Chapter 1: Combinatorial Analysis

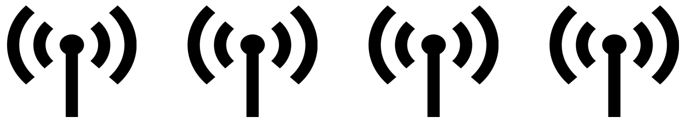
Xiugang Wu

University of Delaware

Spring, 2020

Example Revisited

A communication system consists of four antennas.



How many possibilities are there for exactly two antennas to be defective?
How many possibilities for exactly two consecutive antennas to be defective?

Naive way: List all the possiblities — 0110, 0101, 1010, 0011, 1001, 1100

Chapter 1 is all about how to count effectively and systematically; the mathematical theory of counting is formally known as combinatorial analysis.

Outline

- Basic Principle of Counting
- Permutations
- Combinations
- Multinomial Coefficients

Outline

- Basic Principle of Counting
- Permutations
- Combinations
- Multinomial Coefficients

Basic Principle of Counting

Suppose that two experiments are to be performed. If experiment 1 can result in any one of m possible outcomes, and if for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

$$\begin{array}{cccc}
(1,1) & (1,2) & \cdots & (1,n) \\
(2,1) & (2,2) & \cdots & (2,n) \\
\vdots & \vdots & \ddots & \vdots \\
(m,1) & (m,2) & \cdots & (m,n)
\end{array}$$

Example: A small community has 10 women, each of whom has 3 children. If one woman and one of her children are to be selected as mother and child of the year. How many possibilities?

Solution: $10 \times 3 = 30$

Generalization

Suppose that r experiments are to be performed. If experiment 1 has n_1 possible outcomes; for each of them, there are n_2 possible outcomes of experiment 2; ......; for each of them, there are n_r possible outcomes of experiment r, then there are totally $n_1 n_2 \cdots n_r$ possible outcomes of the r experiments.

Example: How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution: $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$

What if repetition among letters or numbers is prohibited?

Solution: $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$

Outline

- Basic Principle of Counting
- Permutations
- Combinations
- Multinomial Coefficients

Permutations

Example: How many different ordered arrangements of a, b, c?

Solution: $abc; acb; bac; bca; cab; cba$

In the above example, each ordering is known as a permutation. By the principle of counting, there are

$$3 \times 2 \times 1 = 6$$

different permutations. Generally, for n objects, there are

$$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$$

different permutations.

Permutations

Example: Suppose we want to put 10 books on the bookshelf. Of these 10 books, 4 are math, 3 are chemistry, 2 are history, 1 is language. The books of the same subject should be put together. How many different arrangements are possible?

Solution: There are $4! \times 3! \times 2! \times 1! \times 4! = 6912$ different arrangements. (Note that without the restriction that the books of the same subject should be put together, there are $10!$ different arrangements.)

Example: A class consists of 6 men and 4 women. An exam is given and no two students obtain the same score.

(a): How many different rankings are possible? Solution: $10! = 3,628,800$

(b): What if men and women are ranked separately? Solution: $4! \times 6! = 17,280$

Permutations

Example: How many different letter arrangements can be formed using the letters PEPPER?

Solution: $\frac{6!}{3!2!} = 60$.

In general, there are

$$\frac{n!}{n_1!n_2!n_3!\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike. $(n_1 + n_2 + \cdots + n_r = n.)$

Example: How many different signals are possible, by hanging 9 flags in a line, of which 4 are white, 3 red and 2 blue?

Solution: $\frac{9!}{4!3!2!} = 1260$.

Outline

- Basic Principle of Counting
- Permutations
- Combinations
- Multinomial Coefficients

Combinations

Example: How many different groups of 3 can be selected from the 5 items A, B, C, D, E?

Solution: $\frac{5\times4\times3}{3!} = 10$. (Hint: $ABC, ACB, BAC, \ldots$ are the same group.)

In general, if we select r items from n items, then the number of different groups is given by

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \triangleq \binom{n}{r}.$$

Quick facts:

$$\binom{n}{r} = \binom{n}{n-r} = \frac{n!}{(n-r)!r!}$$

For example,

$$\binom{n}{1} = \binom{n}{n-1} = n \quad \text{and} \quad \binom{n}{n} = \binom{n}{0} = 1 = \frac{n!}{n!0!}$$

where $0! = 1$ by convention.

Combinations

Example: From 5 women and 7 men, how many different committees of 2 women and 3 men can be formed?

Solution:

$$\binom{5}{2}\binom{7}{3} = \frac{5\times 4}{2\times 1} \times \frac{7\times 6\times 5}{3\times 2\times 1} = 350.$$

What if 2 of the men refuse to serve together?

Solution: Suppose the 2 men are A and B. Then there are situations:

1) both A and B are in: $\binom{5}{2}\binom{5}{1}$
2) A in, B out: $\binom{5}{2}\binom{5}{2}$
3) A out, B in: $\binom{5}{2}\binom{5}{2}$
4) neither is in: $\binom{5}{2}\binom{5}{3}$

Therefore, the are

$$350 - \binom{5}{2}\binom{5}{1}, \quad \text{or alternatively,} \quad \binom{5}{2}\binom{5}{2}\times 2 + \binom{5}{2}\binom{5}{3}$$

different committees.

Combinations

Example: A communication system with n antennas has m defective. The system will still be functional if no two consecutive antennas are defective. What is the probability that the system is functional?

Solution:

1) The total number of possibilities: $\binom{n}{m}$
2) The number of cases where no two consecutive antennas are defective:

$$\sqcup 1 \sqcup 1 \sqcup 1 \cdots \sqcup 1 \sqcup \Rightarrow \binom{n-m+1}{m}$$

Therefore, the probability of being functional is

$$\frac{\binom{n-m+1}{m}}{\binom{n}{m}},$$

where the probability is 0 if $n - m + 1 < m$.

A Combinatorial Identity

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Proof:

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!} = \frac{(n-1)!r}{r!(n-r)!} + \frac{(n-1)!(n-r)}{r!(n-r)!}$$

$$= \frac{(n-1)!(r+n-r)}{r!(n-r)!} = \frac{(n-1)!n}{r!(n-r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Combinatorial Argument: Suppose we want to choose r items from n items. Consider the following two cases:

1) The 1st item is chosen: $\binom{n-1}{r-1}$

2) The 1st item is not chosen: $\binom{n-1}{r}$

Therefore, it follows that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Example: How many subsets are there of a set consisting of n items $\{1, 2, \ldots, n\}$?

Solution:

$$\sum_{k=1}^{n} \binom{n}{k} = \sum_{k=1}^{n} \binom{n}{k} 1^k 1^{n-k} = 2^n.$$

Alternatively, the answer 2^n can be obtained directly by considering whether an item is in the subset or not, and using the basic principle of counting.

Outline

- Basic Principle of Counting
- Permutations
- Combinations
- Multinomial Coefficients

Multinomial Coefficient

Example: Suppose that n distinct items are to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$. How many different divisions are possible?

Solution: We fill the r groups one by one. There are $\binom{n}{n_1}$ possible choices for the first group, $\binom{n-n_1}{n_2}$ possible choices for the second group, ..., and $\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$ choices for the rth group. By the basic principle of counting, the number of different divisions is

$$
\begin{aligned}
& \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r} \\
= & \frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!n_3!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!n_r!} \\
= & \frac{n!}{n_1!n_2!n_3!\cdots n_r!} \\
\triangleq & \binom{n}{n_1, n_2, \ldots, n_r},
\end{aligned}
$$

where $\binom{n}{n_1,n_2,\ldots,n_r}$ is known as multinomial coefficient.

Question: Can the answer be r^n? Why?

Multinomial Coefficient

Example: Suppose that n distinct items are to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$. How many different divisions are possible?

Alternative Reasoning: Think of the problem as throwing n items into n holes:

$$\underbrace{\sqcup\sqcup\sqcup\cdots\sqcup}_{n_1}\ \underbrace{\sqcup\sqcup\sqcup\cdots\sqcup}_{n_2}\ \ldots\ldots\ \underbrace{\sqcup\sqcup\sqcup\cdots\sqcup}_{n_r}$$

Each permutation corresponds to a way to group them, but if we take $n!$ as the total number of possible ways of grouping, we have over-counted because it doesn't matter how we permute the items within the same group. Therefore, there are $\frac{n!}{n_1!n_2!\cdots n_r!}$ different possible divisions.

Or, we can think of the problem as assigning n labels to n items. Of the n labels, n_1 are alike, n_2 are alike, ..., n_r are alike. If one item receives the kth label, then it is contained by the kth group. There are $\frac{n!}{n_1!n_2!\cdots n_r!}$ possible ways to assign the labels.

Multinomial Coefficient

Example: A class of 68 students are to be divided into 3 sections of sizes $21, 22, 25$. How many different divisions are possible?

Solution: $\frac{68!}{21!22!25!} = \binom{68}{21,22,25}$.

Example: 10 Children are to be divided into 2 basketball teams, each of 5 to play against each other. How many different divisions are possible?

Solution: $\frac{10!}{5!5!} \times \frac{1}{2}$.

The Multinomial Theorem

The Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

The Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, n_2, \ldots, n_r): \\ n_1 + n_2 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$