Chapter 2: Axioms of Probability

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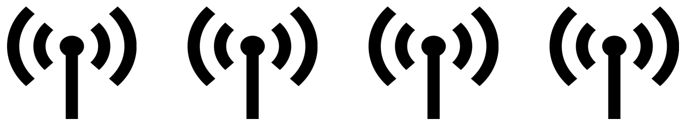
Spring, 2020

Example Re-Revisited

A communication system consists of four antennas. Assume that this system will be functional if no two consecutive antennas are defective.



Question: If there are exactly two antennas defective, what is the probability that the resulting system will be functional?

Thinking process:

1) List all the possiblities: 0110, 0101, 1010, 0011, 1001, 1100
2) If all the cases are equally likely, then the desired probability is $\frac{3}{6} = \frac{1}{2}$.

Outline

- Sample Space and Events
- Axioms of Probability
- Some Simple Propositions
- Sample Space with Equally Likely Outcomes

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Sample Space

S — Sample space: the set of all possible outcomes of an experiment

1) Flip a coin. $S = \{\text{tails}, \text{heads}\}$

2) Flip two coins. $S = \{\text{hh}, \text{tt}, \text{ht}, \text{th}\}$

3) Toss two dices. $S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$

4) The order of finish in a race among 7 horses. $S = \{\text{all } 7! \text{ permutations of } (1,2,3,4,5,6,7)\}$

5) Measuring (in hours) the life time of a transistor. $S = \{x : 0 \leq x < \infty\}$

Event

Event: any subset E of the sample space is known as an event; an event is a set of some possible outcome. If the outcome of the experiment is contained in E, then we say that E has occurred.

1) Flip a coin. $S = \{\text{tails}, \text{heads}\}$.
$E = \{\text{heads}\}, F = \{\text{tails}\}$.

2) Flip two coins. $S = \{\text{hh}, \text{tt}, \text{ht}, \text{th}\}$.
$E = \{\text{hh}, \text{ht}\}$, $F = \{\text{th}, \text{ht}\}$.

3) Toss two dices. $S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$.
$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

4) The order of finish in a race among 7 horses. $S = \{\text{all } 7! \text{ permutations of } (1,2,3,4,5,6,7)\}$.
$E = \{\text{all permutations starting with } 3\}$.

5) Measuring (in hours) the life time of a transistor. $S = \{x : 0 \leq x < \infty\}$.
$E = \{x : 0 \leq x \leq 5\}$

Set Operations

1) Union: $E \cup F$; $\cup_{i=1}^{n} E_i$; $\cup_{i=1}^{\infty} E_i$

2) Intersection: $E \cap F$ or EF; $\cap_{i=1}^{n} E_i$; $\cap_{i=1}^{\infty} E_i$
E and F are said to be mutually exclusive or disjoint if $EF = \emptyset$

3) Complement: $E^c = \{\text{all outcomes not in } E\}$; $S^c = \emptyset$

4) Subset: $E \subseteq F$ iff all elements in E are also in F.
If $E \subseteq F$ and $F \subseteq E$, then $E = F$.

Can you demonstrate these operations by using Venn Diagrams?

Laws of Set Theory

1) Commutative laws: $E \cup F = F \cup E$; $EF = FE$

2) Associative laws: $(E \cup F) \cup G = E \cup (F \cup G)$; $(EF)G = E(FG)$

3) Distributive laws: $(E \cup F)G = (EG) \cup (FG)$; $(EF) \cup G = (E \cup G)(F \cup G)$

4) DeMorgan's law: $\left(\cup_{i=1}^{n} E_i\right)^c = \cap_{i=1}^{n} E_i^c$; $\left(\cap_{i=1}^{n} E_i\right)^c = \cup_{i=1}^{n} E_i^c$

Can you justify these laws by using Venn Diagrams?

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Axioms of Probability

1) Non-negativity: $0 \leq P(E) \leq 1$, for any E

2) Normalization: $P(S) = 1$

3) Additivity: For any sequence of mutually exclusive events $E_1, E_2, \ldots$, i.e. $E_i E_j = \emptyset$ for $i \neq j$,

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

Example: Toss a coin. If heads is as likely as tails, then $P(\{\mathrm{h}\}) = P(\{\mathrm{t}\}) = 0.5$. Or $P(\{\mathrm{h}\}) = 1 - P(\{\mathrm{t}\}) = 0.25$ for a biased coin.

Example: Roll a dice. If all sides are equally likely, then $P(\{i\}) = 1/6$ for any $i = 1, 2, 3, 4, 5, 6$.

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Some Simple Propositions

1) $P(E^c) = 1 - P(E)$

2) If $E \subseteq F$, then $P(E) \leq P(F)$

3) $P(E \cup F) = P(E) + P(F) - P(EF)$

4) $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$

Using Venn Diagrams can help understand these propositions.

Inclusion-Exclusion Identity

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_i P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2})$$

$$+ \cdots + (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r})$$

$$+ \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n)$$

Example

Example: Mike is going to take two course A and B next semester. With probability 0.8 he will like course A; with probability 0.7 he will like B; with probability 0.6 he will like both. What is the probability that he will like neither course?

Solution: Let $A = \{\text{Mike will like course } A\}$ and $B = \{\text{Mike will like course } B\}$. Then we have

$$A \cup B = \{\text{Mike will like course } A \text{ or } B\}$$

which happens with probability

$$P(A \cup B) = P(A) + P(B) - P(AB) = 0.8 + 0.7 - 0.6 = 0.9.$$

Therefore,

$$P(\{\text{Mike will like neither course } A \text{ nor } B\}) = 1 - P(A \cup B) = 1 - 0.9 = 0.1$$

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Sample Space with Equally Likely Outcomes

Suppose $S = \{1, 2, \ldots, N\}$ and $P(\{i\}) = 1/N$ for any $i = 1, 2, \ldots, N$. Then for any E,

$$P(E) = \frac{\text{number of outcomes in } E}{\text{total number of outcomes in } S}$$

Example: If two dice are rolled. What is the probability that the sum of the dice will equal 7?

Solution: The sample space $S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$. The event of interest is

$E = \{\text{the sum of the dice will equal 7}\} = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

Therefore,

$$P(E) = \frac{6}{36} = \frac{1}{6}.$$

Example

Example: Suppose we randomly draw 3 balls from an urn containing 6 white and 5 black balls. What is the probability that of the 3 balls, one is white and two are black?

Solution:

$$\frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}$$

What is the probability that of the 3 balls, one is black and two are white?

Solution:

$$\frac{\binom{6}{2}\binom{5}{1}}{\binom{11}{3}} = \frac{15}{33}$$

Example

Example: There are n people in a room. What is the probability that no two of them celebrate their birthday on the same day of the year?

Solution:

$$p_n = \frac{365 \times 364 \times 363 \times \cdots \times (365 - n + 1)}{365^n}$$

How big should n be such that this probability is less than 1/2?

Solution: Obviously p_n is decreasing with n. When $n \geq 23$, p_n is less than 1/2. Also when $n = 50$, $p_n \approx 3\%$; when $n = 100$, $p_n \approx \frac{1}{3{,}000{,}000}$.