( f_i(x_1) \leq 0, i = 1, 2, \ldots, m \)

where \( \tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2) \)
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Quasiconvex Optimization

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, i = 1, 2, \ldots, m \)
\( Ax = b \)

- quasiconvex \( f_0 \) and convex \( f_i, i \in [1 : m] \)

- can have locally optimal points that are not globally optimal
Quasiconvex Optimization

convex representation of sublevel sets of $f_0$:

if $f_0$ is quasiconvex, there exists a family of functions $\phi_t$ such that
- $\phi_t$ is convex in $x$ for fixed $t$
- $t$-sublevel set of $f_0$ is 0-sublevel set of $\phi_t$, i.e., $f_0(x) \leq t \iff \phi_t(x) \leq 0$

example: $f_0(x) = \frac{p(x)}{q(x)}$, $p$ convex, $q$ concave, $p(x) \geq 0$, $q(x) > 0$ on $\text{dom} f_0$

can take $\phi_t(x) = p(x) - tq(x)$:
- for $t \geq 0$, $\phi_t$ is convex in $x$
- $p(x)/q(x) \leq t$ iff $\phi_t(x) \leq 0$
Quasiconvex Optimization

quasiconvex optimization via convex feasibility problems:

\[ \phi_t(x) \leq 0, \ f_i(x) \leq 0, \ i = 1, \ldots, m, \ Ax = b \]  \hspace{1cm} (1)

- for fixed \( t \), a convex feasibility problem in \( x \)
- if feasible, we can conclude that \( t \geq p^* \); otherwise, \( t \leq p^* \)

\[ Bisection \ method \ for \ quasiconvex \ optimization \]

\begin{itemize}
  \item given \( l \leq p^*, \ u \geq p^*, \) tolerance \( \epsilon > 0 \).
  \item repeat
    \begin{itemize}
      \item 1. \( t := (l + u)/2 \).
      \item 2. Solve the convex feasibility problem (1).
      \item 3. \textbf{if} (1) is feasible, \( u := t \); \textbf{else} \( l := t \).
    \end{itemize}
  \item \textbf{until} \( u - l \leq \epsilon \).
\end{itemize}

requires exactly \( \lceil \log_2((u - l)/\epsilon) \rceil \) iterations
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Linear Program

minimize  \( c^T x + d \)
subject to  \( Gx \leq h \)
\( Ax = b \)

- the objective and constraint functions are all affine
- feasible set is a polyhedron
Examples

diet problem: choose quantities $x_1, x_2, \ldots, x_n$ of $n$ foods
- one unit of food $j$ costs $c_j$, contains amount $a_{ij}$ of nutrient $i$
- healthy diet requires nutrient $i$ in quantity at least $b_i$
- to find cheapest healthy diet, solve the following LP

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b, \ x \geq 0
\end{align*}$$

piecewise-linear minimization:

$$\begin{align*}
\text{minimize} & \quad \max_{i \in [1:m]} (a_{i}^T x + b_i) \\
\text{equivalent to an LP}
\end{align*}$$

$$\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad a_{i}^T x + b_i \leq t, \ i = 1, \ldots, m
\end{align*}$$
find Chebyshev center of a polyhedron $\mathcal{P} = \{ x | a_i^T x \leq b_i, i = 1, \ldots, m \}$:

- Chebyshev center is the center of largest inscribed ball

$$B(x_c, r) = \{ x_c + u \mid \|u\|_2 \leq r \}$$

- $B(x_c, r) \subseteq \mathcal{P}$ iff

$$\sup \{ a_i^T (x_c + u) \mid \|u\|_2 \leq r \} \leq b_i, \forall i \in [1 : m]$$

$$= a_i^T x_c + r \|a_i\|_2$$

- hence $x_c, r$ can be determined by solving the following LP

$$\begin{align*}
\text{maximize} & \quad r \\
\text{subject to} & \quad a_i^T x_c + r \|a_i\|_2 \leq b_i, \ i = 1, \ldots, m
\end{align*}$$
Linear-Fractional Program

Linear fractional programming is to minimize a ratio of affine functions over a polyhedron:

\[
\text{minimize } \frac{c^T x + d}{e^T x + f} \\
\text{subject to } Gx \preceq h \\
A x = b
\]

where the objective function is quasiconvex with its domain as \( \{x | e^T x + f > 0\} \).

- a quasiconvex optimization problem; can be solved by bisection
- transformable to LP: Think of \( y = x/e^T x + f \) and \( z = 1/e^T x + f \). The above problem is equivalent to the following LP (variables \( y, z \))

\[
\text{minimize } c^T y + dz \\
\text{subject to } G y \preceq h z \\
A y = b z \\
e^T y + f z = 1 \\
z \geq 0
\]
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Quadratic Program

In a QP, we minimize a convex quadratic function over a polyhedron

\[
\begin{align*}
\text{minimize} \quad & (1/2)x^T Px + q^T x + r \\
\text{subject to} \quad & Gx \leq h \\
& Ax = b
\end{align*}
\]

where \( P \succeq 0 \). QP includes LP as special case by taking \( P = 0 \).
Examples

least-squares:

\[
\text{minimize } \|Ax - b\|^2_2 = x^T A^T A x - 2b^T A x + b^T b
\]

- can have linear constraints, e.g, \(l \leq x \leq u\)

linear program with random cost:

\[
\text{minimize } \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}(c^T x) + \gamma \text{var}(c^T x)
\]

subject to \(Gx \leq h, Ax = b\)

- \(c\) is random vector with mean \(\bar{c}\) and covariance matrix \(\Sigma\)
- \(c^T x\) is random variable with mean \(\bar{c}^T x\) and variance \(x^T \Sigma x\)
- parameter \(\gamma\) controls the trade-off between expected cost and variance (risk)
In QCQP, we have

\[
\begin{align*}
\text{minimize} & \quad (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & \quad (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- \( P_i \succeq 0 \); objective and constraints are convex quadratic
- if \( P_1, \ldots, P_m > 0 \), feasible set is intersection of \( m \) ellipsoids and an affine set
- QCQP includes QP (and LP) as special case, by taking \( P_i = 0, \ i \in [1 : m] \)
In SOCP, we have

\[
\begin{align*}
\text{minimize} & \quad f^T x \\
\text{subject to} & \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i, i = 1, \ldots, m \\
& \quad Fx = g
\end{align*}
\]

where we call \(\|A_i x + b_i\|_2 \leq c_i^T x + d_i\) a second-order cone constraint, since it is the same as requiring \((A_i x + b_i, c_i^T x + d_i)\) to lie in the second-order cone. SOCP reduces to QCQP if \(c_i = 0\), and reduces to LP if \(A_i = 0\).
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Geometric Programming

- monomial function

\[ f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \]

where \( \text{dom} f = \mathbb{R}^n_{++}, c > 0, \) and exponents \( a_i \) can be any real number

- posynomial function: sum of monomials

\[ f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad \text{dom} f = \mathbb{R}^n_{++} \]

- geometric program (GP)

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 1, i = 1, 2, \ldots, m \\
& \quad h_i(x) = 1, i = 1, 2, \ldots, p
\end{align*}
\]

with \( f_i \) posynomial, \( h_i \) monomial. The domain of the problem is \( D = \mathbb{R}^n_{++}; \) the constraint \( x > 0 \) is implicit.

- GP’s are not convex in their natural form, but can be transformed to convex problems.
Geometric Program in Convex Form

Change variables: $y_i = \log x_i$; take log of objective and constraint functions

- monomial $f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ transforms to
  \[
  \log f(x) = a^T y + b \quad (b = \log c)
  \]

- posynomial $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to
  \[
  \log f(x) = \log \left( \sum_{k=1}^K e^{a_{k}^T y + b_k} \right) \quad (b_k = \log c_k)
  \]

- posynomial form geometric program transforms to convex form:
  \[
  \begin{align*}
  \text{minimize} & \quad \tilde{f}_0(y) = \log \left( \sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}} \right) \\
  \text{subject to} & \quad \tilde{f}_i(y) = \log \left( \sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \right) \leq 0, \; i = 1, 2, \ldots, m \\
  & \quad \tilde{h}_i(y) = g_i^T y + h_i = 0, \; i = 1, 2, \ldots, p
  \end{align*}
  \]
  - if the posynomial objective and constraint functions all have only one term, i.e. are monomials, then the convex form GP reduces to LP
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convex problem with generalized inequality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq_{K_i} 0, \, i = 1, 2, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

with \( f_0 : \mathbb{R}^n \to \mathbb{R} \) convex, \( K_i \subseteq \mathbb{R}^{k_i} \) proper cones, \( f_i : \mathbb{R}^n \to \mathbb{R}^{k_i} \) being \( K_i \)-convex w.r.t. proper cone \( K_i \).

- Many properties of standard convex problems also hold for convex problems with generalized inequality constraints, e.g., convex feasible set, local optimum is global optimum, etc. We will also see that convex problems with generalized inequality constraints can often be solved as easily as standard convex problems.

- conic form problem (cone program): affine objective and constraints

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Fx + g \leq_K 0, \\
& \quad Ax = b
\end{align*}
\]

extends linear programming \((K = \mathbb{R}^m_{+})\) to nonpolyhedral cones
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Semidefinite Program (SDP)

- SDP is a special case of conic form problem when \( K = S^k_+ \):

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0, \quad \text{(LMI constraints)} \\
& \quad Ax = b
\end{align*}
\]

with \( F_i, G \in S^k \).

- includes problems with multiple LMI constraints: for example,

\[
\begin{align*}
& x_1 \hat{F}_1 + x_2 \hat{F}_2 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0 \quad \& \quad x_1 \tilde{F}_1 + x_2 \tilde{F}_2 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0
\end{align*}
\]

is equivalent to single LMI

\[
x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0
\]
LP and SOCP as SDP

- LP and equivalent SDP

  LP: \[ \text{minimize } c^T x \quad \text{subject to } Ax \leq b \]

  SDP: \[ \text{minimize } c^T x \quad \text{subject to } \text{diag}(Ax - b) \preceq 0 \]

  (note difference interpretation of generalized inequality \( \preceq \))

- SOCP and equivalent SDP

  SOCP: \[ \text{minimize } f^T x \]

  subject to \[ \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, 2, \ldots, m \]

  SDP: \[ \text{minimize } f^T x \]

  subject to \[ \begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, 2, \ldots, m \]
Eigenvalue Minimization

minimize $\lambda_{\text{max}}(A(x))$

where $A(x)$ is the linear matrix function

$$A(x) = A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n, \quad A_i \in S^k$$

equivalent SDP

minimize $t$

subject to $A(x) \preceq tI$

- optimization variable $(x, t)$
- follows from

$$\lambda_{\text{max}}(A) \leq t \iff A \preceq tI$$
Matrix Norm Minimization

\[
\text{minimize } \|A(x)\|_2 = (\lambda_{\text{max}}(A(x)^TA(x)))^{1/2}
\]

where \(A(x)\) is the linear matrix function

\[
A(x) = A_0 + x_1A_1 + x_2A_2 + \cdots + x_nA_n, \quad A_i \in \mathbb{R}^{p \times q}
\]

equivalent SDP

\[
\text{minimize } t
\]
\[
\text{subject to } \begin{bmatrix}
  tI & A(x) \\
  A(x)^T & tI
\end{bmatrix} \succeq 0
\]

- optimization variable \((x, t)\)
- follows from
\[
A^TA \leq t^2I, \quad t \geq 0 \iff \begin{bmatrix}
  tI \\
  A^T
\end{bmatrix}
\begin{bmatrix}
  A \\
  tI
\end{bmatrix} \succeq 0
\]
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Vector Optimization

general vector optimization problem

\[
\begin{align*}
\text{minimize (w.r.t. } K) & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \ i = 1, 2, \ldots, m \\
& \quad h_i(x) = 0, \ i = 1, 2, \ldots, p
\end{align*}
\]

vector objective \( f_0 : \mathbb{R}^n \to \mathbb{R}^q \), minimized w.r.t. proper cone \( K \subseteq \mathbb{R}^q \).

convex vector optimization problem

\[
\begin{align*}
\text{minimize (w.r.t. } K) & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \ i = 1, 2, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

with \( f_0 \) being \( K \)-convex, \( f_1, f_2, \ldots, f_m \) convex
Optimal and Pareto Optimal Points

set of achievable objective (vector) values

\[ \mathcal{O} = \{ f_0(x) | x \text{ feasible} \} \]

- \( x \) is optimal if \( f_0(x) \) is the minimum value of \( \mathcal{O} \)
- \( x \) is Pareto optimal if \( f_0(x) \) is a minimal value of \( \mathcal{O} \)

multicriterion optimization: \( K = \mathbb{R}^q_+ \)

\[ f_0(x) = (F_1(x), \ldots, F_q(x)) \]

- \( q \) different objectives \( F_i \); roughly speaking we want all \( F_i \) to be small
- if there exists an optimal point, the objectives are noncompeting; if there are multiple Pareto optimal values, there is a tradeoff between the objectives
Scalarization

to find Pareto optimal points, choose $\lambda \succeq_{K^*} 0$ and solve scalar problem

$$\begin{align*}
\text{minimize} & \quad \lambda^T f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \ i = 1, 2, \ldots, m \\
& \quad h_i(x) = 0, \ i = 1, 2, \ldots, p
\end{align*}$$

- if $x$ is optimal for scalar problem then it is Pareto optimal for vector problem
- for convex vector problem, can find (almost) all Pareto optimal points by varying $\lambda \succeq_{K^*} 0$

scalarization for multicriterion problems: minimize over feasible set

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_n F_n(x)$$

for $\lambda > 0$
**Regularized Least-Squares**

minimize (w.r.t. $\mathbb{R}_+^2$) \((\|Ax - b\|_2^2, \|x\|_2^2)\)

- example for $A \in \mathbb{R}^{100 \times 10}$; heavy line formed by Pareto optimal points

- to determine Pareto optimal points, take $\lambda = (1, \gamma)$ with $\gamma > 0$ and minimize

\[
\|Ax - b\|_2^2 + \gamma \|x\|_2^2
\]

- for fixed $\gamma$, a LS problem