Spherical Waves

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ELEG 648—Spherical Coordinates

Outline

1. Wave Functions
   - Separation of Variables
   - The Special Functions
   - Vector Potentials

Outline

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The Scalar Helmholtz Equation

- Maxwell's equations are complicated in spherical coordinates.
- We postpone examining them for a moment and begin with a scalar Helmholtz equation.

The Scalar Helmholtz Equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}+k^2\psi=0$$

As usual we substitute

$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$$

Separation of Variables

Multiplying the resulting equation by $r^2 \sin^2\theta$, and dividing by ψ gives

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+k^2r^2\sin^2\theta=0$$

We now let

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}=-m^2,$$

substitute, and divide by $\sin^2\theta$. This gives

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)-\frac{m^2}{\sin^2\theta}+k^2r^2=0$$

Separation of Variables

- We may now separate the θ equation.
- The strange separation constant is $-n(n+1)$ is chosen because the form of $\Theta(\theta)$ depends on whether or not $n \in \mathbb{Z}$.

$$\frac{1}{\Theta \sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) - \frac{m^2}{\sin^2\theta} = -n(n+1)$$

Substituting this into the previous equation, we find

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - n(n+1) + k^2r^2 = 0$$

Separation of Variables

Thus, after separation, we are left with the

Spherical Separated Equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left[k^2r^2-n(n+1)\right]R &= 0\\
\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta &= 0\\
\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+m^2\Phi &= 0
\end{aligned}
$$

Note that there is no separation equation here since two of the independent variables refer to angles!

Outline

The Harmonic Equation

The only familiar equation above is the harmonic equation

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + m^2\Phi = 0$$

- As usual, its solutions are $e^{\pm jm\phi}$, $\cos m\phi$, and $\sin m\phi$.
- Again, m must be an integer if $\Phi(\phi)$ must be periodic.

The Spherical Bessel Equation

The radial equation is of the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left[k^2r^2-n(n+1)\right]R=0$$

- The solutions of this equation are called spherical bessel functions $b_n(kr)$.
- This term is general; there are Bessel functions $j_n(kr)$, Neumann functions $y_n(kr)$, and Hankel Functions $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$.
- The spherical functions are related to the cylindrical functions by

$$b_n(kr)=\sqrt{\frac{\pi}{2kr}}B_{n+\frac{1}{2}}(kr)$$

The Spherical Bessel Equation

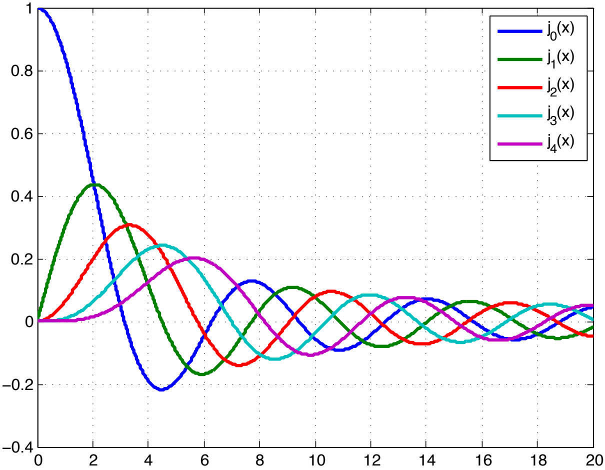
- Each function has the same properties as the corresponding cylindrical function:
  - j_n is the only function regular at the origin.
  - j_n and y_n represent standing waves.
  - $h_n^{(2)}$ is an outgoing wave, $h_n^{(1)}$ is an incoming wave.
- Spherical wave functions are actually expressible in terms of more familiar functions:

$$j_0(kr) = \frac{\sin kr}{kr} \qquad y_0(kr) = -\frac{\cos kr}{kr}$$

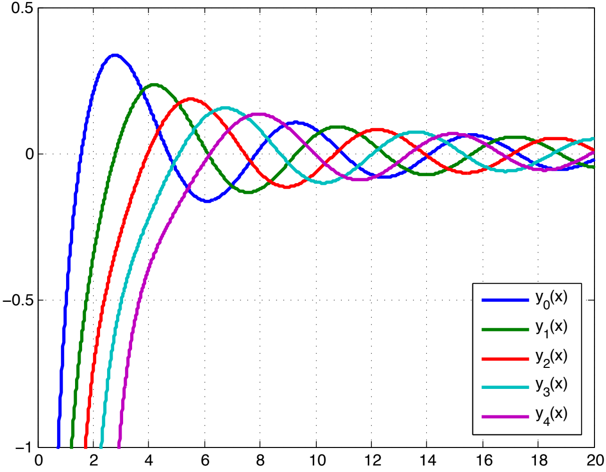
$$h_0^{(1)} = \frac{e^{jkr}}{jkr} \qquad h_0^{(2)} = \frac{e^{-jkr}}{jkr}$$

- The higher order functions can be found from the recurrence formula.

Spherical Bessel Functions



Spherical Neumann Functions



The Associated Legendre Equation

The final equation is of the form

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0$$

Defining $x=\cos\theta$ (and $y=\Theta$) we find

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]y=0.$$

- Since $0\leq\theta\leq\pi$, $-1\leq x\leq 1$.
- One set of solutions is regular for $n\in\mathbb{Z}$, we call them associated Legendre functions of the first kind $P_n^m(x)$.
- The other set, associated Legendre functions of the second kind, $Q_n^m(x)$, are singular at $|x|=1$.

Legendre Polynomials

If $m = 0$, and $n \in \mathbb{Z}$ the solutions become orthogonal polynomials; Legendre Polynomials of degree n. These are given by the well-known

Rodrigues Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n$$

- These polynomials are not orthonormal; they are normalized so $P_n(1) = 1$.
- The first few are: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$... How can I find the rest?

Associated Legendre Facts

- If $\nu \notin \mathbb{Z}$, the functions $P_\nu(x)$ and $P_\nu(-x)$ are independent.
- Therefore, if $\nu \in \mathbb{Z}$, the Q must be used.
- The functions $P_\nu(x)$ (and, for that matter, $P_\nu^m(x)$) for $\nu \notin \mathbb{Z}$ are not regular at $|x| = 1$ either.
- When $m \neq 0$, they are called Associated Legendre Functions.
- Of course, in practice, we use $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$. (We call these $L_n^m(\cos\theta)$.)
- Finally, it is important to note that

$$L_n^m(x) = 0 \qquad \text{for } m > n.$$

Outline

The Problem

- Our normal approach uses the scalar Helmholtz equation to solve for a vector potential component.
- This generally requires choosing a Cartesian component.
- In spherical coordinates, there is no Cartesian component!

One approach is to set fields to be, say, TM_z anyway. Then

$$\mathbf{A} = \mathbf{u}_z \mu\psi = \mathbf{u}_r \mu\psi \cos\theta - \mathbf{u}_\theta \mu\psi \sin\theta,$$

where ψ is a solution to the Helmholtz equation in spherical coordinates. Why is this approach unpopular?

A Better Approach

- Our approach will look for fields TM_r and TE_r by letting $\mathbf{A} = \mathbf{u}_r \mu A_r$ and $\mathbf{F} = \mathbf{u}_r \epsilon F_r$.
- Our problem is that

$$\nabla^2 A_r \neq (\nabla^2 \mathbf{A})_r$$

so that A_r is not a solution of the Helmholtz equation.
- To find the equation the vector potential solves, we go back to the beginning.

Spherical Vector Potential

Because $\nabla \cdot \mathbf{B} = 0$ and $\mathbf{B} = \mu\mathbf{H}$, we define $\mathbf{A}$ through

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}$$

From Faraday's Law,

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} = -j\omega\nabla \times \mathbf{A}$$

This implies

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0$$

or

$$\mathbf{E} + j\omega\mathbf{A} = -\nabla\Phi$$

Spherical Vector Potential

Plugging this into the source-free Ampère-Maxwell law, we find

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = j\omega\epsilon \left(-\nabla\Phi - j\omega\mathbf{A} \right)$$

or

$$\nabla \times \nabla \times \mathbf{A} - k^2\mathbf{A} = -j\omega\mu\epsilon\nabla\Phi.$$

Now assume

$$\mathbf{A} = A_r \mathbf{u}_r.$$

Then

$$\nabla \times \mathbf{A} = \mathbf{u}_\theta \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \mathbf{u}_\phi \frac{1}{r}\frac{\partial A_r}{\partial\theta}$$

Spherical Vector Potential

Now

$$
\begin{aligned}
(\nabla \times \nabla \times \mathbf{A})_r &= -\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A_r}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 A_r}{\partial^2\phi}\right] \\
(\nabla \times \nabla \times \mathbf{A})_\theta &= \frac{1}{r}\frac{\partial^2 A_r}{\partial r \partial\theta} \\
(\nabla \times \nabla \times \mathbf{A})_\phi &= \frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r \partial\phi}
\end{aligned}
$$

and

$$
-j\omega\mu\epsilon\nabla\Phi = -j\omega\mu\epsilon\left[\mathbf{u}_r \frac{\partial\Phi}{\partial r} + \mathbf{u}_\theta \frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \mathbf{u}_\phi \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right]
$$

Spherical Vector Potential

We can equate the $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$ components immediately by choosing

$$-j\omega\mu\epsilon\Phi = \frac{\partial A_r}{\partial r}.$$

Plugging this into the $\mathbf{u}_r$ equation gives

$$\begin{aligned}
\frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 A_r}{\partial^2\phi}\right] + k^2 A_r &= 0 \\
\frac{1}{r}\frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\frac{A_r}{r}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial^2\phi}\frac{A_r}{r}\right] + k^2 \frac{A_r}{r} &= 0 \\
\frac{\partial^2}{\partial r^2}\frac{A_r}{r} + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\frac{A_r}{r}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial^2\phi}\frac{A_r}{r}\right] + k^2 \frac{A_r}{r} &= 0
\end{aligned}$$

Spherical Vector Potential

This equation is recognized as the

Scalar Helmholtz Equation (for TM$_r$ waves)

$$\left[\nabla^2 + k^2\right]\left(\frac{A_r}{r}\right) = 0$$

It is not to hard to imagine the

Scalar Helmholtz Equation (for TE$_r$ waves)

$$\left[\nabla^2 + k^2\right]\left(\frac{F_r}{r}\right) = 0$$

with the gauge condition

$$\frac{\partial F_r}{\partial r} = -j\omega\epsilon\mu\Phi^{\mathrm{m}}.$$

Spherical Vector Potential

Thus, in short, if ψ^{a} and ψ^{f} solve the Helmholtz equation, we have

$$\begin{aligned}\mathbf{A} &= \mu\psi^{\mathrm{a}}\mathbf{r} \\ \mathbf{F} &= \epsilon\psi^{\mathrm{f}}\mathbf{r}\end{aligned}$$

where $\mathbf{r} = r\mathbf{u}_r$. In terms of these we have

The Fields

$$\begin{aligned}\mathbf{E} &= -\nabla\times\mathbf{r}\psi^{\mathrm{f}} + \frac{1}{j\omega\mu}\nabla\times\nabla\times\mathbf{r}\psi^{\mathrm{a}} \\ \mathbf{H} &= \nabla\times\mathbf{r}\psi^{\mathrm{a}} + \frac{1}{j\omega\epsilon}\nabla\times\nabla\times\mathbf{r}\psi^{\mathrm{f}}\end{aligned}$$

Spherical Vector Potential

The components may be written

$$
\begin{aligned}
\epsilon E_r &= \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial r^2}+k^2\right)A_r \\
\epsilon E_\theta &= -\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi}+\frac{1}{j\omega\mu r}\frac{\partial^2 A_r}{\partial r\partial\theta} \\
\epsilon E_\phi &= \frac{1}{r}\frac{\partial F_r}{\partial\theta}+\frac{1}{j\omega\mu r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} \\
\mu H_r &= \frac{1}{j\omega\epsilon}\left(\frac{\partial^2}{\partial r^2}+k^2\right)F_r \\
\mu H_\theta &= \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}+\frac{1}{j\omega\epsilon r}\frac{\partial^2 F_r}{\partial r\partial\theta} \\
\mu H_\phi &= -\frac{1}{r}\frac{\partial A_r}{\partial\theta}+\frac{1}{j\omega\epsilon r\sin\theta}\frac{\partial^2 F_r}{\partial r\partial\phi}
\end{aligned}
$$

The Schelkunoff Bessel Functions

- A_r and F_r do not solve the Helmholtz equation.
- If ψ solves the Helmholtz equation, the vector potential may be $r\psi$.
- It is therefore helpful to define new radial eignefunctions.
- The most common (and obvious) definition is that due to S. A. Schelkunoff

Schelkunoff Bessel Functions

$$\hat{B}_n(kr) \triangleq krb_n(kr) = \sqrt{\frac{\pi kr}{2}} B_{n+\frac{1}{2}}(kr)$$

The Schelkunoff Bessel Functions

The Schelkunoff Bessel Functions solve the equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \frac{n(n+1)}{r^2}\right]\hat{B}_n = 0$$

The solutions for either vector potential can now be written as

$$\sum_m \sum_n C_{mn}\hat{B}_n(kr)L_n^m(\cos\theta)h(m\phi).$$

Outline

Transverse Electric Modes

- Consider a spherical resonator of radius a.
- The field must be finite at the origin.
- The field must be finite at the poles (i.e. $\theta = 0$ and $\theta = \pi$.
- The field must be 2π-periodic in azimuth.

Thus we choose the

Modal Vector Potentials

$$F_r = \epsilon \hat{J}_n(kr) P_n^m(\cos\theta) \left\{ \begin{array}{c} \cos m\phi \\ \sin m\phi \end{array} \right\}$$

(In the future we ignore the $\sin m\phi$ for simplicity.)

Transverse Electric Modes

Now E_θ and E_ϕ are not proportional to derivatives with respect to r, so we need

$$\hat{J}_n(ka) = 0$$

This will be satisfied if

$$ka = u_{np}$$

where, as might be expected,

$$\hat{J}_n(u_{np}) = 0,$$

in particular, u_{np} is the p^{th} root of $\hat{J}_n$

Transverse Magnetic Modes

Transverse magnetic modes are similar. We start with

Modal Vector Potentials

$$A_r = \mu \hat{J}_n(kr) P_n^m(\cos\theta) \left\{ \begin{array}{c} \cos m\phi \\ \sin m\phi \end{array} \right\}$$

In this case, E_θ and E_ϕ are proportional to derivatives with respect to r, so we need

$$\hat{J}_n'(kr) = 0$$

Therefore

$$k = \frac{u_{np}'}{a}$$

Resonant Frequencies

Given the values for k, it is easy to find the

Resonant Frequencies

$$
\begin{aligned}
(f_r)_{mnp}^{\mathrm{TE}} &= \frac{u_{np}}{2\pi a\sqrt{\mu\epsilon}} \\
(f_r)_{mnp}^{\mathrm{TM}} &= \frac{u'_{np}}{2\pi a\sqrt{\mu\epsilon}}
\end{aligned}
$$

It is important to note that many of these modes are degenerate (i.e. have the same cutoff frequency.)

Degeneracies

- Cutoff frequencies in spherical waveguides do not depend on m or on whether they are even or odd.
- The lowest order TE modes have $n = p = 1$, $u_{np} = 4.493$

The following modes have the same cutoff:

$$
\begin{aligned}
(F_r)_{0,1,1} &= \hat{J}_1\left(4.493\frac{r}{a}\right)\cos\theta \\
(F_r)^{\text{even}}_{1,1,1} &= \hat{J}_1\left(4.493\frac{r}{a}\right)\cos\theta\cos\phi \\
(F_r)^{\text{odd}}_{1,1,1} &= \hat{J}_1\left(4.493\frac{r}{a}\right)\cos\theta\sin\phi
\end{aligned}
$$

In this case, the reason for the degeneracy is clear; these are all the same mode, rotated in space. How many modes have $n = 2$, $p = 1$?

Outline

Conical Waveguide

Consider the cone $\theta = \theta_1$. Let's find the field:

- The field should be outwardly travelling.
- The field should be 2π-periodic in azimuth.
- The field should be finite at the poles:
  - If $\theta_1 > \pi/2$, the field must be finite at $\theta = 0$.
  - If $\theta_1 < \pi/2$, the field must be finite at $\theta = \pi$.

Either way, if we choose TM$_r$ functions and assume outward propagation and $\cos m\phi$ dependence,

$$(A_r)_{m\nu} = P_\nu^m(\cos\theta)\cos(m\phi)\hat{H}_\nu^{(2)}(kr)$$

Conical Waveguide

- To satisfy the boundary condition, we need $E_r = E_\phi = 0$.
- Neither E_r nor E_ϕ is proportional to a derivative of A_r with respect to θ.

To make the field vanish at $\theta = \theta_1$ we need

$$P_\nu^m(\cos\theta_1) = 0.$$

This is an equation in ν!

Conical Waveguide

- To satisfy the boundary condition, we need $E_r = E_\phi = 0$.
- Neither E_r nor E_ϕ is proportional to a derivative of A_r with respect to θ.

To make the field vanish at $\theta = \theta_1$ we need

$$P_\nu^m(\cos\theta_1) = 0.$$

This is an equation in ν! For TE modes, we have

$$(F_r)_{m\nu} = P_\nu^m(\cos\theta)\cos(m\phi)\hat{H}_\nu^{(2)}(kr)$$

subject to

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}P_\nu^m(\cos\theta)\right]_{\theta=\theta_1} = 0$$

Biconical Waveguide

- Consider now the waves in the region between $\theta = \theta_1$ and $\theta = \theta_2$.
- Here, neither $\theta = 0$ or $\theta = \pi$ is involved, so we can (and must) use either $Q_n(x)$ or $P_n(-x)$.
- We still have azimuthal 2π-periodicity, so $m \in \mathbb{Z}$.
- We assume outward travel, and $\cos m\phi$ dependence.

Then we may write

$$(A_r)_{m\nu} = [P_\nu^m(\cos\theta)P_\nu^m(-\cos\theta_1) - P_\nu^m(-\cos\theta)P_\nu^m(\cos\theta_1)] \times \cos(m\phi)\hat{H}_\nu(kr)$$

Note that this function vanishes already at $\theta = \theta_1$ by design.

Biconical Waveguide

The vector potential must also vanish at $\theta = \theta_2$. This implies

$$P_\nu^m(\cos\theta_2)P_\nu^m(-\cos\theta_1) - P_\nu^m(-\cos\theta_2)P_\nu^m(\cos\theta_1) = 0$$

This is an (awful, transcendental, obscure) equation for ν. Similar considerations apply to the TE modes.

The TEM Mode

- The bicone is a two-conductor waveguide, and therefore supports a TEM mode.
- The easiest way to study it is to consider a special formulation of the TM_{00} mode.

Thus, let

$$(A_r)_{00} = -j\mu Q_0(\cos\theta)\hat{H}_0^{(2)}(kr) = \ln\cot\frac{\theta}{2}e^{-jkr}.$$

(The extra factor of j is included just to make the expression real.)

The TEM Mode

Now, we first note that

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\cot\frac{\theta}{2} = \tan\frac{\theta}{2}\csc^2\frac{\theta}{2}\left(-\frac{1}{2}\right) = \frac{-1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\csc\theta$$

This implies

$$E_\theta = -\frac{1}{j\omega\epsilon\mu r}\frac{\partial^2 A_r}{\partial r\partial\theta} = \frac{ke^{-jkr}}{\omega\epsilon r\sin\theta}$$

Similarly,

$$H_\phi = -\frac{1}{r}\frac{\partial A_r}{\partial\theta} = \frac{e^{-jkr}}{r\sin\theta}$$

The TEM Mode Voltage

Notice

$$\frac{E_\theta}{H_\phi} = \eta.$$

This is to be expected from TEM waves.
We can also define

Voltage

$$V(r) = \int_{\theta_1}^{\theta_2} E_\theta r d\theta = \eta \ln \frac{\cot \frac{\theta_1}{2}}{\cot \frac{\theta_2}{2}} e^{-jkr}$$

The TEM Mode Current

Similarly we have

Current

$$I(r) = \int_0^{2\pi} H_\phi r \sin\theta \mathrm{d}\phi = 2\pi e^{-jkr}$$

We can also compute the

Transmission Line Characteristic Impedance

$$Z = \frac{V}{I} = \frac{\eta}{2\pi} \ln \frac{\cot\frac{\theta_1}{2}}{\cot\frac{\theta_2}{2}}$$

Outline

Plane Wave Expansion

A wave traveling in the $-z$-direction must have the expansion

$$e^{jz} = e^{jr\cos\theta} \sum_{n=0}^{\infty} a_n j_n(r) P_n(\cos\theta)$$

Multiplying both sides by $\sin\theta$ and integrating from 0 to π gives

$$\int_0^{\pi} e^{jr\cos\theta} P_n(\cos\theta)\sin\theta d\theta = \frac{2a_n}{2n+1} j_n(r)$$

after invoking orthogonality. Computing the integral yields

$$a_n = j^n(2n+1)$$

Plane Wave Expansion

We thus have the

Spherical Expansion of a Plane Wave

$$e^{jr\cos\theta} = \sum_{n=0}^{\infty} j^n (2n+1) j_n(r) P_n(\cos\theta)$$

Similar theorems can translate cylindrical Bessel functions into spherical harmonics, etc.

Addition Theorem

We know that the solution to

$$\nabla^2\psi + k^2\psi = \delta(\mathbf{r} - \mathbf{r}')$$

is

$$\psi(\mathbf{r}) = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} = \frac{-j}{4\pi} h_0^{(2)}(kr)$$

- We can write this in terms of origin centered spherical harmonics in two ways.
- The difference centers on the special nature of the z-axis.
- The solution must have rotational symmetry with respect to the $\mathbf{r}'$ axis.
  - Let the angle with $\mathbf{r}'$ be ξ.
  - It can be shown geometrically that

$$\cos\xi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$$

Addition Theorem

We also know from

- Our study of spherical coordinate special functions, and
- Symmetry,

that the solution can be written as

$$\psi = \frac{-j}{4\pi} \sum_{n=0}^{\infty} c_n j_n(r_<) h_n^{(2)}(r_>) P_n(\cos \xi)$$

By interpreting ξ as θ and, say, $\mu\psi$ as A_z we find

The Addition Theorem

$$h_0^{(2)}(|\mathbf{r} - \mathbf{r}'|) = \sum_{n=0}^{\infty} (2n+1) j_n(r_<) h_n^{(2)}(r_>) P_n(\cos \xi)$$

The Other Addition Theorem

If we do not invoke the polar symmetry we just used, we can solve the problem using associated Legendre functions in their full glory. This leads to

The Legendre Function Addition Theorem

$$P_n(\cos\xi) = \sum_{m=1}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi - \phi')$$

Outline

The Set Up

Consider a sphere of radius *a*, illuminated by the

Incident Wave

$$
\begin{aligned}
E_x^{\mathrm{i}} &= E_0 e^{-jkz} &= E_0 e^{-jkr\cos\theta} \\
H_y^{\mathrm{i}} &= \frac{E_0}{\eta} e^{-jkz} &= \frac{E_0}{\eta} e^{-jkr\cos\theta}
\end{aligned}
$$

- Our wave is not TE_r or TM_r.
- We need to expand it as a combination of these types of modes.
- To do this, we need the r components of **E** and **H**.

The Incident Field

Let us work with E_r^{i} and find the TM part.

$$E_r^{\mathrm{i}} = \cos\phi \sin\theta E_x^{\mathrm{i}} = E_0 \frac{\cos\phi}{jkr} \frac{\partial}{\partial\theta}\left(e^{-jkr\cos\theta}\right)$$

Plugging in the plane wave expansion we find

$$E_r^{\mathrm{i}} = E_0 \frac{\cos\phi}{jkr} \sum_{n=0}^{\infty} j^{-n} j_n(kr) \frac{\partial}{\partial\theta} P_n(\cos\theta)$$

This can be simplified using the Schelkunoff functions and derivative formula for Legendre functions:

$$E_r^{\mathrm{i}} = -\frac{jE_0\cos\phi}{(kr)^2} \sum_{n=1}^{\infty} j^{-n}(2n+1)\hat{J}_n(kr) P_n^1(\cos\theta)$$

The Incident Field

We want to derive this radial field from A_r. We thus write

$$A_r^{\mathrm{i}} = \frac{E_0}{\omega} \cos\phi \sum_{n=1}^{\infty} a_n \hat{J}_n(kr) P_n^1(\cos\theta)$$

By deriving E_r^{i} from this potential and setting it equal to E_r^{i} from the previous slide, we can find the a_n. We find

$$E_r^{\mathrm{i}} = -\frac{jE_0\cos\phi}{(kr)^2} \sum_{n=1}^{\infty} a_n n(n+1) \hat{J}_n(kr) P_n^1(\cos\theta)$$

Setting this equal to the previous E_r^{i} we find

$$a_n = \frac{j^{-n}(2n+1)}{n(n+1)}$$

The Incident Fields

Using duality, we can thus write the

Incident Wave Vector Potentials

$$
\begin{aligned}
A_r^{\mathrm{i}} &= \frac{E_0}{\omega} \cos\phi \sum_{n=1}^{\infty} a_n \hat{J}_n(kr) P_n^1(\cos\theta) \\
F_r^{\mathrm{i}} &= \frac{\epsilon E_0}{k} \sin\phi \sum_{n=1}^{\infty} a_n \hat{J}_n(kr) P_n^1(\cos\theta)
\end{aligned}
$$

Each of these "incident waves" generates a scattered wave independently.

The Scattered Fields

We can also write expressions for the

Scattered Wave Vector Potentials

$$
\begin{aligned}
A_r^s &= \frac{E_0}{\omega} \cos\phi \sum_{n=1}^{\infty} b_n \hat{H}_n^{(2)}(kr) P_n^1(\cos\theta) \\
F_r^s &= \frac{\epsilon E_0}{k} \sin\phi \sum_{n=1}^{\infty} c_n \hat{H}_n^{(2)}(kr) P_n^1(\cos\theta)
\end{aligned}
$$

Remember: the a_n are known. The two sets of unknowns (b_n and c_n) are from the independent TM_r and TE_r waves.

The Solution

Computing the E_θ and E_ϕ components from the vector potentials given and forcing them to vanish gives the coefficients:

The Solution

$$
\begin{aligned}
b_n &= -a_n \frac{\hat{J}_n'(ka)}{\hat{H}_n^{(2)\prime}(ka)} \\
c_n &= -a_n \frac{\hat{J}_n(ka)}{\hat{H}_n^{(2)}(ka)}
\end{aligned}
$$

Now we can easily find the scattered field. Indeed, we can find the far field from

$$
\hat{H}_n^{(2)}(kr) \stackrel{kr\to\infty}{\longrightarrow} j^{n+1} e^{-jkr}
$$

The Far Scattered Field

Plugging this in and retaining only the fields that decay as r^{-1}, we find

The Far Field

$$E_\theta^s = \frac{jE_0}{kr} e^{-jkr} \cos\phi \sum_{n=1}^{\infty} j^n \left[b_n \sin\theta P_n^{1\prime}(\cos\theta) - c_n \frac{P_n^1(\cos\theta)}{\sin\theta} \right]$$

$$E_\phi^s = \frac{jE_0}{kr} e^{-jkr} \sin\phi \sum_{n=1}^{\infty} j^n \left[b_n \frac{P_n^1(\cos\theta)}{\sin\theta} - c_n \sin\theta P_n^{1\prime}(\cos\theta) \right]$$

We are most interested in the backscattered field

$$E_\theta^s\,(\theta = \pi, \phi = \pi)$$

An Aside: Radar Cross Section

- We can get an idea of the size of something by how much energy it scatters.
- This idea is formalized in the radar cross section (RCS) or echo area A_e.
- This is, or course, a far field quantity.

Definition

The RCS is that area which intercepts an amount of power, which, when reradiated isotropically, produces a power at the receiver equal to that actually observed.

An Aside: Radar Cross Section

- If the power in the incident wave is S^{i}, the power that would be intercepted is $A_{\mathrm{e}} S^{\mathrm{i}}$.
- If this power, reradiated isotropically, creates the power received, then

$$S^{\mathrm{r}} = \frac{A_{\mathrm{e}} S^{\mathrm{i}}}{4\pi r^2}$$

We thus have the

Radar Cross Section

$$A_{\mathrm{e}} = \lim_{r\to\infty} \left(4\pi r^2 \frac{S^{\mathrm{r}}}{S^{\mathrm{i}}} \right) = \lim_{r\to\infty} \left(4\pi r^2 \frac{|E^{\mathrm{s}}|^2}{|E^{\mathrm{i}}|^2} \right)$$

The RCS

Plugging our scattered fields into the RCS formula, and using the wonderfully obscure formulas

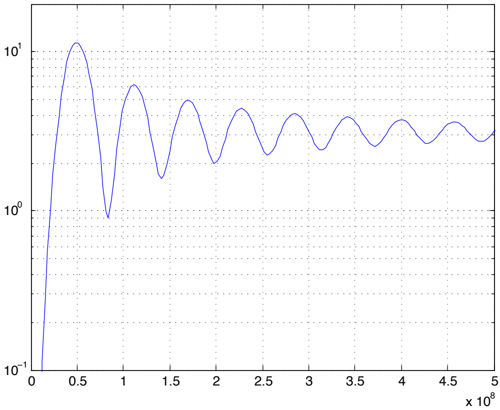
$$\frac{P_n^1(\cos\theta)}{\sin\theta} \xrightarrow{\theta\to\pi} \frac{(-1)^n}{2} n(n+1)$$
$$\sin\theta P_n^{1\prime}(\cos\theta) \xrightarrow{\theta\to\pi} \frac{(-1)^n}{2} n(n+1)$$

and using the Wronskian of Schelkunoff Bessel functions, we find

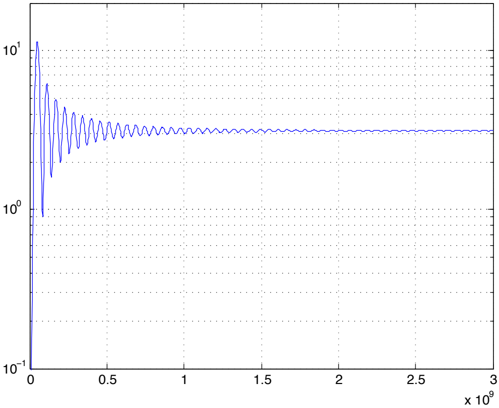
The Radar Cross Section of a Metal Sphere

$$A_e = \frac{\lambda^2}{4\pi}\left|\sum_{n=1}^{\infty} \frac{(-1)^2(2n+1)}{\hat{H}_n^{(2)}(ka)\hat{H}_n^{(2)\prime}(ka)}\right|^2$$

RCS (m^2) of a 1m Radius Sphere vs. Frequency (Hz)



RCS (m^2) of a 1m Radius Sphere vs. Frequency (Hz)



Observations

- At low frequencies

$$A_e \xrightarrow{ka \to 0} \frac{9\lambda^2}{4\pi}(ka)^6$$

- In other words, for small spheres

$$A_e \propto \lambda^{-4}$$

- This law was first invoked by Lord Rayleigh to explain the blueness of the sky.
- What happens for high frequencies?

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- This law was first invoked by Lord Rayleigh to explain the blueness of the sky.
- What happens for high frequencies?

$$A_e \xrightarrow{ka \to \infty} \pi a^2$$