Lecture 2: Convex Sets

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Fall 2019

Recap: Convex Optimization Problems

$$\begin{aligned} &\text{minimize } f_0(x) \\ &\text{subject to } f_i(x) \le b_i, \; i = 1, 2, \ldots, m \end{aligned}$$

where objective and constraint functions are convex

- We can broadly understand and solve convex optimization problems
- In contrast, non-convex problems are mostly treated on a case-by-case basis
- Special property of convex problems: any local minimizer is a global minimizer

Outline

- Affine and Convex Sets
- Important Examples of Convex Sets
- Operations that Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cone and Generalized Inequalities

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Linear, Affine, Convex and Conic Combination

- Linear combination: $\sum_{i=1}^{k} \theta_i x_i$, where $\theta_i \in \mathbf{R}$

- Affine combination: $\sum_{i=1}^{k} \theta_i x_i$, where $\theta_i \in \mathbf{R}, \sum_{i=1}^{k} \theta_i = 1$

- Convex combination: $\sum_{i=1}^{k} \theta_i x_i$, where $\theta_i \in \mathbf{R}, \sum_{i=1}^{k} \theta_i = 1, \theta_i \geq 0$

- Conic combination: $\sum_{i=1}^{k} \theta_i x_i$, where $\theta_i \in \mathbf{R}, \theta_i \geq 0$

Linear Space

- Linear space: $V \subseteq \mathbf{R}^n$ such that $x_1, x_2 \in V \Rightarrow \theta_1 x_1 + \theta_2 x_2 \in V, \forall \theta_1, \theta_2 \in \mathbf{R}$;
contains linear combination of any two points in the set

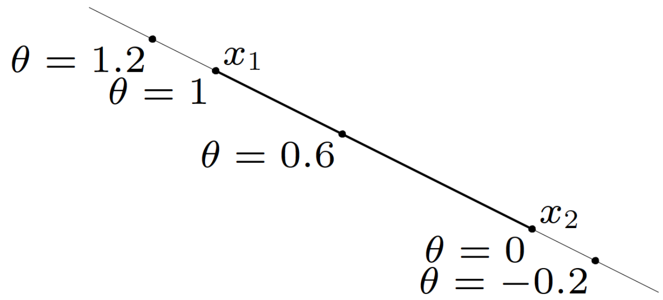
- Linear span: $\mathbf{span}(C) = \left\{ \sum_{i=1}^{k} \theta_i x_i \middle| k \in \mathbf{Z}_+, x_i \in C, \theta_i \in \mathbf{R} \right\}$;
contains all linear combinations of points in C;
smallest linear space that contains C

- Dimension: $\mathbf{dim}(V)$ is the size of a minimal spanning set for V

- Example: $V = \{x | Ax = 0\}, A \in \mathbf{R}^{m \times n}$;
[what about $\{x | Ax = b\}$?]

Affine Set

- Line through x_1, x_2: all points $x = \theta x_1 + (1-\theta)x_2, \theta \in \mathbf{R}$



- Affine set: contains the line through any two points in the set

- Affine hull: $\mathbf{aff}(C) = \left\{ \sum_{i=1}^{k} \theta_i x_i \middle| k \in \mathbf{Z}_+, x_i \in C, \sum_{i=1}^{k} \theta_i = 1 \right\}$;

  smallest affine set that contains C

- Dimension: $V = C - x_0, \forall x_0 \in C$ is a subspace; $\mathbf{dim}(C) = \mathbf{dim}(V)$

- Example: $C = \{x | Ax = b\}, A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m$;

  Subspace associated with C is the null space of A;

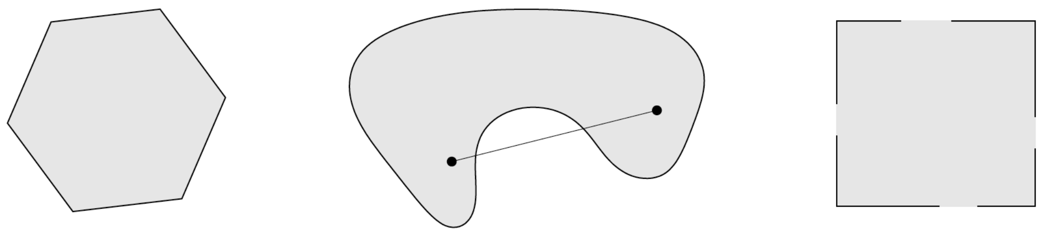
  Conversely, every affine set can be expressed as solution set of linear equations

Convex Set

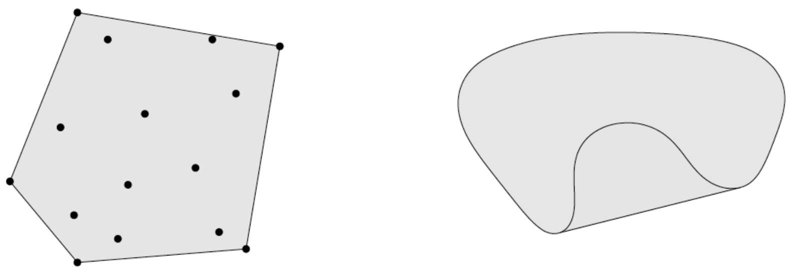
- Line segment between x_1 and x_2: all points $x = \theta x_1 + (1-\theta)x_2, \theta \in [0,1]$

- Convex set: contains line segment between any two points in the set ;

$$x_1, x_2 \in C \Rightarrow \theta x_1 + (1-\theta)x_2 \in C, \forall \theta \in [0,1]$$



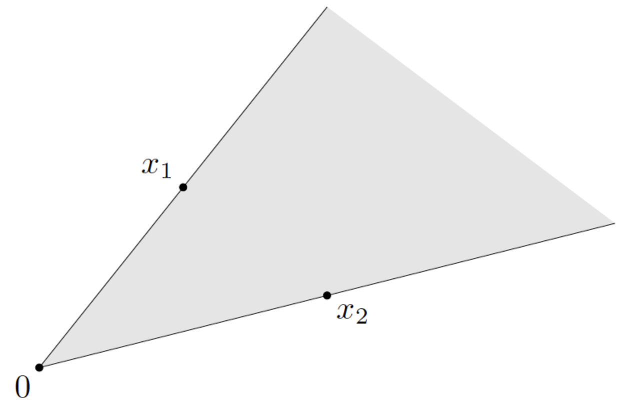
- Convex hull $\mathbf{conv}(C)$: set of all convex combinations of points in C



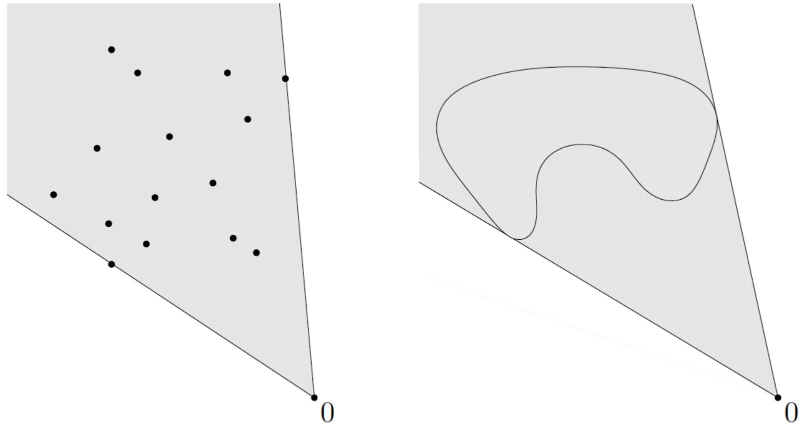
Convex Cone

- Convex cone: contains conic combination of any two points in the set;

$$x_1, x_2 \in C \Rightarrow \theta_1 x_1 + \theta_2 x_2 \in C, \forall \theta_1, \theta_2 \geq 0$$



- Conic hull **cone**(C): set of all conic combinations of points in C



Summary

- Linear combination; linear space; linear span
- Affine combination; affine set; affine hull
- Convex combination; convex set; convex hull
- Conic combination; convex cone; conic hull
- $\textbf{span}(C) \supseteq \textbf{aff}(C), \textbf{cone}(C) \supseteq \textbf{conv}(C)$
- Subspace $\Rightarrow$ affine set, convex cone $\Rightarrow$ convex set

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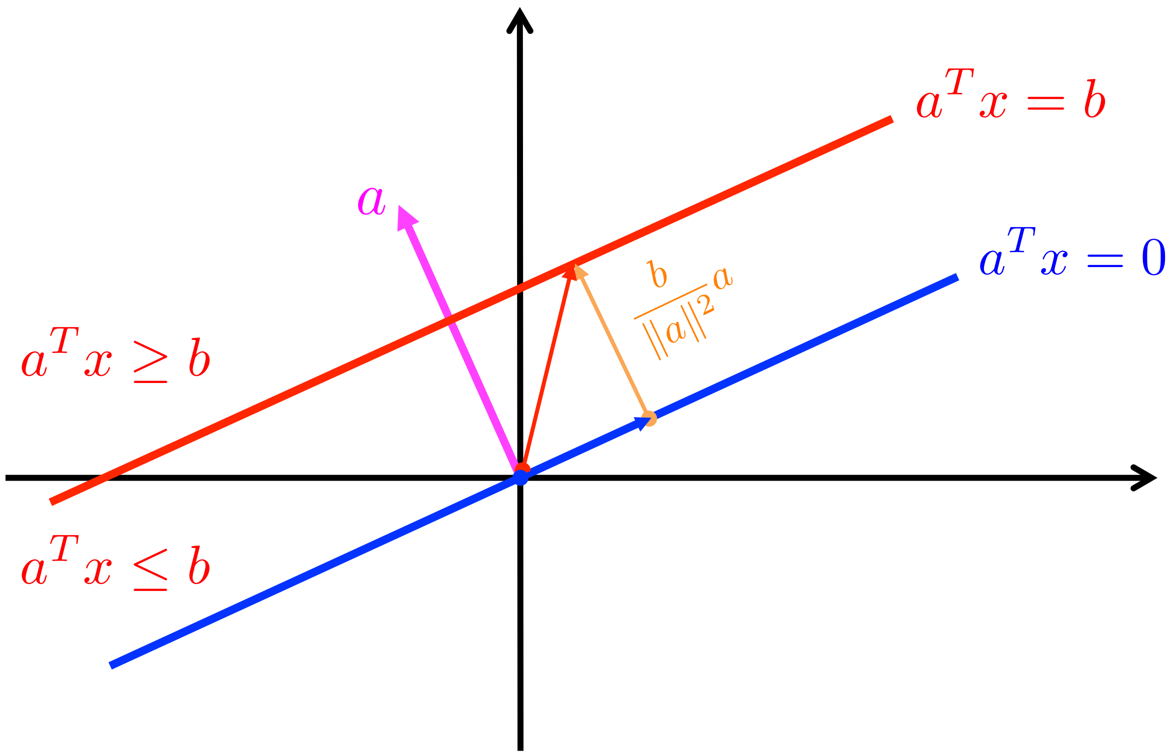
Some Simple Examples

- empty set $\emptyset$, singleton $\{x_0\}$, and the whole space $\mathbf{R}^n$ are affine (therefore convex); empty set $\emptyset$ and singleton $\{x_0\}$ are NOT subspace though
- Any line is affine and hence convex; if it passes through zero, then it is also subspace and hence convex cone
- Any line segment is convex, but not affine, not cone, not subspace
- A ray $\{x_0 + \theta v | \theta \geq 0\}$ is convex, but not affine, not subspace; it becomes a convex cone if $x_0 = 0$
- Any subspace is affine and convex cone, and hence convex

Hyperplanes and Halfspaces

- Hyperplane: $\{x|a^T x = b\}, a \neq 0$; halfspace: $\{x|a^T x \leq b\}, a \neq 0$
- Hyperplanes are affine and convex; halfspaces are convex

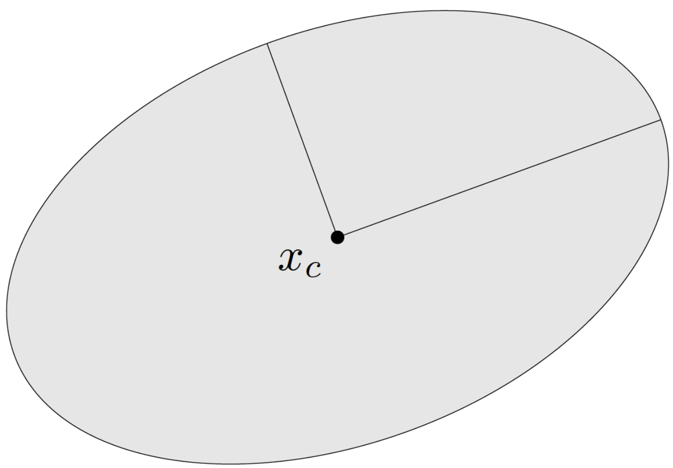


Euclidean Balls and Ellipsoids

- (Euclidean) ball: $B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}$ or $x_c + rB(0,1)$

- Ellipsoid: $\mathcal{E} = x_c + AB(0,1), A = P^{1/2}, P \succ 0;$

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$



Norm Balls and Norm Cones

- Norm: a function $\|\cdot\|$ that satisfies
  - $\|x\| \geq 0$; $\|x\| = 0$ iff $x = 0$
  - $\|tx\| = |t|\|x\|, \forall t \in \mathbf{R}$
  - $\|x + y\| \leq \|x\| + \|y\|$

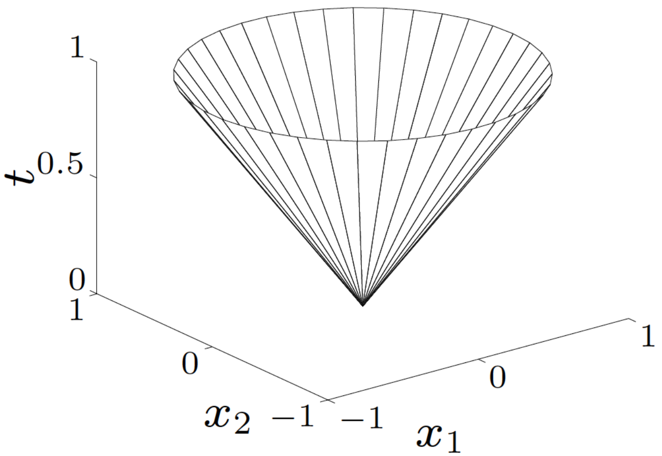
Notation: $\|\cdot\|$: general (unspecified) norm

$\|\cdot\|_{\text{symb}}$: particular norm

- Norm ball: $\{x | \|x - x_c\| \leq c\}$

- Norm cone: $\{(x, t) \mid \|x\| \leq t\} \subseteq \mathbf{R}^{n+1}$

  Euclidean norm cone is called second-order cone



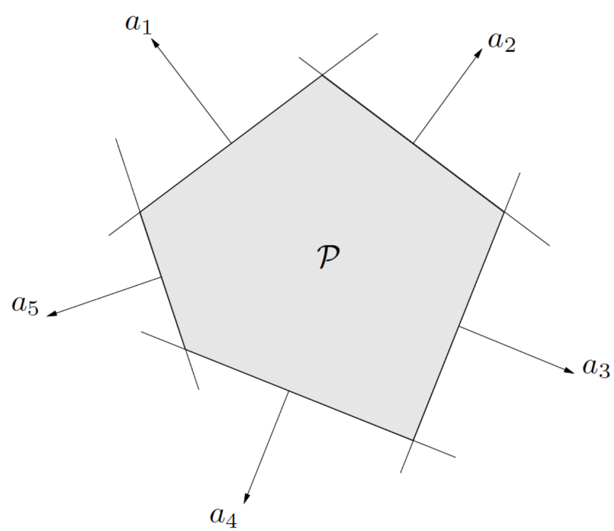
- Norm balls and norm cones are convex

Polyhedra

- Polyhedron: solution set of a finite number of linear equalities and inequalities;

$$\mathcal{P} = \{x \mid a_j^T x \leq b_j, j \in [1:m], c_j^T x = d_j, j \in [1:p]\};$$

$$\mathcal{P} = \{x \mid Ax \preceq b, Cx = d\}, A \in \mathbf{R}^{m \times n}, C \in \mathbf{R}^{p \times n}$$



- Polyhedron is intersection of finite number of halfspaces and hyperplanes

- Also called polytope if bounded

Positive Semidefinite Cone

- set of symmetric matrices, i.e. $\mathbf{S}^n = \{X \in \mathbf{R}^{n \times n} \mid X^T = X\}$

- set of symmetric positive semidefinite matrices: $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$

  – $X \in \mathbf{S}^n_+ \Leftrightarrow z^T X z \geq 0, \forall z$
  – $\mathbf{S}^n_+$ is convex cone

- set of symmetric positive definite matrices: $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$

  – $\mathbf{S}^n_{++}$ is convex but not convex cone

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A Calculus of Convex Sets

Practical methods for establishing convexity of set C:

- Apply definition: $x_1, x_2 \in C \Rightarrow \theta x_1 + (1-\theta) x_2 \in C$

- Show that C is obtained from simple convex sets (hyperplanes, half spaces, norm balls, etc) by operations that preserve convexity
  - intersection
  - affine functions
  - perspective function
  - linear-fractional functions

Intersection

Intersection of (any number of) convex sets is convex

- polyhedron is the intersection of halfspaces and hyperplanes, and hence convex

- positive semidefinite cone $\mathbf{S}_+^n = \cap_{z \neq 0} \{A \in \mathbf{S}^n | z^T A z \geq 0\}$; the intersection of an infinite number of halfspaces in $\mathbf{S}^n$

- conversely, every closed convex set is the intersection of halfspaces
  - in particular, every closed convex set C is the intersection of all halfspaces that contain C

Affine Functions

Function $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine if $f(x) = Ax + b, A \in \mathbf{R}^{m\times n}, b \in \mathbf{R}^m$

If S is convex and f is affine, then $f(S)$ and $f^{-1}(S)$ are affine

- scaling αS, translation $a + S$, projection $T = \{x_1 | (x_1, x_2) \in S\}$

- sum of sets $S_1 + S_2$ is convex if S_1, S_2 are convex; follows by applying $f(x_1, x_2) = x_1 + x_2$ to product set $S_1 \times S_2$

- solution set of linear matrix inequality $\{x | x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \preceq B\}$ where $B, A_i \in \mathbf{S}^m$, $x \in \mathbf{R}^n$; it is the inverse image of the positive semidefinite cone under the affine function $f : \mathbf{R}^n \to \mathbf{S}^m$ given by $f(x) = B - A(x)$

- polyhedron $\mathcal{P} = \{x | Ax \preceq b, Cx = d\}$ is the inverse image of $\mathbf{R}^m_+ \times \{0\}$ under the affine mapping $f(x) = (b - Ax, d - Cx)$; i.e., $\mathcal{P} = \{x | f(x) \in \mathbf{R}^m_+ \times \{0\}\}$

- ellipsoid $\mathcal{E} = \{x | (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$ is convex; it is the image of $B(0,1) = \{u | u^T u \leq 1\}$ under the affine mapping $f(u) = P^{1/2} u + x_c$; it is also the inverse image of $B(0,1)$ under the affine mapping $g(x) = P^{-1/2}(x - x_c)$

Perspective and Linear-Fractional Functions

- perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$

$$P(x,t) = x/t, \ \mathbf{dom}\ P = \{(x,t) | t > 0\}$$

images and inverse images of convex sets under perspective are convex

- linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$

$$f(x) = \frac{Ax+b}{c^T x + d}, \ \mathbf{dom}\ f = \{x | c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional are convex

- Example: Suppose u and v are random variables that take on values in $[1:m]$ and $[1:n]$ respectively. Let $P_{u,v} = \{p_{u,v}(i,j)\}_{(i,j)\in[1:m]\times[1:n]}$ and let $P_{u|v=j} = \{p_{u|v}(i|j)\}_{i\in[1:m]}$ for any $j \in [1:n]$. Note that $p_{u|v}(i|j) = \frac{p_{u,v}(i,j)}{\sum_{i=1}^m p_{u,v}(i,j)}$, i.e. $P_{u|v=j}$ is obtained by a linear-fractional mapping from $P_{u,v}$. Therefore, if C is a convex set of joint distributions for (u,v), then the set of conditional distributions of u given $v=j$ for any j is also convex.

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Generalized Inequalities

A convex cone $K \subseteq \mathbf{R}^n$ is a proper cone if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

Examples:

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n | x_i \geq 0, i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$

Generalized inequality defined by proper cone K:

$$x \preceq_K y \Leftrightarrow y - x \in K, \quad x \prec_K y \Leftrightarrow y - x \in \textbf{int } K$$

Examples:

- componentwise inequality: $x \preceq_{\mathbf{R}^n_+} y \Leftrightarrow x_i \leq y_i, \forall i$
- matrix inequality: $X \preceq_{\mathbf{S}^n_+} Y \Leftrightarrow Y - X$ positive semidefinite

Many properties of $\preceq_K$ are similar to $\leq$ on $\mathbf{R}$

- e.g., $x \preceq_K y, u \preceq_K v \Rightarrow x + u \preceq_K y + v$

Minimum and Minimal Elements

Generally $\preceq_K$ is not a linear ordering: may have neither $x \preceq_K y$ nor $y \preceq_K x$

$x \in S$ is the minimum element of S w.r.t. $\preceq_K$ if $x \preceq_K y, \forall y \in S$, or equivalently, $S \subseteq x + K$

- if a set has a minimum element, then it is unique

$x \in S$ is the minimal element of S w.r.t. $\preceq_K$ if there does not exist $y \in S, y \neq x$ such that $y \preceq_K x$, or equivalently, $(x - K) \cap S = \{x\}$

- a set can have many different minimal elements

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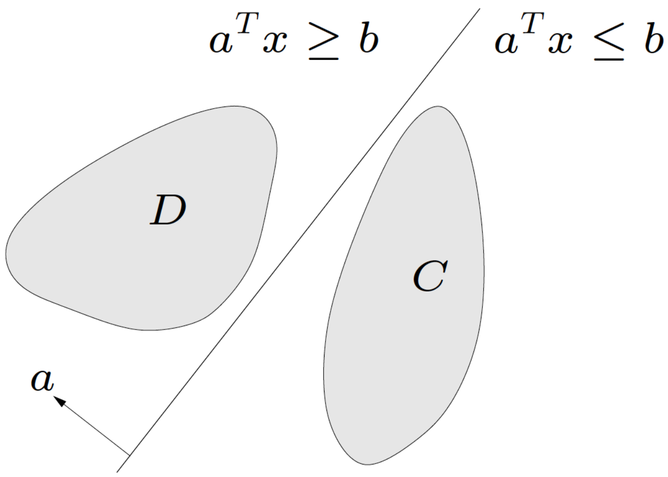
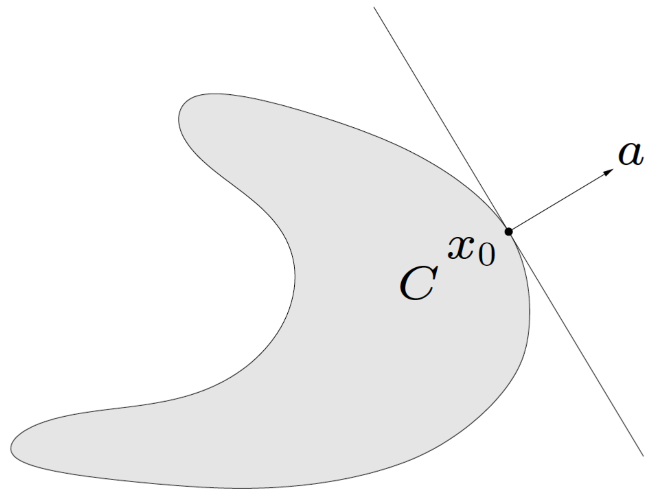
Separating Hyperplane Theorem

If C and D are nonempty disjoint convex sets, then there exists $a \neq 0, b$ such that $a^T x \leq b$ for $x \in C$ and $a^T x \geq b$ for $x \in D$. The hyperplane $\{x | a^T x = b\}$ is called a separating hyperplane for the sets C and D.



Supporting Hyperplane Theorem

Any hyperplane $\{x | a^T x = a^T x_0\}$ with $a \neq 0$ such that $a^T x \leq a^T x_0$ for all $x \in C$ is called a supporting hyperplane to set C at boundary point x_0.



Supporting Hyperplane Theorem: If C is convex, then there exists a supporting hyperplane at every boundary point of C.

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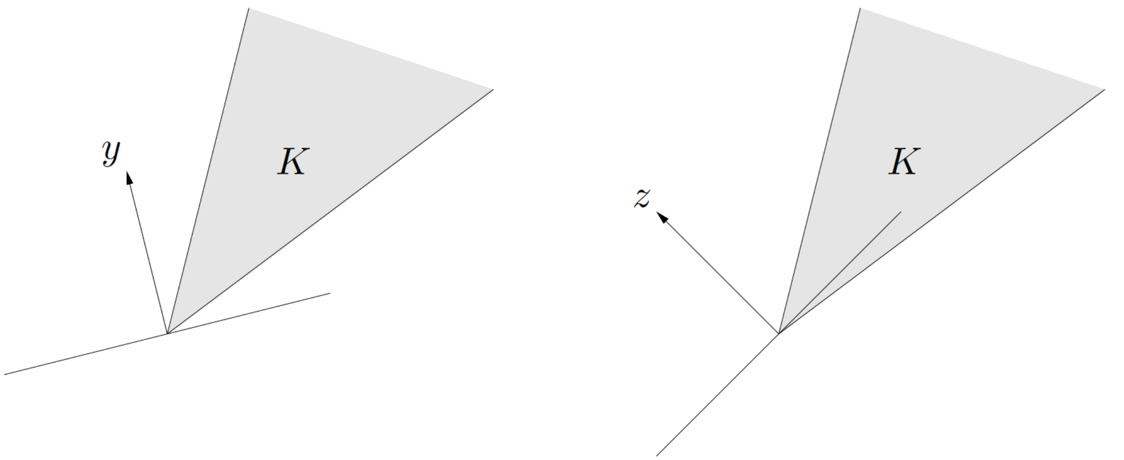
Dual Cone

Dual cone of a cone K: $K^* = \{y | y^T x \geq 0, \forall x \in K\}$



Examples:

- $K = \mathbf{R}^n_+, K^* = \mathbf{R}^n_+$
- $K = \mathbf{S}^n_+, K^* = \mathbf{S}^n_+$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}, K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}, K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

First three examples are self-dual cones

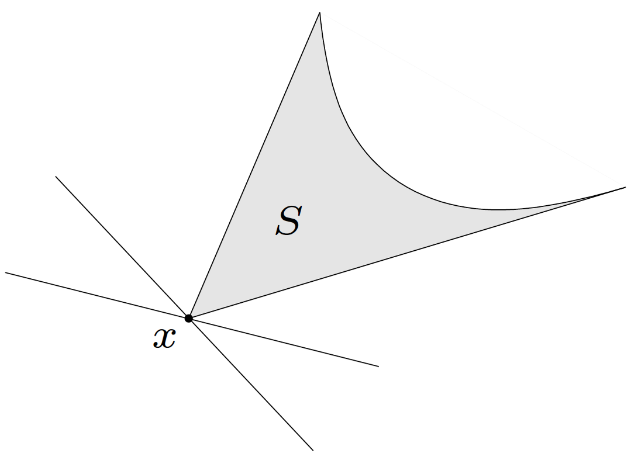
Dual cones of proper cones are proper, hence define generalized inequalities:

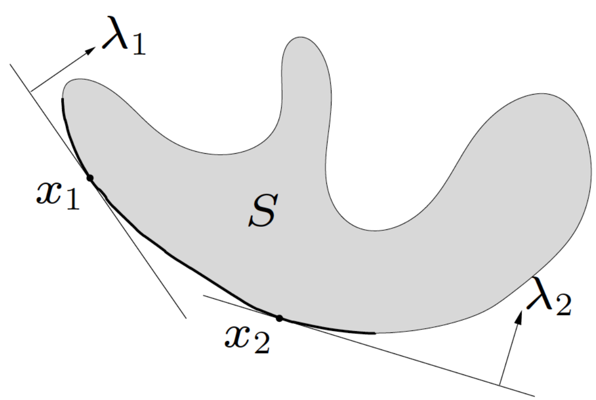
$$y \succeq_{K^*} 0 \Leftrightarrow y^T x \geq 0 \ \forall x \in K$$

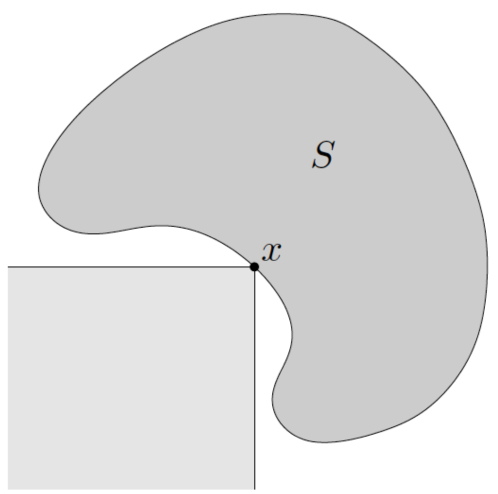
Minimum and Minimal Elements via Dual Inequalities

minimum element w.r.t. $\preceq_K$: x is minimum element of S iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S

minimal elements w.r.t. $\preceq_K$:
- if x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$ then x is minimal
- if x is minimal element of a convex set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S







Efficient Production Frontier

Consider manufacturing a product with n resources:

- different production methods use different amounts of resources $x \in \mathbf{R}^n$
- production set P: resource vectors x for all possible production methods
- efficient (pareto optimal) methods correspond to resource vectors x that are minimal w.r.t. $\mathbf{R}^n_+$

Example ($n = 2$):

- x_1, x_2, x_3 are efficient
- x_4, x_5 are not efficient
- x_1 minimizes $\lambda^T z$ over P for the shown $\lambda \in \mathbf{R}^2_{++}$
- x_2 is efficient but cannot be found by minimizing $\lambda^T z$ for some $\lambda \in \mathbf{R}^2_{++}$
- λ_i can be interpreted as the price of resource i