Chapter 7: Properties of Expectation

Xiugang Wu

University of Delaware

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Outline

- Expectation of Sums of RVs
- Covariance, Variance of Sums, and Correlation
- Conditional Expectation

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Expectation of Sums of RVs

General Formula: functions $g(X)$ of r.v.'s

Discrete: $E[g(X)] = \sum_x g(x) \cdot p(x)$

Continuous: $E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$

Two r.v.'s: $E[g(X,Y)] = \begin{cases} \sum_{x,y} g(x,y) \cdot p(x,y) & \text{Discrete case} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dx\, dy & \text{Continuous Case} \end{cases}$

In particular, if $g(X,Y) = X+Y$, then

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) \cdot f(x,y)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left[\underbrace{\int_{-\infty}^{\infty} f(x,y)\, dy}_{f_X(x)} \right] dx + \int_{-\infty}^{\infty} y \cdot \left[\underbrace{\int_{-\infty}^{\infty} f(x,y)\, dx}_{f_Y(y)} \right] dy = E[X] + E[Y]$$

By induction, generally, $E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$

Example

Example: (The sample mean). Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed (i, i, d) r.v.'s having distribution F and Expectation μ.

Such a sequence of r.v.'s is said to constitute a sample from the distribution F. The $\bar{X}$, defined by $\bar{X} = \sum_{i=1}^{n} X_i/n$, is called the sample mean.

$$E[\bar{X}] = \frac{E[X_1+X_2+\cdots+X_n]}{n} = \frac{E[X_1]+E[X_2]+\cdots+E[X_n]}{n} = \frac{n\mu}{n} = \mu$$

When the distribution mean μ is unknown, the sample mean is often used in statistics to estimate it.

Example: Expectation of a binomial r.v.. Let $X \sim$ binomial (n,p), What's $E[X]$?

Recall that such a r.v. represents the # of successes in n independent trials, when each trial has prob. p of being a success, so we have $X = X_1 + X_2 + \cdots + X_n$, where $X_i = \begin{cases} 1 & \text{if the ith trial is a success} \\ 0 & \text{if the ith trial is a failure} \end{cases}$

Hence, X_i is a Bernolli r.v. having expectation $E[X_i] = 1\cdot p + 0\cdot(1-p) = p$

Thus, $E[X] = E[X_1] + \cdots + E[X_n] = np$

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Covariance

Proposition 4.1 If X and Y are independent, then for any functions g and h,
$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Proof: $$E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\underbrace{f(x,y)}_{f_X(x)\cdot f_Y(y)}\,dx\,dy = \underbrace{\int_{-\infty}^{\infty} g(x)f_X(x)dx}_{E[g(X)]}\cdot\underbrace{\int_{-\infty}^{\infty} h(y)f_Y(y)dy}_{E[h(Y)]}$$
$$= E[g(X)]E[h(Y)]$$

Definition: The covariance between X and Y is defined as $cov(X,Y) = E[(X-EX)(Y-EY)]$

$$Cov(X,X) = E[(X-EX)(X-EX)] = E[(X-EX)^2] = Var(X)$$

we have known that $Var(X) = E[X^2] - (E[X])^2$

$$Cov(X,Y) = E[(X-EX)(Y-EY)] = E[XY - X\,EY - Y\,EX + EX\cdot EY]$$
$$= E[XY] - \underbrace{E[X\cdot EY]}_{E[Y]\cdot E[X]} - \underbrace{E[Y\cdot EX]}_{E[X]\cdot E[Y]} + \underbrace{E[EX\cdot EY]}_{E[X]\cdot E[Y]} = E[XY] - E[X]\cdot E[Y]$$

Covariance

If X and Y are independent, $cov(X,Y) = E[XY] - E[X]\cdot E[Y] = E[X]\cdot E[Y] - E[X]\cdot E[Y] = 0$

Therefore, independence implies the zero covariance, but the converse doesn't neccessarily hold.
A counter example to show the zero covariance doesn't imply independence:

Let X be

$$P\{X=0\} = P\{X=1\} = P\{X=-1\} = \frac{1}{3}$$

and define

$$Y = \begin{cases} 0 & \text{if } X \neq 0 \\ 1 & \text{if } X = 0 \end{cases}$$

$$\left.\begin{array}{l} \text{Now, } X\cdot Y \equiv 0, \text{ so } E[X\cdot Y] = 0 \\ E[X] = 0, \text{ so } E[X]\cdot E[Y] = 0 \end{array}\right\} \Rightarrow cov(X,Y) = E[XY] - E[X]E[Y] = 0$$

But obviously, X and Y are not independent.

Properties of Covariance

Proposition 4.2.

(i) $cov(X,Y) = cov(Y,X)$

(ii) $cov(X,X) = Var(X)$

(iii) $cov(aX,Y) = a\,cov(X,Y)$

(iv) $cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} cov(X_i, Y_j)$

we have known that $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$, what's $Var(\sum_{i=1}^{n} X_i)$?

$$Var(\sum_{i=1}^{n} X_i) = cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} cov(X_i, X_j)$$

$$= \sum_{i=j=1}^{n} cov(X_i, X_j) + \sum_{i\neq j} cov(X_i, X_j) = \sum_{i=1}^{n} Var(X_i) + \sum_{i\neq j} cov(X_i, X_j)$$

If $X_i, (i=1,\cdots,n)$ are independent, since $cov(X_i, X_j) = 0$ for $i \neq j$.

therefore $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$

Example

Example: Let $X_1, \cdots, X_n$ be i.i.d having expectation μ, and variance σ^2, Consider the sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The quantities $X_i - \bar{X}$, $i=1,2,\cdots,n$, are called deviations. We have the following results:

Last time: $E[\bar{X}] = \mu$.

This time: $Var(\bar{X}) = \frac{\sigma^2}{n} \longrightarrow 0$ as $n \to \infty$.

Proof: $$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \left(\frac{1}{n}\right)^2 Var\left(\sum_{i=1}^{n} X_i\right) \overset{\text{independence}}{=} \left(\frac{1}{n}\right)^2 \cdot \sum_{i=1}^{n} Var(X_i)$$

$$= \left(\frac{1}{n}\right)^2 \cdot \sum_{i=1}^{n} \sigma^2 = \left(\frac{1}{n}\right)^2 \cdot n\sigma^2 = \sigma^2/n$$

Example: Variance of a binomial r.v.

Let $X \sim binomial(n, p)$, then $X = X_1 + X_2 + \cdots + X_n$, where $X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ trial is a success} \\ 0 & \text{otherwise} \end{cases}$

$$Var(X) = Var(X_1) + Var(X_2) + \cdots Var(X_n)$$

Since $Var(X_i) = E[X_i^2] - (E[X_i])^2 = p - p^2$, $Var(X) = n\,Var(X_1) = np(1-p)$

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Conditional Expectation

Discrete: Conditional PMF: $p_{X|Y}(x|y) = P\{X=x \mid Y=y\} = \dfrac{p(x,y)}{p_Y(y)}$

Conditional Expectation: $E[X|Y=y] = \sum_x x P\{X=x|Y=y\}$

$= \sum_x x\, p_{X|Y}(x|y)$

Continuous: Conditional PDF: $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

Conditional Expectation: $E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$

Example: Suppose $f(x,y) = \dfrac{e^{-x/y}e^{-y}}{y}$, $0<x<\infty$, $0<y<\infty$, Compute $E[X|Y=y]$

Solution: $f_{X|Y}(x|y) = \frac{1}{y} e^{-x/y}$, $x>0$, $y>0$.

$E[X|Y=y] = \int_{-\infty}^{\infty} x \frac{1}{y} e^{-x/y} dx = \int_0^{\infty} x \frac{1}{y} e^{-x/y} dx = y$

Similar Formulas: $E[g(X)|Y=y] = \begin{cases} \sum_x g(x)\, p_{X|Y}(x|y) & \text{for discrete case} \\ \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\,dx & \text{for continuous case} \end{cases}$

and, $E\left[\sum_{i=1}^{n} X_i \mid Y=y\right] = \sum_{i=1}^{n} E[X_i|Y=y]$

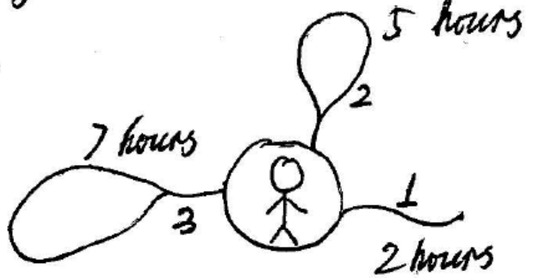
Computing Expectation by Conditioning

Note that $E[X|Y]$ is a r.v., since $E[X|Y] = \begin{cases} E[X|Y=y_1] & \text{when } Y=y_1 \\ E[X|Y=y_2] & \text{when } Y=y_2 \\ \vdots & \vdots \end{cases}$

Proposition 5.1 $\quad E[X] = E[E[X|Y]]$

Discrete case: $E[X] = \sum_y E[X|Y=y]\, P\{Y=y\}$.

Continuous case: $E[X] = \int_{-\infty}^{\infty} E[X|Y=y]\, f_Y(y)\, dy$



Example: A miner is trapped as the figure shows. What's the expected length of time until he gets out?

Let X be the # of hours to get out, and let Y be the path chosen the first time and take value 1, 2, or 3.

$E[X] = E[E[X|Y]]$, and $E[X|Y] = \begin{cases} E[X|Y=1] = 2 \\ E[X|Y=2] = 5 + E[X] \\ E[X|Y=3] = 7 + E[X] \end{cases}$

Hence, $E[X] = E[E[X|Y]] = 2 \cdot \frac{1}{3} + (5 + E[X]) \cdot \frac{1}{3} + (7 + E[X]) \cdot \frac{1}{3} = \frac{14}{3} + \frac{2}{3} E[X]$

$\therefore E[X] = 14$