Lecture 4: Convex Optimization Problems

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Outline

- Standard Form Optimization Problem
- Convex Optimization Problem
- Quasiconvex Optimization
- Linear Optimization
- Quadratic Optimization
- Geometric Optimization
- Generalized Inequality Constraints
- Semidefinite Programming
- Vector Optimization

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Standard Form Optimization

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & h_i(x) = 0, i = 1, 2, \ldots, p \end{array}$$

Optimal value:

$$p^* = \inf\{f_0(x) | f_i(x) \leq 0, i = 1, \ldots, m; h_i(x) = 0, i = 1, \ldots, p\}$$

- $p^* = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^* = -\infty$ if problem is unbounded below
- a feasible x is optimal if $f_0(x) = p^*$; X_{opt} is the set of optimal points
- x is locally optimal if there is $R > 0$ such that

$$f_0(x) = \inf\{f_0(z) | f_i(z) \leq 0, i = 1, \ldots, m; h_i(z) = 0, i = 1, \ldots, p; \|z - x\|_2 \leq R\}$$

Implicit Constraints

The standard form optimization problem has an implicit constraint:

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\ f_i \cap \bigcap_{i=0}^{p} \mathbf{dom}\ h_i$$

- we call $\mathcal{D}$ the domain of the problem
- the constraints $f_i(x) \leq 0, h_i(x) = 0$ are the explicit constraints
- a problem is unconstrained if it has no explicit constraints

Example:

$$\text{minimize } f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility Problem

$$\begin{array}{rl} \text{find} & x \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & h_i(x) = 0, i = 1, 2, \ldots, p \end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{rl} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & h_i(x) = 0, i = 1, 2, \ldots, p \end{array}$$

- $p^* = 0$ if constraints are feasible; any feasible x is optimal
- $p^* = \infty$ if constraints are infeasible

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Convex Optimization Problem

Standard form convex optimization problem

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, i = 1, 2, \ldots, m \\ & a_i^T x = b_i, i = 1, 2, \ldots, p \end{array}$$

- $f_1, f_2, \ldots, f_m$ are convex; equality constraints are affine
- problem is quasiconvex if f_0 is quasiconvex (and $f_1, f_2, \ldots, f_m$ are convex)

can be more compactly written as

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

- feasible set of a convex optimization problem is convex

Example

$$\begin{array}{rl} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1 + x_2^2) \le 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{array}$$

- f_0 is convex; feasible set $\{(x_1, x_2) | x_1 = -x_2 \le 0\}$ is convex
- not a convex problem: f_1 is not convex, h_1 is not affine
- equivalent (not identical) to the convex problem

$$\begin{array}{rl} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \le 0 \\ & x_1 + x_2 = 0 \end{array}$$

Local and Global Optima

any locally optimal point of a convex problem is globally optimal

Proof: Since x is locally optimal, there exists an $R > 0$ such that $f_0(z) \geq f_0(x)$ for any z feasible and $\|z - x\|_2 \leq R$. Consider an arbitrary feasible y that is not necessarily in $B(x, R)$. There must exist some $\alpha > 0$ s.t. $(1-\alpha)x+\alpha y \in B(x, R)$ and therefore

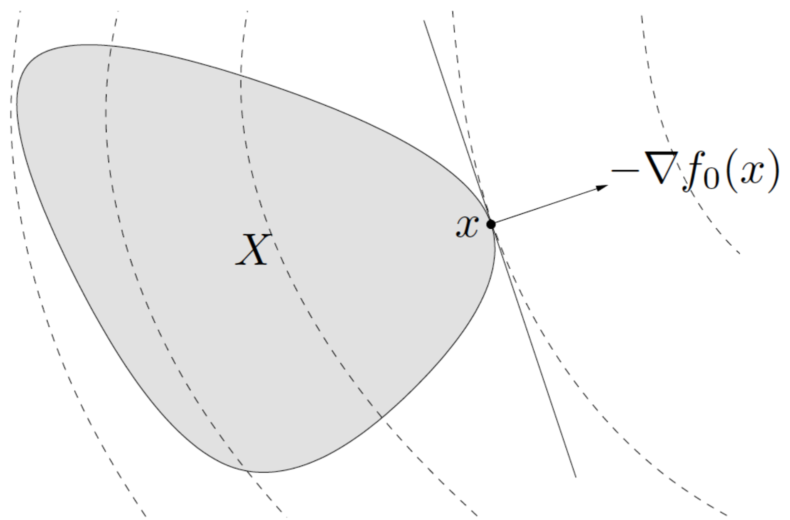
$$f(x) \leq f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y).$$

This immediately implies that $f(x) \leq f(y)$ for any feasible y.

Optimality Criterion For Differentiable Objective

x is optimal iff it is feasible and

$$\nabla f_0(x)^T (y - x) \geq 0, \text{ for all feasible } y$$



if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

Examples

- unconstrained problem: x is optimal iff

$$x \in \mathbf{dom} f_0, \nabla f_0(x) = 0$$

- equality constrained problem

$$\text{minimize } f_0(x) \text{ subject to } Ax = b$$

x is optimal iff there exists a ν such that

$$x \in \mathbf{dom} f_0, \ Ax = b, \ \nabla f_0(x) + A^T \nu = 0$$

- minimization over nonnegative orthant

$$\text{minimize } f_0(x) \text{ subject to } x \succeq 0$$

x is optimal iff

$$x \in \mathbf{dom} f_0, \ x \succeq 0, \ \nabla f_0(x)_i \geq 0 \text{ for } x_i = 0, \ \nabla f_0(x)_i = 0 \text{ for } x_i > 0$$

Equivalent Convex Problems

Informally, two problems are equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa

Common transformations that preserve convexity:

eliminating equality constraints:

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{rl} \text{minimize} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, i = 1, 2, \ldots, m \end{array}$$

where F and x_0 are such that

$$Ax = b \Leftrightarrow x = Fz + x_0,$$

i.e. $\mathcal{R}(F) = \mathcal{N}(A)$.

Equivalent Convex Problems

introducing equality constraints:

$$\begin{array}{ll} \text{minimize} & f_0(A_0x + b) \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, i = 1, 2, \dots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, i = 1, 2, \dots, m \\ & y_i = A_ix + b_i, i = 0, 1, 2, \dots, m \end{array}$$

introducing slack variables for linear inequalities:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, i = 1, 2, \dots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, i = 1, 2, \dots, m \\ & s_i \geq 0, i = 1, 2, \dots, m \end{array}$$

Equivalent Convex Problems

epigraph form: standard form convex problem is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x) - t \le 0 \\ & f_i(x) \le 0, i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

minimizing over some variables:

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \le 0, i = 1, 2, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \le 0, i = 1, 2, \ldots, m \end{array}$$

where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

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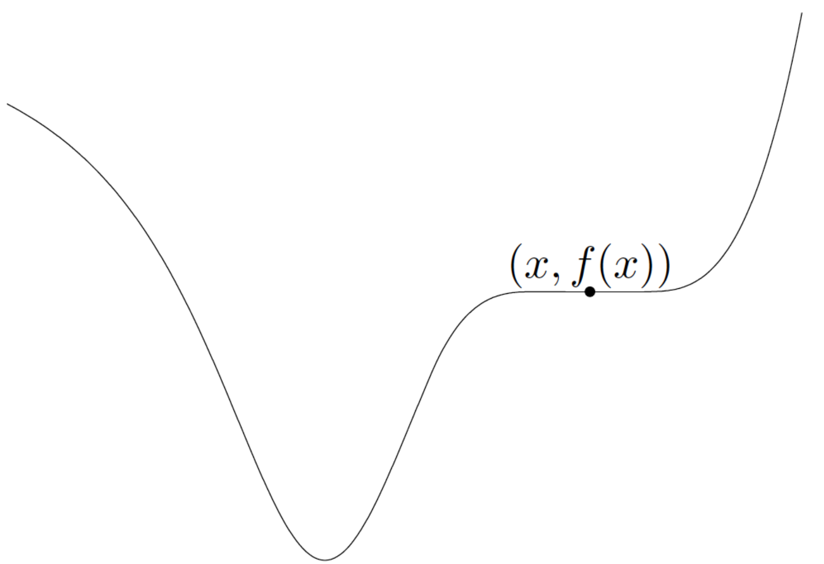
Quasiconvex Optimization

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

- quasiconvex f_0 and convex $f_i, i \in [1 : m]$

- can have locally optimal points that are not globally optimal



Quasiconvex Optimization

convex representation of sublevel sets of f_0:

if f_0 is quasiconvex, there exists a family of functions ϕ_t such that

- ϕ_t is convex in x for fixed t
- t-sublevel set of f_0 is 0-sublevel set of ϕ_t, i.e., $f_0(x) \le t \Leftrightarrow \phi_t(x) \le 0$

example: $f_0(x) = \frac{p(x)}{q(x)}$, p convex, q concave, $p(x) \ge 0$, $q(x) > 0$ on $\mathbf{dom} f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \ge 0$, ϕ_t is convex in x
- $p(x)/q(x) \le t$ iff $\phi_t(x) \le 0$

Quasiconvex Optimization

quasiconvex optimization via convex feasibility problems:

$$\phi_t(x) \leq 0, \;\; f_i(x) \leq 0, \;\; i = 1, \ldots, m, \;\; Ax = b \tag{1}$$

- for fixed t, a convex feasibility problem in x
- if feasible, we can conclude that $t \geq p^*$; otherwise, $t \leq p^*$

Bisection method for quasiconvex optimization

given $l \leq p^\star$, $u \geq p^\star$, tolerance $\epsilon > 0$.
repeat
1. $t := (l + u)/2$.
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, $u := t$; **else** $l := t$.

until $u - l \leq \epsilon$.

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requires exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations

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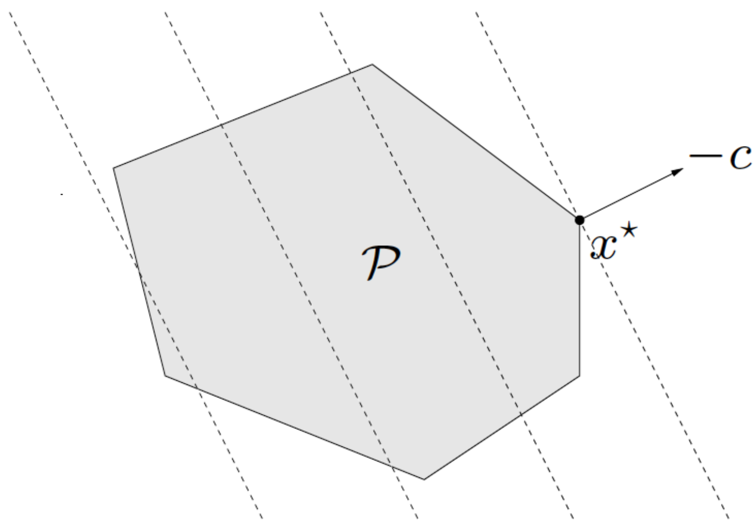
Linear Program

$$
\begin{array}{ll}
\text{minimize} & c^T x + d \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- the objective and constraint functions are all affine
- feasible set is a polyhedron



Examples

diet problem: choose quantities $x_1, x_2, \ldots, x_n$ of n foods
- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i
- to find cheapest healthy diet, solve the following LP

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, x \succeq 0 \end{array}$$

piecewise-linear minimization:

$$\text{minimize} \quad \max_{i \in [1:m]} (a_i^T x + b_i)$$

equivalent to an LP

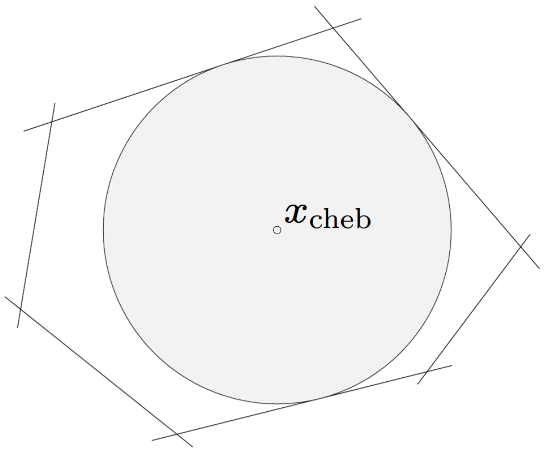
$$\begin{array}{rl} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \; i = 1, \ldots, m \end{array}$$

Examples

find Chebyshev center of a polyhedron $\mathcal{P} = \{x | a_i^T x \leq b_i, i = 1, \ldots, m\}$:

- Chebyshev center is the center of largest inscribed ball

$$B(x_c, r) = \{x_c + u \mid \|u\|_2 \leq r\}$$



- $B(x_c, r) \subseteq \mathcal{P}$ iff

$$\underbrace{\sup\{a_i^T(x_c + u) \mid \|u\|_2 \leq r\}}_{= a_i^T x_c + r\|a_i\|_2} \leq b_i, \ \forall i \in [1:m]$$

- hence x_c, r can be determined by solving the following LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \leq b_i, \ i = 1, \ldots, m \end{array}$$

Linear-Fractional Program

Linear fractional programming is to minimize a ratio of affine functions over a polyhedron:

$$
\begin{aligned}
\text{minimize} \quad & \frac{c^T x + d}{e^T x + f} \\
\text{subject to} \quad & Gx \preceq h \\
& Ax = b
\end{aligned}
$$

where the objective function is quasiconvex with its domain as $\{x | e^T x + f > 0\}$.

- a quasiconvex optimization problem; can be solved by bisection
- transformable to LP: Think of $y = x/e^T x + f$ and $z = 1/e^T x + f$. The above problem is equivalent to the following LP (variables y, z)

$$
\begin{aligned}
\text{minimize} \quad & c^T y + dz \\
\text{subject to} \quad & Gy \preceq hz \\
& Ay = bz \\
& e^T y + fz = 1 \\
& z \geq 0
\end{aligned}
$$

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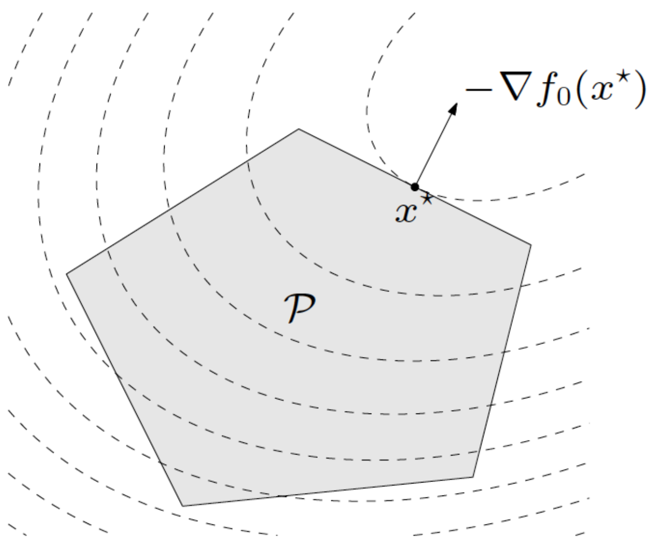
Quadratic Program

In a QP, we minimize a convex quadratic function over a polyhedron

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

where $P \succeq 0$. QP includes LP as special case by taking $P = 0$.



Examples

least-squares:

$$\text{minimize} \quad \|Ax - b\|_2^2 = x^T A^T A x - 2b^T A x + b^T b$$

- can have linear constraints, e.g, $l \preceq x \preceq u$

linear program with random cost:

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}(c^T x) + \gamma \mathbf{var}(c^T x) \\ \text{subject to} & Gx \preceq h, \ Ax = b \end{array}$$

- c is random vector with mean $\bar{c}$ and covariance matrix Σ
- $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- parameter γ controls the trade-off between expected cost and variance (risk)

Quadratically Constrained Quadratic Program

In QCQP, we have

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \;\; i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \succeq 0$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \succ 0$, feasible set is intersection of m ellipsoids and an affine set
- QCQP includes QP (and LP) as special case, by taking $P_i = 0, \;\; i \in [1 : m]$

Second-Order Cone Program

In SOCP, we have

$$\begin{aligned} \text{minimize} \quad & f^T x \\ \text{subject to} \quad & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, i = 1, \ldots, m \\ & Fx = g \end{aligned}$$

where we call $\|A_i x + b_i\|_2 \leq c_i^T x + d_i$ a second-order cone constraint, since it is the same as requiring $(A_i x + b_i, c_i^T x + d_i)$ to lie in the second-order cone. SOCP reduces to QCQP if $c_i = 0$, and reduces to LP if $A_i = 0$.

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Geometric Programming

- monomial function

$$f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

where $\mathbf{dom} f = \mathbf{R}_{++}^n$, $c > 0$, and exponents a_i can be any real number

- posynomial function: sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \ \mathbf{dom} f = \mathbf{R}_{++}^n$$

- geometric program (GP)

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, i = 1, 2, \ldots, m \\ & h_i(x) = 1, i = 1, 2, \ldots, p \end{array}$$

with f_i posynomial, h_i monomial. The domain of the problem is $\mathcal{D} = \mathbf{R}_{++}^n$; the constraint $x \succ 0$ is implicit.

- GP's are not convex in their natural form, but can be transformed to convex problems.

Geometric Program in Convex Form

Change variables: $y_i = \log x_i$; take log of objective and constraint functions

- monomial $f(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ transforms to

$$\log f(x) = a^T y + b \qquad (b = \log c)$$

- posynomial $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(x) = \log \left(\sum_{k=1}^{K} e^{a_k^T y + b_k} \right) \qquad (b_k = \log c_k)$$

- posynomial form geometric program transforms to convex form:

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(y) = \log \left(\sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}} \right) \\ \text{subject to} & \tilde{f}_i(y) = \log \left(\sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \right) \leq 0, i = 1, 2, \ldots, m \\ & \tilde{h}_i(y) = g_i^T y + h_i = 0, i = 1, 2, \ldots, p \end{array}$$

- if the posynomial objective and constraint functions all have only one term, i.e. are monomials, then the convex form GP reduces to LP

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Generalized Inequality Constraints

convex problem with generalized inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

with $f_0 : \mathbf{R}^n \to \mathbf{R}$ convex, $K_i \subseteq \mathbf{R}^{k_i}$ proper cones, $f_i : \mathbf{R}^n \to \mathbf{R}^{k_i}$ being K_i-convex w.r.t. proper cone K_i.

- Many properties of standard convex problems also hold for convex problems with generalized inequality constraints, e.g., convex feasible set, local optimum is global optimum, etc. We will also see that convex problems with generalized inequality constraints can often be solved as easily as standard convex problems.

- conic form problem (cone program): affine objective and constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0, \\ & Ax = b \end{array}$$

extends linear programming ($K = \mathbf{R}^m_+$) to nonpolyhedral cones

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Semidefinite Program (SDP)

- SDP is a special case of conic form problem when $K = \mathbf{S}_+^k$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0, \qquad \text{(LMI constraints)} \\ & Ax = b \end{array}$$

with $F_i, G \in \mathbf{S}^k$.

- includes problems with multiple LMI constraints: for example,

$$x_1 \hat{F}_1 + x_2 \hat{F}_2 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0 \quad \& \quad x_1 \tilde{F}_1 + x_2 \tilde{F}_2 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

is equivalent to single LMI

$$x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0$$

LP and SOCP as SDP

- LP and equivalent SDP

$$\begin{array}{rll} \text{LP:} & \text{minimize } c^T x & \text{subject to } Ax \preceq b \\ \text{SDP:} & \text{minimize } c^T x & \text{subject to } \mathbf{diag}(Ax - b) \preceq 0 \end{array}$$

(note difference interpretation of generalized inequality $\preceq$)

- SOCP and equivalent SDP

$$\begin{array}{rll} \text{SOCP:} & \text{minimize} & f^T x \\ & \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, 2, \ldots, m \end{array}$$

$$\begin{array}{rll} \text{SDP:} & \text{minimize} & f^T x \\ & \text{subject to} & \begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, 2, \ldots, m \end{array}$$

Eigenvalue Minimization

$$\text{minimize } \lambda_{\max}(A(x))$$

where $A(x)$ is the linear matrix function

$$A(x) = A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n, \quad A_i \in \mathbf{S}^k$$

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

- optimization variable (x, t)
- follows from

$$\lambda_{\max}(A) \leq t \Leftrightarrow A \preceq tI$$

Matrix Norm Minimization

$$\text{minimize } \|A(x)\|_2 = (\lambda_{\max}(A(x)^T A(x)))^{1/2}$$

where $A(x)$ is the linear matrix function

$$A(x) = A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n, \quad A_i \in \mathbf{R}^{p \times q}$$

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{array}$$

- optimization variable (x, t)
- follows from

$$A^T A \preceq t^2 I, \; t \geq 0 \Leftrightarrow \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0$$

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Vector Optimization

general vector optimization problem

$$\begin{array}{ll} \text{minimize (w.r.t. } K) & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & h_i(x) = 0, i = 1, 2, \ldots, p \end{array}$$

vector objective $f_0 : \mathbf{R}^n \to \mathbf{R}^q$, minimized w.r.t. proper cone $K \subseteq \mathbf{R}^q$.

convex vector optimization problem

$$\begin{array}{ll} \text{minimize (w.r.t. } K) & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

with f_0 being K-convex, $f_1, f_2, \ldots, f_m$ convex

Optimal and Pareto Optimal Points

set of achievable objective (vector) values

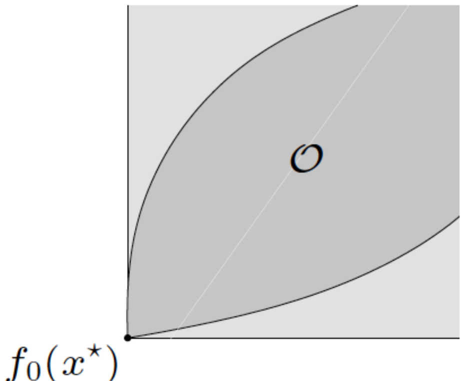
$$\mathcal{O} = \{f_0(x) | x \text{ feasible}\}$$

- x is optimal if $f_0(x)$ is the minimum value of $\mathcal{O}$
- x is Pareto optimal if $f_0(x)$ is a minimal value of $\mathcal{O}$

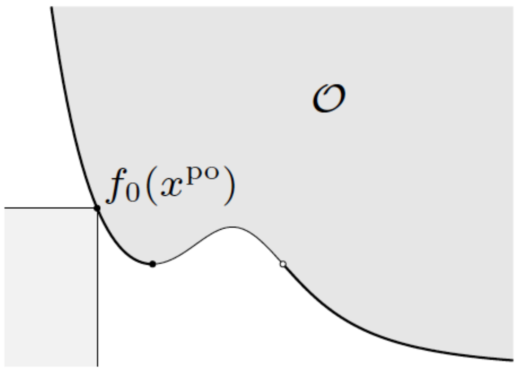
multicriterion optimization: $K = \mathbf{R}^q_+$

$$f_0(x) = (F_1(x), \ldots, F_q(x))$$

- q different objectives F_i; roughly speaking we want all F_i to be small
- if there exists an optimal point, the objectives are noncompeting; if there are multiple Pareto optimal values, there is a tradeoff between the objectives



$x^\star$ is optimal



x^{po} is Pareto optimal

Scalarization

to find Pareto optimal points, choose $\lambda \succ_{K^*} 0$ and solve scalar problem

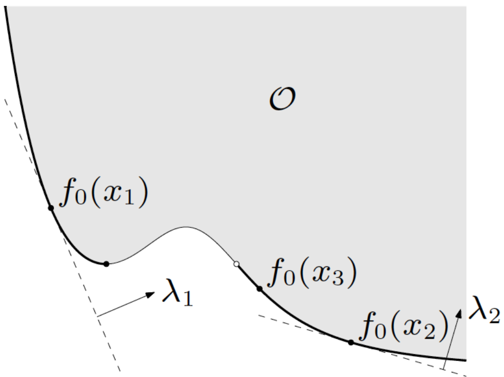
$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & h_i(x) = 0, i = 1, 2, \ldots, p \end{array}$$

- if x is optimal for scalar problem then it is Pareto optimal for vector problem
- for convex vector problem, can find (almost) all Pareto optimal points by varying $\lambda \succ_{K^*} 0$

scalarization for multicriterion problems: minimize over feasible set

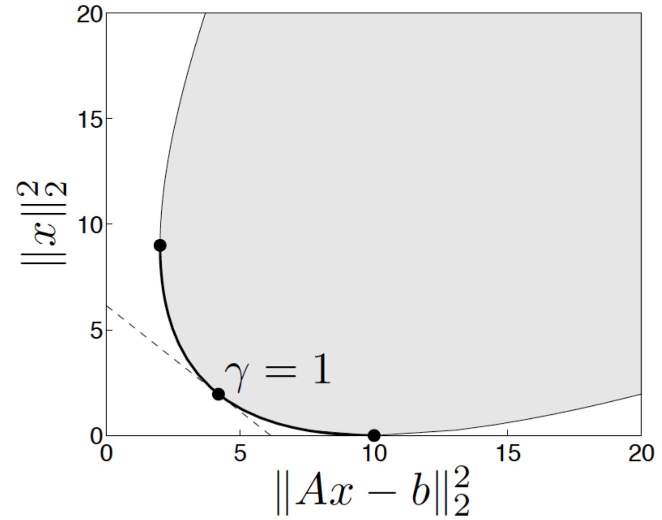
$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_n F_n(x)$$

for $\lambda \succ 0$



Regularized Least-Squares

$$\text{minimize (w.r.t. } \mathbf{R}^2_+) \quad (\|Ax - b\|_2^2, \|x\|_2^2)$$



- example for $A \in \mathbf{R}^{100 \times 10}$; heavy line formed by Pareto optimal points

- to determine Pareto optimal points, take $\lambda = (1, \gamma)$ with $\gamma > 0$ and minimize

$$\|Ax - b\|_2^2 + \gamma \|x\|_2^2$$

- for fixed γ, a LS problem