Chapter 6: Jointly Distributed Random Variables

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Spring, 2020

Outline

- Joint Distribution Functions
- Independent RVs
- Sum of Independent RVs
- Conditional Distribution: Discrete Case
- Conditional Distribution: Continuous Case

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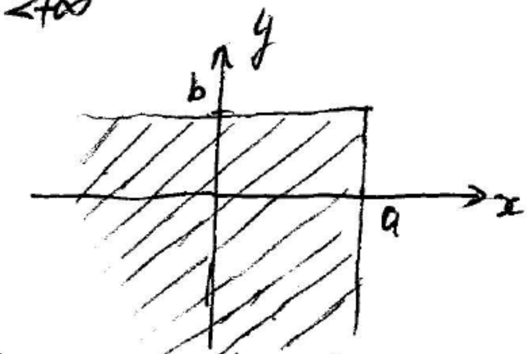
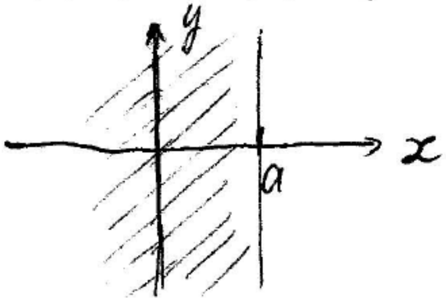
Joint CDF

Joint cumulative distribution function (JCDF) of X and Y,

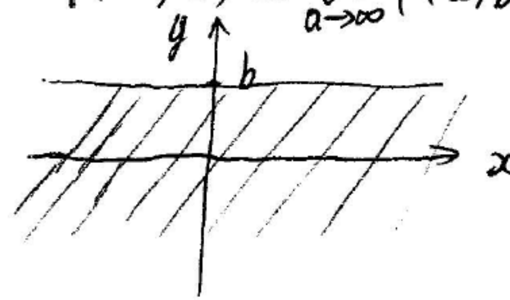
$$F(a,b) = P\{X \le a, Y \le b\}, \quad -\infty < a, b < +\infty$$



$$\left.\begin{aligned} F_X(a) &= P\{X \le a\} = P\{X \le a, -\infty < Y < +\infty\} = F(a, \infty) = \lim_{b\to\infty} F(a,b) \\ F_Y(b) &= P\{Y \le b\} = P\{X < \infty, Y \le b\} = F(\infty, b) = \lim_{a\to\infty} F(a,b) \end{aligned}\right\}$$ Marginal Distribution



$F_X(a) = F(a, \infty) = \lim_{b\to\infty} F(a,b)$

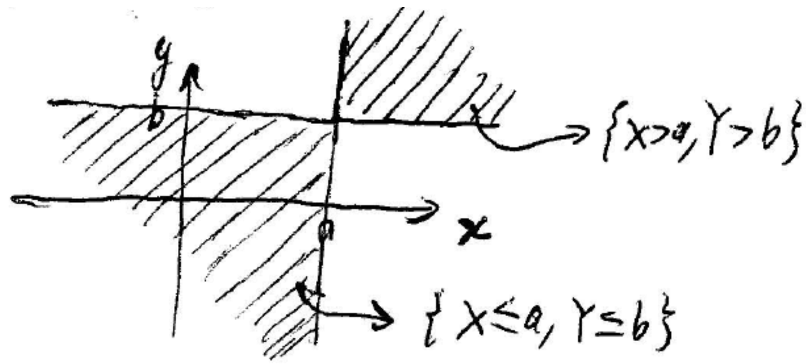


$F_Y(b) = F(\infty, b) = \lim_{a\to\infty} F(a,b)$

Example

For example, $P\{X>a, Y>b\}$

$$= 1 - F_X(a) - F_Y(b) + F(a,b)$$



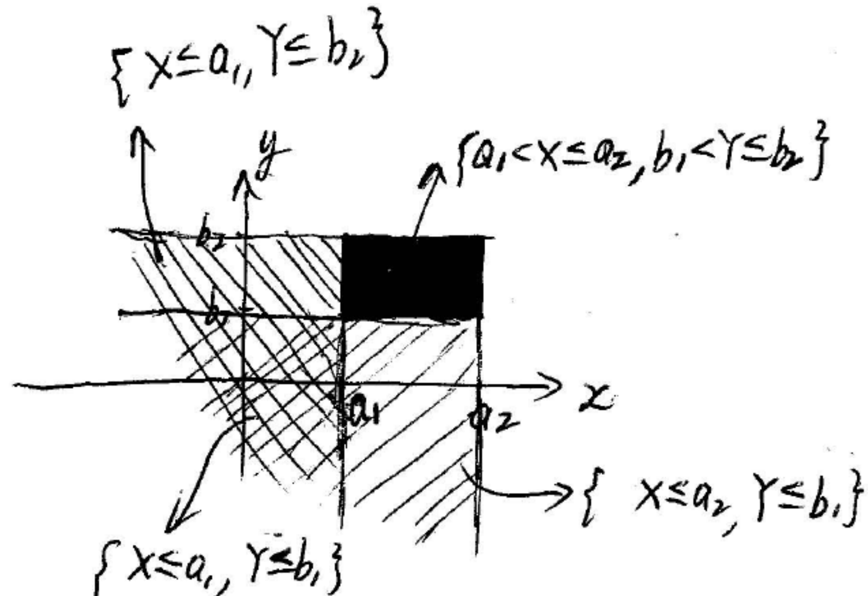
This can also be seen from: $P\{X>a, Y>b\} = 1 - P\{X>a, Y>b\}^c$

$$= 1 - P(\{X>a\}^c \cup \{Y>b\}^c)$$

$$= 1 - P(\{X\le a\} \cup \{Y\le b\})$$

$$= 1 - (P\{X\le a\} + P\{Y\le b\} - P\{X\le a, Y\le b\})$$

$$= 1 - F_X(a) - F_Y(b) + F(a,b)$$



$P\{a_1 < X \le a_2,\ b_1 < Y \le b_2\}$

$$= F(a_2,b_2) - F(a_2,b_1) - F(a_1,b_2) + F(a_1,b_1)$$

Joint PMF

Discrete: Joint probability mass function (JPMF)

$$p(x, y) = P\{X = x, Y = y\}$$

$$p_X(x) = P\{X = x\} = \sum_{y: p(x,y)>0} p(x, y)$$

$$p_Y(y) = P\{Y = y\} = \sum_{x: p(x,y)>0} p(x, y)$$

Example

Example: Suppose 3 balls are randomly selected from 3 red, 4 white and 5 blue balls. Let X and Y denote the # of red and white balls chosen, respectively.

Then, $p(x, y)$ can be computed as following:

$$p(0,0) = \binom{5}{3} \Big/ \binom{12}{3} = \frac{10}{220}.$$

$$p(0,1) = \binom{5}{2} \cdot \binom{4}{1} \Big/ \binom{12}{3} = \frac{40}{220}.$$

$$p(0,2) = \binom{5}{1} \cdot \binom{4}{2} \Big/ \binom{12}{3} = \frac{30}{220}$$

$$p(0,3) = \frac{4}{220}$$

$$p(1,0) = \frac{30}{220},$$

$$p(1,1) = \frac{60}{220}$$

$$p(1,2) = \frac{18}{220},$$

$$p(2,0) = \frac{15}{220}$$

$$p(2,1) = \frac{12}{220}$$

$$p(3,0) = \frac{1}{220}$$

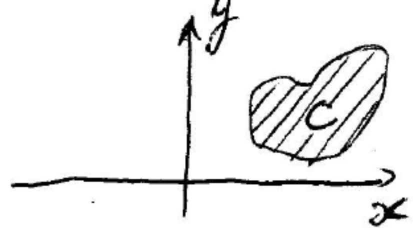
Then, the JPMF of X and Y is $P\{X=i, Y=j\}$

i \ j	0	1	2	3	$P\{X=i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
$P\{Y=j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Joint PDF

Continuous: X and Y are jointly continuous if there exists a function $f(x, y)$, such that for every set $C \subset R^2$:

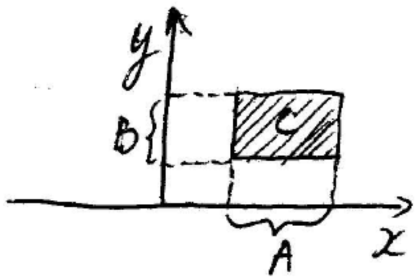
$$P\{(X, Y) \in C\} = \iint_{(x,y)\in C} f(x, y)\, dx\, dy$$



The function $f(x, y)$ is called the JPDF of X and Y.

In the case where $C = A \times B$.

$$P\{(X, Y) \in A\times B\} = P\{X\in A, Y\in B\}$$
$$= \iint_{\substack{x\in A \\ y\in B}} f(x, y)\, dx\, dy$$
$$= \int_{y\in B} \int_{x\in A} f(x, y)\, dx\, dy$$



$C = A \times B$

Example

For example, $A = (-\infty, a]$, $B = (-\infty, b]$.

$$F(a,b) = P\{X \le a, Y \le b\} = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy$$

$$f(a,b) = \frac{\partial^2}{\partial a \partial b} F(a,b)$$

For $B = (-\infty, +\infty)$, $P\{X \in A\} = \int_{-\infty}^{+\infty}\int_A f(x,y)\,dx\,dy = \int_A \int_{-\infty}^{+\infty} f(x,y)\,dy\,dx$,

Compared with $P\{X \in A\} = \int_A f_X(x)\,dx$, it follows that $\underline{f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy}$

Similarly, $\underline{f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx}$

$\rightarrow$ Marginal PDF

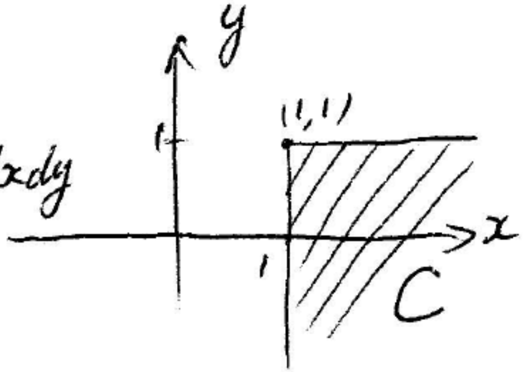
Example

Example: The JPDF of X and Y is $f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$

Compute (a). $P\{X>1,\ Y<1\}$

(b). $P\{X<Y\}$

Solution:



(a) $P\{X>1,\ Y<1\} = \iint_{(x,y)\in C} f(x,y)\,dx\,dy = \int_{-\infty}^{1}\int_{1}^{\infty} 2e^{-x}e^{-2y}\,dx\,dy$

$= \int_0^1\int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy = \underbrace{\int_0^1 2e^{-2y}\,dy}_{(1-e^{-2})} \cdot \underbrace{\int_1^\infty e^{-x}\,dx}_{e^{-1}}$

$= e^{-1}(1-e^{-2})$

(b). $P\{X<Y\} = \iint_{(x,y)\in C} f(x,y)\,dx\,dy$

$= \int_{-\infty}^{+\infty}\int_{x}^{\infty} f(x,y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f(x,y)\,dx\,dy$

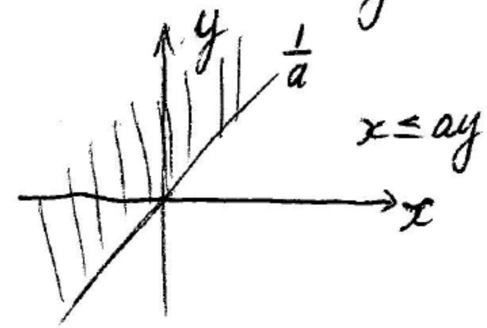
$= \int_0^{+\infty}\int_x^{+\infty} 2e^{-x}e^{-2y}\,dy\,dx = \int_0^{+\infty} e^{-x}\int_x^{+\infty} 2e^{-2y}\,dy\,dx = \frac{1}{3}$

Example

Example: The joint density of X and Y is $f(x,y)=\begin{cases} e^{-(x+y)} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$

Find the density function of the r.v. X/Y.

Solution: Since $f_{X/Y}(a)=\frac{d}{da}F_{X/Y}(a)$, and $F_{X/Y}(a)=P\{X/Y\le a\}$, we start by computing $P\{X/Y\le a\}$. for $a>0$.



$$P\{X/Y\le a\}$$
$$=\iint_{x/y\le a} f(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{ay} f(x,y)\,dx\,dy$$
$$=\int_0^{\infty}\int_0^{ay} e^{-(x+y)}\,dx\,dy = 1-\frac{1}{a+1}$$

Differentiation yields that $f_{X/Y}(a)=\frac{1}{(a+1)^2},\ 0<a<\infty$

Generalization

In general, Joint probability distributions for n random variables

$$F(a_1, a_2, \cdots, a_n) = P\{X_1 \le a_1, X_2 \le a_2, \cdots, X_n \le a_n\}$$

Continuous: if there exists $f(x_1, x_2, \cdots, x_n)$.

such that $P\{(X_1, X_2, \cdots, X_n) \in C\}$, $C \subset R^n$

$$= \iint\cdots\int_{(x_1, x_2, \cdots x_n) \in C} f(x_1, x_2, \cdots x_n)\, dx_1\, dx_2 \cdots dx_n$$

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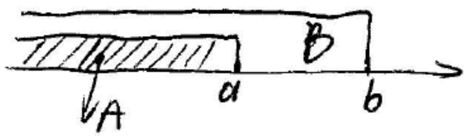
Independent RVs

The r.v.'s X and Y are said to be independent if for any two sets of real numbers A and B, $P\{\underbrace{X\in A}_{E_1}, \underbrace{Y\in B}_{E_2}\} = P\{\underbrace{X\in A}_{E_1}\}\cdot P\{\underbrace{X\in B}_{E_2}\}$.

$$P(E_1E_2) = P(E_1)P(E_2)$$

Based on the three axioms, we only need to consider $A=(-\infty, a]$, $B=(-\infty, b]$

That is, $P\{X\le a, Y\le b\} = P\{X\le a\}\cdot P\{Y\le b\}$, for any $a, b\in R$.



That's equivalent to say $F(a,b) = F_X(a)\cdot F_Y(b)$

For Discrete case, it is equivalent that $p(x,y) = p_X(x)\,p_Y(y)$

For continuous case, it is equivalent that $f(x,y) = f_X(x)\,f_Y(y)$

Interpretation: X and Y are independent if knowing the value of one doesn't change the distribution of the other one.

Example

Example: $n+m$ independent trials with a common success probability p. Let X be the # of successes in the first n trials, Let Y be the # of successes in the last m trials, and Z be the # of successes in the total $n+m$ trials.

Q: Are X and Y independent? Yes

Are X and Z independent? No, since the value of X affects the distribution of Z.

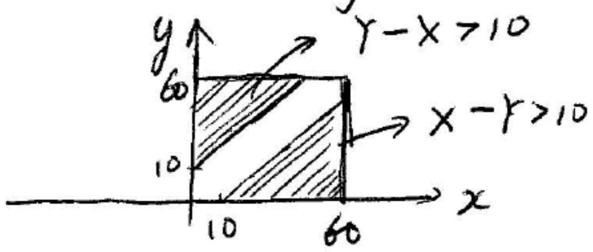
For example, Z can take value 0 with some probability, but if $X > 0$, $P(Z=0) = 0$.

Example: (Ron deezvous)

A man and a woman decide to meet at a certain location. If each person indep-endently arrives at a time distributed between 12 noon and 1 p.m. Find the probability that the first to arrive has to wait longer than 10 minutes.

Example

Solution: Let X and Y denote respectively the time past 12 that the man and the woman arrives, then we have known that X and Y are independent r.v, each of which is uniformly distributed on (0, 60).



$$P\{|X-Y|>10\}$$

$$= \iint_{|x-y|>10} f(x,y)\,dx\,dy$$

$$= 2\iint_{x-y>10} f(x,y)\,dx\,dy$$

$$= 2\iint_{x-y>10} f_X(x) f_Y(y)\,dx\,dy$$

$$= 2\iint_{x-y>10} \left(\frac{1}{60}\right)^2 dx\,dy$$

$$= \frac{2}{(60)^2}\int_0^{50}\int_{y+10}^{60} dx\,dy$$

$$= \frac{2}{(60)^2}\int_0^{50} (50-y)\,dy$$

$$= \frac{25}{36}$$

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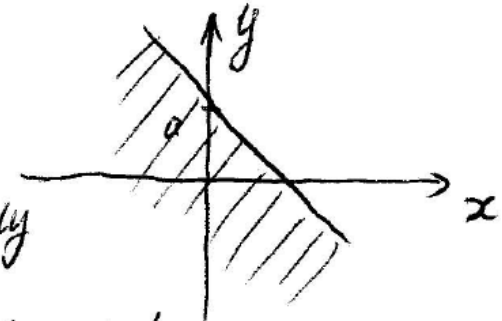
Sum of Independent RVs

Suppose $\left.\begin{array}{l} X \sim f_X(x) \\ Y \sim f_Y(y) \end{array}\right\}$ independent, what's the density of $X+Y$, i.e. f_{X+Y} ?



Solution: $F_{X+Y}(a) = P\{X+Y \le a\} = \iint_{x+y \le a} f(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \underbrace{\int_{-\infty}^{a-y} f_X(x)\,dx}_{P\{X \le a-y\} = F_X(a-y)}\,dy = \int_{-\infty}^{\infty} f_Y(y) \cdot F_X(a-y) \cdot dy$$

$\therefore f_{X+Y}(a) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(a-y)\,dy = f_Y * f_X \sim$ convolution

$$= \int_{-\infty}^{\infty} f_X(x) f_Y(a-x)\,dx$$

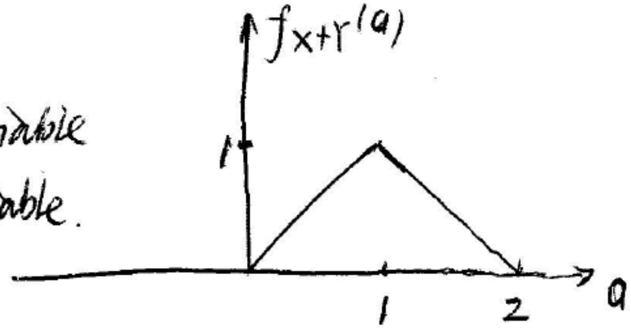
Example

Example: If X and Y are independent r.v.'s. both uniformly distributed on (0, 1). Calculate the probability density of X + Y.

Solution: Since $f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$, $f_Y(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(x) f_Y(a-x)\,dx = \int_0^1 f_Y(a-x)\,dx$$

$$\overset{y=a-x}{=\!=\!=} \int_a^{a-1} f_Y(y)\,(-dy) = \int_{a-1}^{a} f_Y(y)\,dy = \begin{cases} 0 & a \le 0 \\ a & 0 < a \le 1 \\ 2-a & 1 < a \le 2 \\ 0 & a > 2 \end{cases}$$



Because of the shape of its density function, the random variable X+Y is said to be a triangular distributed random variable.

Example

Example: (Sums of independent Poisson r.v.'s).

If $\left.\begin{array}{l} X \sim \text{Poisson}(\lambda_1) \\ Y \sim \text{Poisson}(\lambda_2) \end{array}\right>$ independent, what's $X+Y \sim$?

$$P_X(n) = P\{X=n\} = e^{-\lambda_1}\frac{\lambda_1^n}{n!}, \quad n = 0, 1, 2, \cdots$$

$$P_Y(n) = P\{Y=n\} = e^{-\lambda_2}\cdot\frac{\lambda_2^n}{n!}, \quad n = 0, 1, 2, \cdots$$

$$P_{X+Y}(n) = P\{X+Y=n\} = \sum_{k=0}^{n} P\{X=k, Y=n-k\} = \sum_{k=0}^{n} P\{X=k\}\cdot P\{Y=n-k\}$$

$$= \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}\cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} = e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k}{k!}\cdot\frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\underbrace{\frac{n!}{k!(n-k)!}}_{\binom{n}{k}}\lambda_1^k\lambda_2^{n-k} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k\lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)}\cdot\frac{(\lambda_1+\lambda_2)^n}{n!}$$

$$\sim \text{Poisson}(\lambda_1+\lambda_2)$$

Example

Example: Sum of independent binomial r.v's.

Let X and Y be independent r.v's with respective parameters (n, p) and (m, p). Calculate the distribution of X+Y.

Solution: Without any computation at all, we can immediately conclude, by recalling the interpretation of a binomial r.v., that X+Y is binomial with parameters (n+m, p).

This follows because X represents the # of successes in n independent trials, each of which results in a success with probability p; similarly, Y represents the # of successes in m independent trials, each trial being a success with probability p.

Hence, as X and Y are assumed independent, it follows that X+Y represent the # of successes in n+m independent trials, and $X+Y \sim$ binomial $(n+m, p)$.

Sum of Independent Normal RVs

Proposition 3.2.

If $X_i \sim N(\mu_i, \sigma_i^2)$, $i=1,2,\cdots,n$ are independent, Then $\sum_{i=1}^n X_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$.

Proof:

Start with a simple case, Let $\left.\begin{array}{l} X \sim N(0,\sigma^2) \\ Y \sim N(0,1) \end{array}\right\}$ independent, what's f_{X+Y}?

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\,dy$$

$$f_X(a-y) f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(a-y)^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$$

$$= \frac{1}{2\pi\sigma} \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left[-\left(\underbrace{\frac{-2ay+y^2}{2\sigma^2} + \frac{y^2}{2}}\right)\right]$$

$$\frac{-2ay + (1+\sigma^2) y^2}{2\sigma^2} = \frac{1+\sigma^2}{2\sigma^2}\left[y^2 - \frac{2ay}{1+\sigma^2} + \left(\frac{a}{1+\sigma^2}\right)^2 - \left(\frac{a}{1+\sigma^2}\right)^2\right]$$

$$= \frac{1+\sigma^2}{2\sigma^2}\left[\left(y - \frac{a}{1+\sigma^2}\right)^2 - \left(\frac{a}{1+\sigma^2}\right)^2\right]$$

Sum of Independent Normal RVs

$$\because f_X(a-y)f_Y(y) = \frac{1}{2\pi\sigma}\exp\left[-\frac{a^2}{2\sigma^2}+\frac{1+\sigma^2}{2\sigma^2}\cdot\left(\frac{a}{1+\sigma^2}\right)^2\right]\exp\left[-\frac{1+\sigma^2}{2\sigma^2}\left(y-\frac{a}{1+\sigma^2}\right)^2\right]$$

$$= \frac{1}{2\pi\sigma}\exp\left[\frac{-a^2(1+\sigma^2)+a^2}{2\sigma^2(1+\sigma^2)}\right]\exp\left[-\frac{\left(y-\frac{a}{1+\sigma^2}\right)^2}{\frac{2\sigma^2}{1+\sigma^2}}\right]$$

$$= \frac{1}{2\pi\sigma}\exp\left[-\frac{a^2}{2(1+\sigma^2)}\right]\exp\left[-\frac{\left(y-\frac{a}{1+\sigma^2}\right)^2}{\frac{2\sigma^2}{1+\sigma^2}}\right]$$

$$\therefore f_{X+Y}(a) = \frac{1}{2\pi\sigma}\exp\left[-\frac{a^2}{2(1+\sigma^2)}\right]\int_{-\infty}^{\infty}\exp\left[-\frac{\left(y-\frac{a}{1+\sigma^2}\right)^2}{\frac{2\sigma^2}{1+\sigma^2}}\right]dy$$

$$= \frac{1}{2\pi\sigma}\exp\left[-\frac{a^2}{2(1+\sigma^2)}\right]\cdot\sqrt{2\pi}\sqrt{\frac{\sigma^2}{1+\sigma^2}}\cdot\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma^2}{1+\sigma^2}}}\exp\left[-\frac{\left(y-\frac{a}{1+\sigma^2}\right)^2}{\frac{2\sigma^2}{1+\sigma^2}}\right]dy$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy = 1$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1+\sigma^2}}\exp\left[-\frac{a^2}{2(1+\sigma^2)}\right] \sim N(0,\ 1+\sigma^2)$$

Sum of Independent Normal RVs

$$\therefore f_{X+Y}(a) = \frac{1}{2\pi\sigma} \exp\left[-\frac{a^2}{2(1+\sigma^2)}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\left(y-\frac{a}{1+\sigma^2}\right)^2}{\frac{2\sigma^2}{1+\sigma^2}}\right] dy$$

$$= \frac{1}{2\pi\sigma} \exp\left[-\frac{a^2}{2(1+\sigma^2)}\right] \cdot \sqrt{2\pi}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma^2}{1+\sigma^2}}} \exp\left[-\frac{\left(y-\frac{a}{1+\sigma^2}\right)^2}{\frac{2\sigma^2}{1+\sigma^2}}\right] dy$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = 1$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1+\sigma^2}} \exp\left[-\frac{a^2}{2(1+\sigma^2)}\right] \sim N(0, 1+\sigma^2)$$

Now, suppose $\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\}$ independent.

$$X_1 + X_2 = \sigma_2\Big(\underbrace{\frac{X_1-\mu_1}{\sigma_2}}_{\sim N\left(0, \left(\frac{\sigma_1}{\sigma_2}\right)^2\right)} + \underbrace{\frac{X_2-\mu_2}{\sigma_2}}_{\sim N(0,1)}\Big) + \mu_1 + \mu_2,$$

$$\left(\begin{array}{l} \text{since } E\left[\frac{X_1-\mu_1}{\sigma_2}\right] = \frac{1}{\sigma_2}(E[X_1]-\mu_1) = 0 \\ Var\left(\frac{X_1-\mu_1}{\sigma_2}\right) = \frac{1}{\sigma_2^2} Var(X_1) = \left(\frac{\sigma_1}{\sigma_2}\right)^2 \end{array}\right)$$

$$\therefore \frac{X_1-\mu_1}{\sigma_2} + \frac{X_2-\mu_2}{\sigma_2} \sim N\left(0, 1+\left(\frac{\sigma_1}{\sigma_2}\right)^2\right)$$

$\therefore X_1 + X_2 = \sigma_2\left(\frac{X_1-\mu_1}{\sigma_2} + \frac{X_2-\mu_2}{\sigma_2}\right) + \mu_1 + \mu_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$, using the formula $E[aX+b] = aE[X]+b$, $Var(aX+b) = a^2 Var(X)$ again.

Sum of Independent Normal RVs

Thus, Proposition 3.2 is established when $n=2$, The general case now follows by induction. That is, assume that it is true when there are $n-1$ random variables.

Now, consider the case of n, and write

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n-1} X_i + X_n$$

By the induction hypothesis, $\sum_{i=1}^{n-1} X_i$ is normal with mean $\sum_{i=1}^{n-1} \mu_i$ and Variance $\sum_{i=1}^{n-1} \sigma_i^2$. Therefore, by the result for $n=2$, we can conclude that $\sum_{i=1}^{n} X_i$ is normal with mean $\sum_{i=1}^{n} \mu_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$.

Outline

- Joint Distribution Functions
- Independent RVs
- Sum of Independent RVs
- Conditional Distribution: Discrete Case
- Conditional Distribution: Continuous Case

Conditional Distribution: Discrete Case

Recall conditional probability $P(E|F) = \frac{P(EF)}{P(F)}$

If X and Y are discrete r.v.'s, conditional PMF of X given $Y=y$

$$P_{X|Y}(x|y) = P\{X=x \mid Y=y\} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{P(x,y)}{P_Y(y)}$$

We know that $P(E|F)=P(E) \iff E$ and F are independent

Similarly, $P_{X|Y}(x|y) = P_X(x) \iff X$ and Y are independent, which can be easily seen from that if X and Y are independent, then $P_{X|Y}(x|y) = \frac{P(x,y)}{P_Y(y)} = \frac{P_X(x)P_Y(y)}{P_Y(y)} = P_X(x)$

The Conditional CDF of X given that $Y=y$ is defined as

$$F_{X|Y}(x|y) = P\{X \le x \mid Y=y\} = \sum_{a \le x} P\{X=a \mid Y=y\} = \sum_{a \le x} P_{X|Y}(a|y)$$

Example

Example: Suppose the JPMF of X and Y is $p(0,0)=0.4$, $p(0,1)=0.2$, $p(1,0)=0.1$, $p(1,1)=0.3$

Calculate the conditional PMF of X, given that $Y=1$.

Solution: First, $p_Y(1)=\sum_x p(x,1)=p(0,1)+p(1,1)=0.5$

Hence, $$p_{X|Y}(x|1)=\frac{p(x,1)}{p_Y(1)}=\begin{cases}\frac{p(0,1)}{p_Y(1)}=\frac{0.2}{0.5}=0.4 & \text{for } x=0\\ \frac{p(1,1)}{p_Y(1)}=\frac{0.3}{0.5}=0.6 & \text{for } x=1\end{cases}$$

Besides, $p_X(0)=\sum_y p(0,y)=p(0,0)+p(0,1)=0.6$

$p_X(1)=\sum_y p(1,y)=p(1,0)+p(1,1)=0.4$

Since $p_X(x)\neq p_{X|Y}(x|1)$ for $x=0,1$, X and Y are not independent.

Example

Example: Suppose the JPMF of X and Y is $p(0,0)=0.4$, $p(0,1)=0.2$, $p(1,0)=0.1$, $p(1,1)=0.3$

Calculate the conditional PMF of X, given that $Y=1$.

Alternatively, we can also explore whether X and Y are independent by checking whether the equation $p(x,y)=p_X(x)p_Y(y)$ holds.

$$p_Y(0)=\sum_x p(x,0)=p(0,0)+p(1,0)=0.5.$$

Obviously, $p(x,y)\neq p_X(x)p_Y(y)$, therefore, X and Y are not independent.

Note: To make sure X and Y are independent, $p(x,y)=p_X(x)p_Y(y)$ should hold for all cases of (x,y), i.e. $(0,0),(0,1),(1,0),(1,1)$.

Outline

- Joint Distribution Functions
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Conditional Distribution: Continuous Case

If X and Y have a joint PDF $f(x,y)$, then the conditional PDF of X, given that $Y=y$ is defined as

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}$$, for all values of y, such that $f_Y(y)>0$

Similar to $P(X\in A)=\int_A f_X(x)\,dx$

we have $P(X\in A \mid Y=y)=\int_A f_{X|Y}(x|y)\,dx$

In particular, let $A=(-\infty, a]$, then we have the conditional CDF of X,

$$F_{X|Y}(a|y)=P\{X\le a \mid Y=y\}=\int_{-\infty}^{a} f_{X|Y}(x|y)\,dx$$

If X and Y are independent, $f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{f_X(x)f_Y(y)}{f_Y(y)}=f_X(x)$

Example

Example: Suppose that the joint density of X and Y is

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y} e^{-y}}{y} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X>1 \mid Y=y\}$.

Solution: $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} \dfrac{e^{-x/y} e^{-y}}{y}\,dx$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-x/y} e^{-y}/y}{\int_0^{\infty} \frac{e^{-x/y} e^{-y}}{y}\,dx} = \frac{1}{y} e^{-x/y} \qquad 0<x<\infty,\ 0<y<\infty,$$

$$\therefore P\{X>1 \mid Y=y\} = \int_1^{\infty} f_{X|Y}(x|y)\,dx = \int_1^{\infty} \frac{1}{y} e^{-x/y}\,dx = e^{-\frac{1}{y}}$$