Chapter 5: Continuous Random Variables

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Outline

- Continuous Random Variables
- Expectation and Variance for Continuous RV
- Uniform RV
- Normal (Gaussian) RV
- Exponential RV
- Gamma RV

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Continuous Random Variables

X is called a continuous RV if there is a nonnegative function $f(x)$ such that for any set $B \subseteq \mathbb{R}$,

$$P(X \in B) = \int_B f(x)dx$$

where $f(x)$ is called the probability density function (PDF) of X.

For example, suppose X has PDF $f(x)$. Then we have

$$P(X \in [a,b]) = \int_a^b f(x)dx$$

$$P(X \in \mathbb{R}) = \int_{-\infty}^{\infty} f(x)dx = 1$$

$$P(X = a) = \int_a^a f(x)dx = 0$$

Example

Problem: Suppose that X is a continuous RV with PDF

$$f(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find a) the value of c; and b) $P(X > 1)$.

Solution: a) Since $f(x)$ is a PDF, we have

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^2 c(4x - 2x^2)dx = c \times (2x^2 - \frac{2}{3}x^3)\Big|_0^2 = c \times \frac{8}{3}$$

and therefore $c = \frac{3}{8}$.

b) Plugging the value of c, we have

$$P(X > 1) = \int_1^{\infty} f(x)dx = \int_1^2 \frac{3}{8}(4x - 2x^2)dx = \frac{1}{2}$$

Cumulative Distribution Function

The cumulative distribution function, or CDF, for a continuous RV X is defined as

$$F(a) = P(X \leq a) = P(X < a) = \int_{-\infty}^{a} f(x)dx$$

The CDF and PDF are related by

$$f(a) = F'(a) = \frac{d}{da}F(a)$$

Example: If X is a continuous RV with distribution function F_X and density function f_X. Find the density function of $Y = 2X$.

Solution: By the definition of CDF, we have

$$F_Y(a) = P(Y \leq a) = P(2X \leq a) = P\left(X \leq \frac{a}{2}\right) = F_X\left(\frac{a}{2}\right)$$

Therefore, the density of Y is given by

$$f_Y(a) = \frac{d}{da}F_Y(a) = \frac{d}{da}F_X\left(\frac{a}{2}\right) = \frac{1}{2}f_X\left(\frac{a}{2}\right)$$

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Expectation

Expectation for a continuous RV X:

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Example: For X with PDF given by

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

the expectation $\mathrm{E}[X]$ is

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{1} x \times 2x dx = \frac{2}{3}$$

Expectation

Expectation for function of a continuous RV X:

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$$

and $\mathrm{E}[aX + b] = a\mathrm{E}[X] + b$.

Example: For X with PDF given by

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

the expectation $\mathrm{E}[e^X]$ is

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} e^x f(x) dx = \int_0^1 e^x dx = e - 1$$

Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X-\mu)^2] = \mathrm{E}[X^2] - \mu^2, \quad \mu = \mathrm{E}[X]$$

and

$$\mathrm{Var}(aX+b) = a^2 \mathrm{Var}(X)$$

Example: For X with PDF given by

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

the second moment $\mathrm{E}[X^2]$ is

$$\mathrm{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^1 x^2 \times 2x dx = \frac{1}{2}$$

and $\mathrm{Var}(X) = \mathrm{E}[X^2] - \mu^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

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Uniform RV

A RV $X \sim \text{Unif}(\alpha, \beta)$ is said to be uniformly distributed over (α, β) if its PDF is

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

Its CDF is given by

$$F(a) = \int_{-\infty}^{a} f(x)dx = \begin{cases} 0 & a \leq \alpha \\ \frac{a-\alpha}{\beta-\alpha} & \alpha \leq a \leq \beta \\ 1 & a \geq \beta \end{cases}$$

The expectation $\text{E}[X]$ is

$$\text{E}[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_{\alpha}^{\beta} x \times \frac{1}{\beta - \alpha} dx = \frac{\beta + \alpha}{2}$$

The second moment $\text{E}[X^2]$ is

$$\text{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{\alpha}^{\beta} x^2 \times \frac{1}{\beta - \alpha} dx = \frac{\beta^3 - \alpha^3}{3(\beta - \alpha)} = \frac{\beta^2 + \alpha^2 + \beta\alpha}{3}$$

So we have $\text{Var}(X) = \text{E}[X^2] - \mu^2 = \frac{(\beta-\alpha)^2}{12}$

Uniform RV

Example: Bus arrives at a stop at 7, 7:15, 7:30, 7:45 and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30.
Find the probability that he waits

(a). less than 5 minutes for a bus.

(b). more than 10 minutes for a bus.

Solution:

(a). P{wait for a bus less than 5 minutes}

= P{arrive on (7:10, 7:15) or (7:25, 7:30)}

= P{arrive on (7:10, 7:15)} + P{arrive on (7:25, 7:30)}

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

(b). P{wait for more than 10 minutes}

= P{arrive on (7:00, 7:05) or (7:15, 7:20)} $= \frac{1}{3}$

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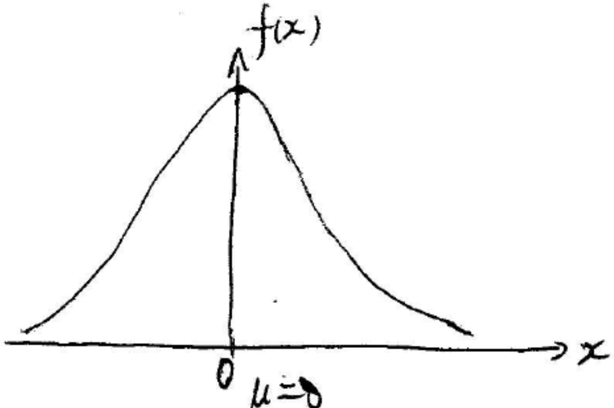
Normal RV: PDF

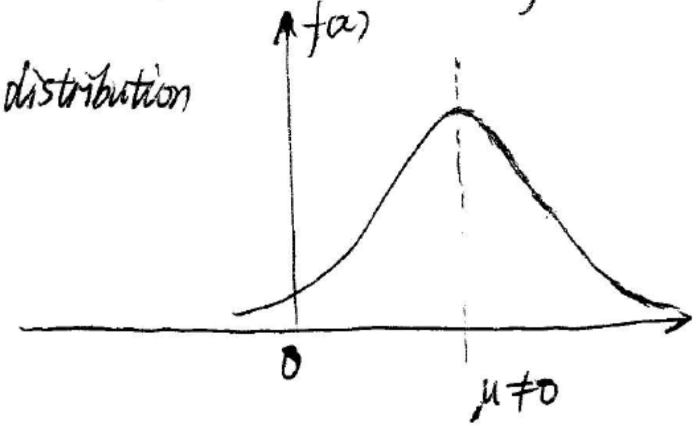
X is a normal (Gaussian) r.v. if its PDF is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$$

where μ and σ^2 are the parameters and we can show in fact μ is the expectation and σ^2 is the variance of the normal r.v.



Bell-shaped distribution



Usually, we use $X \sim N(\mu, \sigma^2)$ to denote that X is a normal r.v. with parameters μ and σ^2.

Normal RV: PDF

Now, To show that $f(x)$ is indeed a PDF, we need to show that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

By making the substitution $y = (x-\mu)/\sigma$, we see that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$, denoted as I.

Then, to show $I=1$ is equivalent to show $I^2 = 1$.

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dy\, dx$$

change to polar coordinates

$$= \frac{1}{2\pi} \cdot \int_{0}^{\infty}\int_{0}^{2\pi} e^{-\frac{r^2}{2}} \cdot r \cdot d\theta\, dr$$

$$= \int_{0}^{+\infty} e^{-\frac{r^2}{2}} r\, dr = -e^{-\frac{r^2}{2}}\Big|_{0}^{\infty} = 1$$

Normal RV: Expectation and Variance

In order to verify that $E[X]=\mu$ and $Var(X)=\sigma^2$, we need to calculate

$$E[X]=\int_{-\infty}^{\infty} x\cdot f(x)\,dx, \text{ and } Var(X)=E[(X-\mu)^2]=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx.$$

Let's first consider the special case when $\mu=0, \sigma^2=1$, which is called standard normal distribution. Its PDF is given by

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},$$ and we can denote it as $Z\sim N(0,1)$.

Now, $E[Z]=\int_{-\infty}^{\infty} x\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}d\left(\frac{x^2}{2}\right)=\frac{1}{\sqrt{2\pi}}\left(-e^{-\frac{x^2}{2}}\right)\Big|_{-\infty}^{\infty}=0\ (=\mu)$

$$Var(Z)=E[(Z-\mu)^2]=E[Z^2]=\int_{-\infty}^{\infty}x^2\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x\cdot e^{-\frac{x^2}{2}}d\left(\frac{x^2}{2}\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x\,d\left(-e^{-\frac{x^2}{2}}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left[-xe^{-\frac{x^2}{2}}\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{\infty}\left(-e^{-\frac{x^2}{2}}\right)dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx=1\ (=\sigma^2)$$

Normal RV: Expectation and Variance

In order to show, in the general case, $E[X]=\mu$, and $Var(X)=\sigma^2$, we construct another r.v. $Y=\sigma Z+\mu$, where $Z\sim N(0,1)$.

Then, $Y=\sigma Z+\mu$ is normally distributed with parameters μ and σ^2, i.e. $Y\sim N(\mu,\sigma^2)$

To show this, we know that

$$F_Y(y)=P(Y\le y)=P(\sigma Z+\mu\le y)=P\left(Z\le\frac{y-\mu}{\sigma}\right)=F_Z\left(\frac{y-\mu}{\sigma}\right)$$

$$\therefore\quad f_Y(y)=f_Z\left(\frac{y-\mu}{\sigma}\right)\cdot\frac{1}{\sigma}=\frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\mu)^2}{2\sigma^2}},$$

$$\therefore\quad Y\sim N(\mu,\sigma^2)$$

Besides, it's obvious that $E[Y]=E[\sigma Z+\mu]=\sigma E[Z]+\mu=\mu\quad(E[Z]=0)$

$$Var(Y)=Var(\sigma Z+\mu)=\sigma^2 Var(Z)=\sigma^2\quad(Var(Z)=1)$$

Combining all the above, we have $Y\sim N(\mu,\sigma^2)$ and $E[Y]=\mu$, $Var(Y)=\sigma^2$, which is exactly what we desire to show.

Normal RV: CDF

$X \sim N(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad E[X] = \mu, \; Var(X) = \sigma^2$$

standard $Z \sim N(0, 1)$.

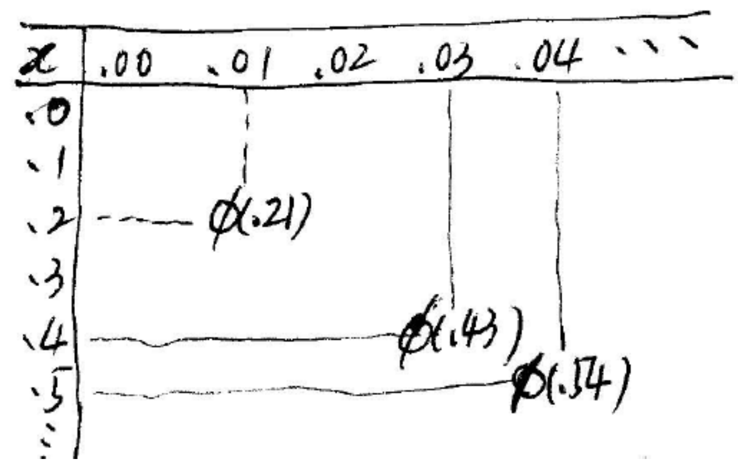
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Distribution Function $F(x) = P\{X \le x\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$

Distribution function for standard Normal r.v.

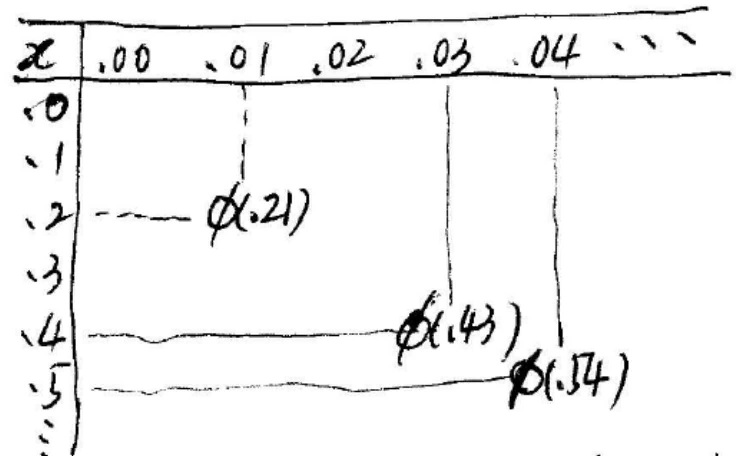
$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

The values of $\Phi(x)$ for nonnegative x are given in the Table 5.1 in textbook. Because of symmetry, the table only provide $\Phi(x)$ for nonnegative x.

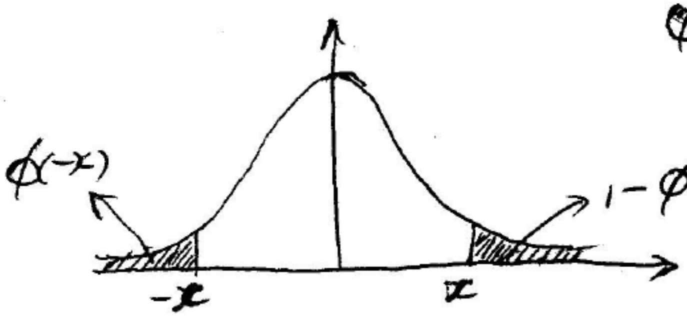


Normal RV: CDF

The values of $\Phi(x)$ for nonnegative x are given in the Table 5.1 in textbook. Because of symmetry the table only provide $\phi(x)$ for nonnegative x.



Obviously, $\phi(-x) = 1 - \phi(x)$, since $\underbrace{P\{X \le -x\}}_{\phi(-x)} = \underbrace{P\{X \ge x\}}_{1-\phi(x)}$ for standard normal distribution



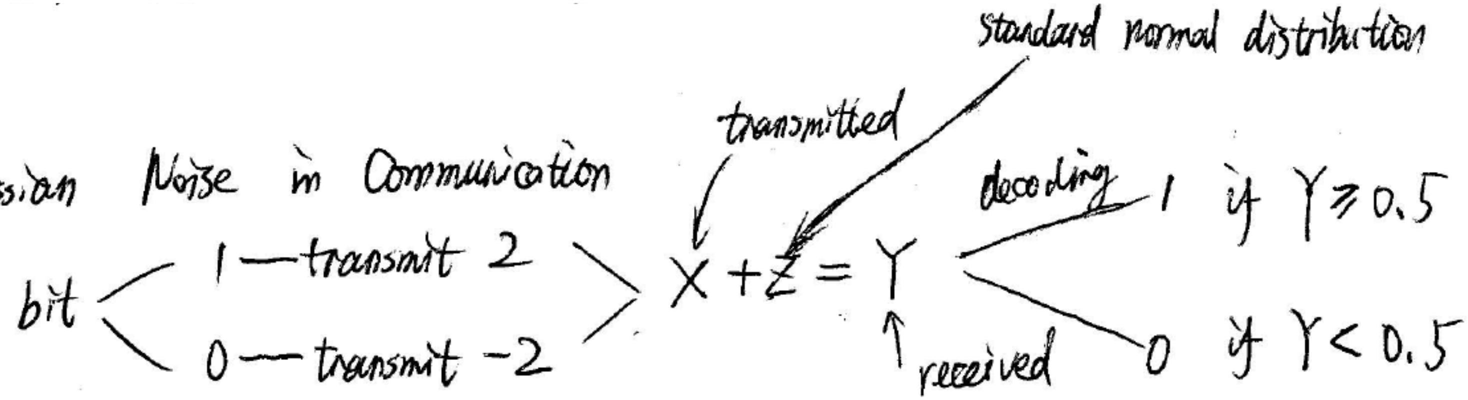
In general case,

$$F(x) = P\{X \le x\} = P\{\sigma Z + \mu \le x\} = P\{Z \le \frac{x-\mu}{\sigma}\} = \phi(\frac{x-\mu}{\sigma})$$

Example

Example:

Gaussian Noise in Communication



$$P\{\text{error} \mid \text{message is } 1\} = P\{Z < -1.5\} = \Phi(-1.5) = 1 - \Phi(1.5) = 0.0668$$

$$P\{\text{error} \mid \text{message is } 0\} = P\{Z > 2.5\} = 1 - P\{Z \leq 2.5\} = 1 - \Phi(2.5) = 0.0062$$

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Exponential RV

Exponential Random Variables

An exponential r.v. with parameter $\lambda > 0$, has the PDF $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

$F(x) = 1 - e^{-\lambda x}$, for $x \geq 0$

$E[X] = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$

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Gamma RV

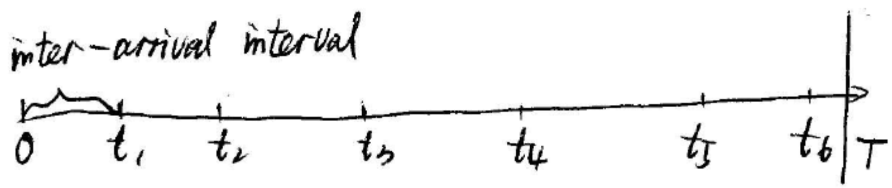
A gamma distribution with parameters (α, λ), $\alpha > 0$ and $\lambda > 0$, has PDF

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$, if $\alpha = n$, $\Gamma(n) = (n-1)!$

$E[X] = \frac{\alpha}{\lambda}$, $Var(X) = \frac{\alpha}{\lambda^2}$

In Queueing Systems,



inter-arrival interval $t_i - t_{i-1} \sim$ exponential (λ)

$t_6 \sim$ Gamma $(6, \lambda)$

# of customers $\sim$ Poisson $(\lambda) \sim e^{-(\lambda T)} \frac{(\lambda T)^n}{n!}$