Maxwell's Equations

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ELEG 648—Maxwell's Equations

Outline

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Introduction

- Maxwell's equations are a macroscopic theory. We can ignore the atomic structure of matter if
  - Linear dimensions much greater than atomic dimensions.
  - Charges much greater then electronic charge.
- We use mksc units, that is

m	The meter, a unit of length,
kg	The kilogram, a unit of mass,
s	The second, a unit of time, and
C	The coulomb, a unit of electrical charge.

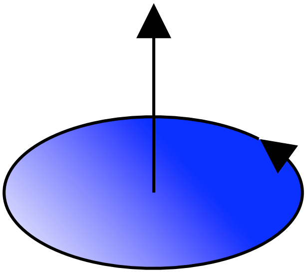
- Technically, the Ampère (A=C/s) is the basic unit for reasons of metrology, not theory.

Variables

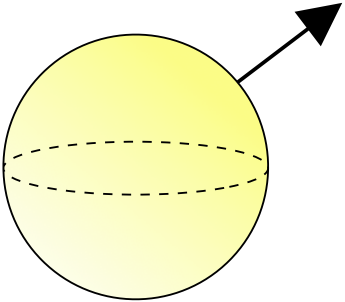
- From mechanics and circuit theory:
  - $1\text{N} = 1\frac{\text{kg}\cdot\text{m}}{\text{s}^2}$
  - $1\text{J} = 1\text{N}\cdot\text{m}$
  - $1\text{V} = 1\frac{\text{J}}{\text{C}}$
  - $1\text{A} = 1\frac{\text{C}}{\text{s}}$
- Our variables are:

$\mathcal{E}$ electric field (V/m)
$\mathcal{H}$ magnetic field (A/m)
$\mathcal{D}$ electric flux density (C/m^2)
$\mathcal{B}$ magnetic flux density ($\text{T} = \text{Wb}/\text{m}^2$)
$\mathcal{J}$ electric current density (A/m^2)
$\mathcal{Q}_v$ electric charge density (C/m^3)

Conventions



The normal to an open surface bounded by a contour is related to the contour by the right hand rule.



The normal to a closed surface points out from the surface.

Outline

Maxwell's Equations in Integral Form

$$\oiint \mathcal{D} \cdot d\mathbf{S} = \iiint \mathcal{Q}_v dv$$

$$\oiint \mathcal{B} \cdot d\mathbf{S} = 0$$

$$\oint \mathcal{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint \mathcal{B} \cdot d\mathbf{S}$$

$$\oint \mathcal{H} \cdot d\mathbf{l} = \iint \mathcal{J} \cdot d\mathbf{S} + \frac{d}{dt} \iint \mathcal{D} \cdot d\mathbf{S}$$

- The first two equations relate integrals over volumes to integrals over the surface bounding them.
- The second two equations relate integrals over surfaces to the contours bounding them. In Faraday's law, the same surface must be used for both flux integrals.

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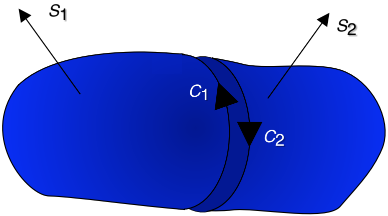
Conservation of Charge



Consider a closed surface cleaved in half by an open surface. Using the Maxwell-Ampère Law in both directions gives

$$\oint_{C_1} \mathcal{H} \cdot d\mathbf{l} = \iint_{S_1} \mathcal{J} \cdot d\mathbf{S} + \frac{d}{dt} \iint_{S_1} \mathcal{D} \cdot d\mathbf{S}$$
$$\oint_{C_2} \mathcal{H} \cdot d\mathbf{l} = \iint_{S_2} \mathcal{J} \cdot d\mathbf{S} + \frac{d}{dt} \iint_{S_2} \mathcal{D} \cdot d\mathbf{S}$$

Conservation of Charge

Adding these equations gives

$$\iint_S \mathcal{J} \cdot d\mathbf{S} = -\frac{d}{dt} \iint_S \mathcal{D} \cdot d\mathbf{S}$$

Substituting Gauß's law for the electric field gives

The Law of Conservation of Charge

$$\iint_S \mathcal{J} \cdot d\mathbf{S} = -\frac{d}{dt} \iiint_V \mathcal{Q}_v dv$$

Outline

The Divergence Theorem

The divergence of a vector is a scalar.

The Divergence Theorem

$$\oiint \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \iiint \nabla \cdot \mathbf{A} \,\mathrm{d}v$$

Note that the divergence theorem tells us that divergence is outward flux per unit volume. (Inward fluxes cancel.)

Divergence in Cartesian Coordinates

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence Equations

Using the divergence theorem and Gauß's Law for the Electric field,

$$\oiint \mathcal{D} \cdot d\mathbf{S} = \iiint \nabla \cdot \mathcal{D} dv = \iiint \mathcal{Q}_v dv$$

Since this must be true over any volume, the integrands must be equal and we have

Gauß's Law for the Electric Field in Differential Form

$$\nabla \cdot \mathcal{D} = \mathcal{Q}_v$$

The differential form for Gauß's Law for the magnetic field and for the Law of Conservation of Charge (i.e., the Equation of Continuity) may be derived similarly.

Stokes's Theorem

The curl of a vector is a vector.

Stokes's Theorem

$$\oint \mathbf{A} \cdot d\mathbf{l} = \iint \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

Note that the curl is the rotation per unit area, with direction given by the right-hand rule. (Internal circulation cancels.)

Curl in Cartesian Coordinates

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{a}_z$$

Curl Equations

Using Stokes's Theorem in Faraday's Law and assuming the surface does not move

$$\oint \mathcal{E} \cdot \mathrm{d}\mathbf{l} = \iint \nabla \times \mathcal{E} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint \mathcal{B} \cdot \mathrm{d}\mathbf{S} = -\iint \frac{\partial \mathcal{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}$$

Since this must be true over any surface, we have

Faraday's Law in Differential Form

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$$

The Maxwell-Ampère Law can be similarly converted.

Maxwell's Equations in Differential Form

Maxwell's Equations

$$\nabla \cdot \mathcal{D} = \mathcal{Q}_v$$

$$\nabla \cdot \mathcal{B} = 0$$

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$$

$$\nabla \times \mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t}$$

Continuity Equation

$$\nabla \cdot \mathcal{J} = -\frac{\partial \mathcal{Q}_v}{\partial t}$$

Outline

Constitutive Relationships

- Maxwell's Equations as they stand are not complete.
- The missing ingredient is the influence of matter, most generally of the form

$$\begin{aligned}\mathcal{D} &= \mathcal{D}\left(\mathcal{E},\mathcal{B}\right)\\ \mathcal{H} &= \mathcal{H}\left(\mathcal{E},\mathcal{B}\right)\\ \mathcal{J} &= \mathcal{J}\left(\mathcal{E},\mathcal{B}\right)\end{aligned}$$

- The exact form of these can be deduced by experiment or analysis of molecular structure.

Free Space

In vacuum (or, for all practical purposes, air) the constitutive relationships are

$$\begin{aligned} \mathcal{D} &= \epsilon_0 \mathcal{E} \\ \mathcal{B} &= \mu_0 \mathcal{H} \\ \mathcal{J} &= 0 \end{aligned}$$

We will see later that c, the speed of light in vacuum, is given by the formula

The Speed of Light

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.99792458 \times 10^8 \text{m/s}$$

Free Space

- The value of the speed of light is set by international agreement, and serves to define the meter. (The second is defined by another standard.)
- A useful approximation is $c = 3 \times 10^8$m/s
- The internationally agreed upon value for the permeability of free space is

$$\mu_0 = 4\pi \times 10^{-7}\text{H/m}$$

  (By definition, 1H=1V-s/A.)
- The above implies

$$\epsilon_0 \approx 8.854 \times 10^{-12}\text{F/m} \approx \frac{10^{-9}}{36\pi}\text{F/m}$$

  (By definition, 1F = 1 C/V)

Simple Matter

For many materials excited by weak fields, the constitutive relationships take a simple form over large frequency bands

$$\begin{aligned} \mathcal{D} &= \epsilon \mathcal{E} \\ \mathcal{B} &= \mu \mathcal{H} \\ \mathcal{J} &= \sigma \mathcal{E} \end{aligned}$$

- ϵ is called permittivity (F/m).
- μ is called permeability (H/m).
- σ is called conductivity (S/m).

(By definition 1 S = 1 A/V.)

Simple Matter Terminology

- A material with $\sigma = \infty$ is called a perfect electric conductor or PEC.
- A material with $\sigma = 0$ is a perfect dielectric.
- The idea of a "good conductor" or "good dielectric" is intuitive, but will be defined more carefully later.
- The value $\epsilon_{\mathrm{r}} = \epsilon/\epsilon_0$ is called the relative permittivity or the dielectric constant.
- The value $\mu_{\mathrm{r}} = \mu/\mu_0$ is called the relative permeablity.

Why is all matter not simple matter?

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Why is all matter not simple matter?

Complicated Matter

- In general linear matter, the constitutive parameter is a causal function of time to be convolved with the appropriate variable, i.e.

$$\mathcal{D}(t) = \int_{-\infty}^{t} \epsilon(t-\tau)\mathcal{E}(\tau)\mathrm{d}\tau$$

- In nonlinear matter, the constitutive parameters are functions of the field variables, i.e.

$$\epsilon = \epsilon(\mathcal{E}).$$

In short, in such media, the fields cannot be computed by convolution in time.

Complicated Matter

- In anisotropic matter, the constitutive parameter is a matrix so that, for instance, $\mathcal{E}$ and $\mathcal{D}$ are not parallel. Normal matter is called isotropic.
- Finally, in chiral matter, $\mathcal{D}$ is a function (generally linear) of both $\mathcal{E}$ and $\mathcal{B}$, with a similar relation for $\mathcal{H}$.
- In this class, we will never deal with anything more complicated then general linear matter.

Outline

Generalized Current

- Ampère's law was originally

$$\nabla \times \mathcal{H} = \mathcal{J}.$$

- Maxwell amended this to include the "displacement current"

$$\mathcal{J}^{\mathrm{d}} = \frac{\partial \mathcal{D}}{\partial t}$$

He did this to ensure conservation of charge, and envisioned it as a real current flow in the ether. This view is incorrect, but the definition is useful.

Generalized Current

- In addition, there is the regular conduction current, i.e., the flow of electrons. This is usually given by Ohm's Law:

Ohm's Law

$$\mathcal{J}^{\mathrm{c}} = \sigma \mathcal{E}$$

- Finally, there is impressed current $\mathcal{J}^{\mathrm{i}}$. Impressed currents are those we think of as sources.
- We may thus define total current:

Total Current

$$\mathcal{J}^{\mathrm{t}} = \mathcal{J}^{\mathrm{d}} + \mathcal{J}^{\mathrm{c}} + \mathcal{J}^{\mathrm{i}}$$

Generalized Current

- In a similar vein,
$$\mathcal{M}^{\mathrm{d}} = \frac{\partial \mathcal{B}}{\partial t}$$
can be though of as a magnetic displacement current.
- "Voltage" sources can be envisioned as impressed magnetic current $\mathcal{M}^{\mathrm{i}}$.
- Total magnetic current is then

Total Magnetic Current

$$\mathcal{M}^{\mathrm{t}} = \mathcal{M}^{\mathrm{d}} + \mathcal{M}^{\mathrm{i}}$$

Generalized Current

In terms of this generalized current, the curl equations become

$$\nabla \times \mathcal{E} = -\mathcal{M}^{\mathrm{t}}$$
$$\nabla \times \mathcal{H} = \mathcal{J}^{\mathrm{t}}$$

A Vector Identity

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

A Theorem

$$\nabla \cdot \mathcal{M}^{\mathrm{t}} = \nabla \cdot \mathcal{J}^{\mathrm{t}} = 0$$

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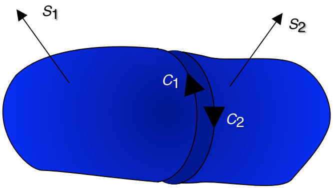
Proof of a Vector Identity

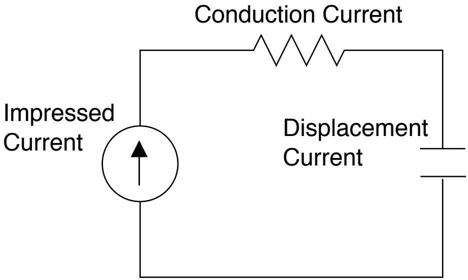


Consider any arbitrary volume V, and a (several times differentiable) vector $\mathbf{A}$

$$\begin{aligned}
\iiint_V \nabla \cdot \nabla \times \mathbf{A} dv &= \iint_{S_1} \nabla \times \mathbf{A} \cdot d\mathbf{S} + \iint_{S_2} \nabla \times \mathbf{A} \cdot d\mathbf{S} \\
&= \oint_{C_1} \mathbf{A} \cdot d\mathbf{l} + \oint_{C_2} \mathbf{A} \cdot d\mathbf{l} \\
&= 0. \qquad \square
\end{aligned}$$

Generalized Current

We thus see that total current is solenoidal; that is it has no sources and sinks. Here is a circuit that demonstrates all three types:



Outline

Poynting's Theorem

Consider the standard vector identity

$$\nabla \cdot (\mathcal{E} \times \mathcal{H}) = \mathcal{H} \cdot \nabla \times \mathcal{E} - \mathcal{E} \cdot \nabla \times \mathcal{H}$$

(This is just a form of the product rule of differentiation.)
Substituting Maxwell's (curl) Equations gives

$$\nabla \cdot (\mathcal{E} \times \mathcal{H}) = -\mathcal{H} \cdot \mathcal{M}^{\mathrm{t}} - \mathcal{E} \cdot \mathcal{J}^{\mathrm{t}}$$

Define

The Poynting Vector

$$\mathcal{S} = \mathcal{E} \times \mathcal{H}$$

Contributions of Electric Currents to Poynting's Theorem

$$\mathcal{E} \cdot \mathcal{J}^{\mathrm{t}} = \mathcal{E} \cdot \mathcal{J}^{\mathrm{d}} + \mathcal{E} \cdot \mathcal{J}^{\mathrm{c}} + \mathcal{E} \cdot \mathcal{J}^{\mathrm{i}}$$

- Change in stored electrical energy:

$$\mathcal{E} \cdot \mathcal{J}^{\mathrm{d}} = \epsilon \mathcal{E} \cdot \frac{\partial \mathcal{E}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon \mathcal{E}^2 \right) = \frac{\partial}{\partial t} w_{\mathrm{e}}$$

- Conductive loss:

$$p_{\mathrm{d}} = \mathcal{E} \cdot \mathcal{J}^{\mathrm{c}} = \sigma \mathcal{E}^2$$

- Supplied electrical power:

$$p_{\mathrm{s,e}} = -\mathcal{E} \cdot \mathcal{J}^{\mathrm{i}}$$

Magnetic Contributions

By the same token, magnetic contributions are

- Change in stored magnetic energy:

$$\mathcal{H} \cdot \mathcal{M}^{\mathrm{d}} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\mathcal{H}^2\right) = \frac{\partial}{\partial t} w_{\mathrm{h}}$$

- Supplied magnetic power:

$$p_{\mathrm{s,h}} = -\mathcal{H} \cdot \mathcal{M}^{\mathrm{i}}$$

We can now define the total supplied power

$$p_{\mathrm{s}} = p_{\mathrm{s,e}} + p_{\mathrm{s,h}},$$

and total stored energy,

$$w = w_{\mathrm{e}} + w_{\mathrm{h}}.$$

Poynting's Theorem

Substituting into the expression

$$\nabla \cdot (\mathcal{E} \times \mathcal{H}) = \mathcal{H} \cdot \nabla \times \mathcal{E} - \mathcal{E} \cdot \nabla \times \mathcal{H}$$

gives

Poynting's Theorem (Microscopic)

$$p_{\mathrm{s}} = \nabla \cdot \mathcal{S} + p_{\mathrm{d}} + \frac{\partial w}{\partial t}$$

Integrating this over a volume and defining

$$P_{\mathrm{s}} = \iiint_V p_{\mathrm{s}} \mathrm{d}v \qquad \text{(power supplied)}$$

$$P_{\mathrm{d}} = \iiint_V p_{\mathrm{d}} \mathrm{d}v \qquad \text{(power dissipated)}$$

Poynting's Theorem

$$W = \iiint_V w \, \mathrm{d}v \qquad \text{(energy stored)}$$

$$P_\mathrm{f} = \oiint_S \mathcal{S} \cdot \mathrm{d}\mathbf{S} \qquad \text{(outward power flux)}$$

Collecting terms again leads to

Poynting's Theorem (Macroscopic)

$$P_\mathrm{s} = P_\mathrm{f} + P_\mathrm{d} + \frac{\mathrm{d}W}{\mathrm{d}t}$$

In what way is $\mathcal{S}$ power flux density? In what way is it not?

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Phasors

The phasor idea is based on

Euler's Identity

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \qquad (\alpha \in \mathbb{R}), j \equiv \sqrt{-1}$$

Using Euler's Identity, we can represent sinusoids by complex expressions

$$v(t) = a\cos(\omega t + \alpha) \Leftrightarrow ae^{j\alpha}.$$

Vectors can be converted to phasors in precisely the same way

$$\mathcal{E}(t) = \mathbf{E}\cos(\omega t + \alpha) \Leftrightarrow \mathbf{E}e^{j\alpha}.$$

Note that such a phasor cannot easily be interpreted directly in 3-D, though it can be at any instant t.

Time and Frequency Domain

To get back from the frequency domain to the time domain, multiply by $e^{j\omega t}$ and take the real part:

$$\mathbf{E}e^{j\alpha} \Leftrightarrow \text{Re}\left\{\mathbf{E}e^{j\alpha}e^{j\omega t}\right\} = \mathbf{E}\cos\left(\omega t + \alpha\right)$$

(Here we have assumed $\mathbf{E} \in \mathbb{R}^3$.) Also note (capitals are complex; lowercase, real)

$$\begin{aligned}
\text{Re}(A) + \text{Re}(B) &= \text{Re}(A + B) \\
\text{Re}(aA) &= a\text{Re}(A) \\
\frac{\partial}{\partial x}\text{Re}(A) &= \text{Re}\left(\frac{\partial A}{\partial x}\right) \\
\int \text{Re}(A)dx &= \text{Re}\left(\int A dx\right)
\end{aligned}$$

A Justification

Theorem

Suppose $A, B \in \mathbb{C}$. Then

$$Re(Ae^{j\omega t}) = Re(Be^{j\omega t}) \quad \forall t \Rightarrow A = B$$

Proof.

Inserting $\omega t = 0$ into the assumption reveals that

$$\text{Re}(A) = \text{Re}(B).$$

Similarly, inserting $\omega t = \frac{\pi}{2}$ gives

$$\text{Im}(A) = \text{Im}(B).$$

□

Maxwell's Equations in the Phasor Domain

Here, nonscript letters are complex numbers (i.e. phasors)

$$\begin{aligned}
\nabla \times \mathcal{H} &= \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t} \Rightarrow \\
\nabla \times \text{Re}\left\{\mathbf{H}e^{j\omega t}\right\} &= \text{Re}\left\{\mathbf{J}e^{j\omega t}\right\} + \frac{\partial}{\partial t}\text{Re}\left\{\mathbf{D}e^{j\omega t}\right\} \\
\text{Re}\left\{\nabla \times \mathbf{H}e^{j\omega t}\right\} &= \text{Re}\left\{\mathbf{J}e^{j\omega t} + \frac{\partial}{\partial t}\left(\mathbf{D}e^{j\omega t}\right)\right\} \\
\text{Re}\left\{\nabla \times \mathbf{H}e^{j\omega t}\right\} &= \text{Re}\left\{\mathbf{J}e^{j\omega t} + j\omega\mathbf{D}e^{j\omega t}\right\} \\
\nabla \times \mathbf{H} &= \mathbf{J} + j\omega\mathbf{D}
\end{aligned}$$

Maxwell's Equations in the Phasor Domain

1. Faraday's Law:

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

2. Ampère-Maxwell Law:

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D}$$

3. Gauß's Law for the Electric Field:

$$\nabla \cdot \mathbf{D} = q$$

4. Gauß's Law for the Magnetic Field:

$$\nabla \cdot \mathbf{B} = 0$$

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More About Phasors

Of course, in general, field are not time-harmonic, but can be written as a sum of time-harmonic fields using Fourier analysis. In this case,

$$\mathcal{E}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\omega) e^{j\omega t} \mathrm{d}\omega$$

If we are in a general linear medium, then

$$\begin{aligned} \mathcal{D}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau \epsilon(t-\tau) \int_{-\infty}^{\infty} \mathrm{d}\omega \mathbf{E}(\omega) e^{j\omega\tau} \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \mathbf{E}(\omega) \left[\int_{-\infty}^{\infty} \mathrm{d}\tau \epsilon(t-\tau) e^{j\omega\tau} \right] \end{aligned}$$

Constitutive Parameters in the Phasor Domain

The integral in brackets can be simplified with the substitution $\xi = t - \tau$.

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \epsilon(t-\tau) e^{j\omega\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau \epsilon(\xi) e^{j\omega(t-\xi)}$$
$$= e^{j\omega t} \hat{\epsilon}(\omega).$$

Here, $\hat{\epsilon}(\omega)$ is the Fourier transform of $\epsilon(t)$. Substituting this back into the definition of $\mathcal{D}$ gives

$$\mathcal{D}(t) = \int_{-\infty}^{\infty} \hat{\epsilon}(\omega)\mathbf{E}(\omega) e^{j\omega t} \mathrm{d}\omega$$

Constitutive Parameters in the Phasor Domain

Given that this says that $\mathcal{D}(t)$ is the inverse transform of $\hat{\epsilon}(\omega)\mathbf{E}(\omega)$, we have

Constitutive Relations in the Frequency Domain

$$\begin{aligned} \mathbf{D}(\omega) &= \hat{\epsilon}(\omega)\mathbf{E}(\omega) \\ \mathbf{B}(\omega) &= \hat{\mu}(\omega)\mathbf{H}(\omega) \\ \mathbf{J}^{\mathrm{c}}(\omega) &= \hat{\sigma}(\omega)\mathbf{E}(\omega) \end{aligned}$$

Therefore, in the frequency domain, all linear media are simple linear media. Also, notice that the constitutive parameters here can all be complex.

Constitutive Parameters in the Phasor Domain

Both conductive and displacement currents are induced by the field (as opposed to impressed). Thus, induced currents are given by

$$\begin{aligned} \mathbf{J} &= \left[\hat{\sigma}(\omega) + j\omega\hat{\epsilon}(\omega)\right]\mathbf{E} &= \hat{y}(\omega)\mathbf{E} \\ \mathbf{M} &= j\omega\hat{\mu}(\omega)\mathbf{H} &= \hat{z}(\omega)\mathbf{H} \end{aligned}$$

Note that the difference between $\hat{\sigma}$ and $\hat{\epsilon}$ is primarily philosophical and difficult to measure. Since they are combined in a way they such that they need not be separated, it it also irrelevant.

Final Form of the Curl Equations

Given all of these definitions, we can write the curl equations in the form

$$\begin{aligned} -\nabla \times \mathbf{E} &= \hat{z}(\omega)\mathbf{H} + \mathbf{M}^{\mathrm{i}} \\ \nabla \times \mathbf{H} &= \hat{y}(\omega)\mathbf{E} + \mathbf{J}^{\mathrm{i}} \end{aligned}$$

This is the most important form of these equations since it clearly separates sources from field effects.

Outline

Poynting's Theorem Revisited

Because $\mathcal{S} = \mathcal{E} \times \mathcal{H}$, the computation of the Poynting vector is nonlinear, so more care is needed in bring it into the frequency domain. To simplify matters, we can first look at how power is computed in circuit theory. Suppose

$$\begin{aligned} \mathcal{V}(t) &= \text{Re}\left\{ |V|\, e^{j\phi_\text{V}} e^{j\omega t} \right\} \\ \mathcal{I}(t) &= \text{Re}\left\{ |I|\, e^{j\phi_\text{I}} e^{j\omega t} \right\} \end{aligned}$$

Now

$$\begin{aligned} \mathcal{V}(t)\mathcal{I}(t) &= |V||I| \cos(\omega t + \phi_\text{V}) \cos(\omega t + \phi_\text{I}) \\ &= \frac{1}{2} |V||I| \left[\cos(\phi_\text{V} - \phi_\text{I}) + \cos(2\omega t + \phi_\text{V} + \phi_\text{I}) \right] \end{aligned}$$

Poynting's Theorem Revisited

From this expression, it is clear that the average power is given by

Circuit Theory Average Power

$$\overline{P} = \frac{1}{2}|V||I|\cos(\phi_V - \phi_I) = \frac{1}{2}\text{Re}\left\{VI^*\right\}$$

Since the Poynting vector is merely a list of differences of products like the above, we define the

Complex Poynting Vector

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$$

Frequency Domain Poynting Theorem

In terms of this the average "power flow at a point" is given by $\overline{\mathcal{S}} = \text{Re}\{\mathbf{S}\}$. Taking

1. the dot product of the Ampère-Maxwell law with $\mathbf{H}^*$, and
2. subtracting the dot product of the Faraday law with $\mathbf{E}$, and
3. applying a vector identity gives the

Microscopic Complex Poynting Theorem

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -\mathbf{E} \cdot \mathbf{J}^{t*} - \mathbf{H}^* \cdot \mathbf{M}^{t}$$

Integrating over volume givens the

Macroscopic Complex Poynting Theorem

$$\oiint (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} = -\iiint \mathbf{E} \cdot \mathbf{J}^{t*} + \mathbf{H}^* \cdot \mathbf{M}^{t} dv$$

Complex Poynting Theorem in Simple Media

In simple media (i.e., assuming for simplicity that σ, ϵ, and μ are real

$$\begin{aligned}\mathbf{E}\cdot\mathbf{J}^{\mathrm{t}*} &= \sigma|\mathbf{E}|^2 - j\omega\epsilon|\mathbf{E}|^2 + \mathbf{E}\cdot\mathbf{J}^{\mathrm{i}*}\\ \mathbf{H}^*\cdot\mathbf{M}^{\mathrm{t}} &= j\omega\mu|\mathbf{H}|^2 + \mathbf{H}^*\cdot\mathbf{M}^{\mathrm{i}}\end{aligned}$$

Now define

$$\begin{aligned}\overline{p_{\mathrm{d}}} &= \tfrac{1}{2}\sigma|\mathbf{E}|^2\\ \overline{w_{\mathrm{e}}} &= \tfrac{1}{4}\epsilon|\mathbf{E}|^2\\ \overline{w_{\mathrm{m}}} &= \tfrac{1}{4}\mu|\mathbf{H}|^2\\ \overline{p_{\mathrm{s,e}}} &= -\tfrac{1}{2}\mathbf{E}\cdot\mathbf{J}^{\mathrm{i}*}\\ \overline{p_{\mathrm{s,m}}} &= -\tfrac{1}{2}\mathbf{H}^*\cdot\mathbf{M}^{\mathrm{i}}\end{aligned}$$

Complex Poynting Theorem in Simple Media

Finally, defining $p_\mathrm{f} = \nabla \cdot \mathbf{S}$, we can finally interpret Poynting's Theorem

Poynting's Theorem Interpreted

$$\overline{p_\mathrm{s}} = \overline{p_\mathrm{f}} + \overline{p_\mathrm{d}} + 2j\omega(\overline{w_\mathrm{m}} - \overline{w_\mathrm{e}})$$

All but the last term represent real average power flow. The last term represents reactive power, that is, the movement of power back and forth between electric and magnetic form. How do I know this?

Finally, the integral form of this theorem is trivial to derive and interpret.

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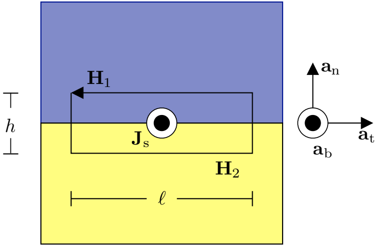
Outline

Boundary Conditions

At an abrupt change in material parameters, Maxwell's equations in differential form do not apply. We can consider the integral form to see what happens at such a boundary. Let $\mathbf{J}_s$ be a surface current (A/m) in the boundary. (It cannot be displacement current.)

$$\lim_{h\to 0}\oint \mathbf{H}\cdot d\mathbf{l} = \lim_{h\to 0}\iint \mathbf{J}^t\cdot d\mathbf{S}$$

$$\begin{aligned}
(\mathbf{H}_2-\mathbf{H}_1)\cdot \ell\mathbf{a}_t &= \mathbf{J}_s\cdot \ell\mathbf{a}_b\\
(\mathbf{H}_2-\mathbf{H}_1)\cdot(\mathbf{a}_n\times\mathbf{a}_b) &= \mathbf{J}_s\cdot\mathbf{a}_b\\
-[\mathbf{a}_n\times(\mathbf{H}_2-\mathbf{H}_1)]\cdot\mathbf{a}_b &= \mathbf{J}_s\cdot\mathbf{a}_b\\
\mathbf{a}_n\times(\mathbf{H}_1-\mathbf{H}_2) &= \mathbf{J}_s
\end{aligned}$$



General Boundary Conditions

Faraday's Law works the same way. General boundary conditions can be derived from the Gauss Laws by looking at a small volume. The results are

General Boundary Conditions

$$
\begin{aligned}
\mathbf{a}_\text{n} \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{J}_\text{s} \\
(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_\text{n} &= \mathbf{M}_\text{s} \\
\mathbf{a}_\text{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= q_\text{S} \\
\mathbf{a}_\text{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= q_{\text{m,S}}
\end{aligned}
$$

Finite Conductivity Boundary

If neither material has infinite conductivity, there can be no charge in the boundary. This leads to the

Dielectric Boundary Conditions

$$\begin{aligned}
\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) &= 0 \\
\mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\
\mathbf{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= 0 \\
\mathbf{a}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0
\end{aligned}$$

Conductor Boundary Conditions

If medium 2 is a perfect electric conductor (PEC), the fields vanish there and there can be electric current and charge in the interface. This gives the

PEC Boundary Conditions

$$\begin{aligned} \mathbf{a}_n \times \mathbf{H}_1 &= \mathbf{J}_s \\ \mathbf{a}_n \times \mathbf{E}_1 &= 0 \\ \mathbf{a}_n \cdot \mathbf{D}_1 &= q_s \\ \mathbf{a}_n \cdot \mathbf{B}_1 &= 0 \end{aligned}$$

PMC boundary conditions are dual.