Guided Waves

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ELEG 648—Guided Waves

Outline

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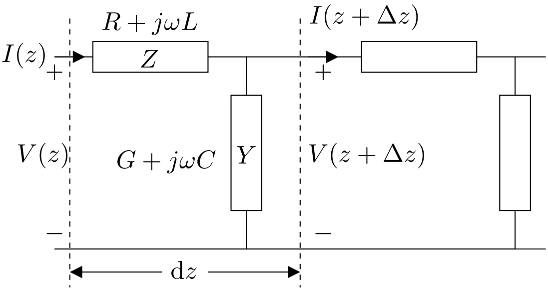
The Circuit Model of Transmission Lines



All circuit quantities (R, L, G, C, Z and Y) are per unit length.

$$
\begin{aligned}
V(z + \Delta z) - V(z) &= -I(z) Z \Delta z \\
I(z + \Delta z) - I(z) &= -V(z) Y \Delta z
\end{aligned}
$$

Telegrapher's Equations

This leads to the

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$$\frac{\mathrm{d}V}{\mathrm{d}z} = -ZI \qquad \frac{\mathrm{d}I}{\mathrm{d}z} = -YV$$

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$$\frac{d^2V}{dz^2} - ZYV = 0 \qquad \frac{d^2I}{dz^2} - ZYI = 0$$

Now define

$$\gamma = jk = \sqrt{ZY}$$

For a lossless transmission line $R = G = 0$, so $ZY < 0$, and γ is imaginary.

Solution of the Telegrapher's Equations

The $+z$-directed voltage wave is

$$V^+(z) = V_0 e^{-\gamma z}$$

From the Telegrapher's Equation we have

$$I^+(z) = -\frac{1}{Z}\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{\gamma}{Z} V_0 e^{-\gamma z}$$

Thus, define the

Characteristic Impedance

$$Z_0 = \frac{V^+(z)}{I^+(z)} = \frac{Z}{\gamma} = \sqrt{\frac{Z}{Y}}$$

Outline

Relationship to Maxwell's Equations

Implicit in the transmission line model is the fact that there is no coupling (mutual impedance or shunt current flow) between sections. This is equivalent to the assumption that

$$E_z = H_z = 0$$

We are also interested in traveling wave solutions, so we assume that the remaining fields have z-dependence $e^{-\gamma z}$. We can thus make the substitution

$$\frac{\partial}{\partial z} \to -\gamma$$

Relationship to Maxwell's Equations

$$
\begin{aligned}
-\hat{z}\mathbf{H} &= \nabla \times \mathbf{E} \\
&= \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -\gamma \\ E_x & E_y & 0 \end{vmatrix}
\end{aligned}
$$

Thus, writing this out

$$
\begin{aligned}
\gamma E_y &= -\hat{z} H_x \\
\gamma E_x &= \hat{z} H_y \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= 0
\end{aligned}
$$

The first two equations (and their duals) can be written in vector form as

$$\mathbf{E} = \eta \mathbf{H} \times \mathbf{u}_z \qquad \eta \mathbf{H} = \mathbf{u}_z \times \mathbf{E}$$

Defining the

Transverse Laplacian

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The last equation and its dual give

Transverse Laplace Equations

$$\nabla_t^2 \mathbf{E} = 0 \qquad \nabla_t^2 \mathbf{H} = 0$$

Relationship to Maxwell's Equations

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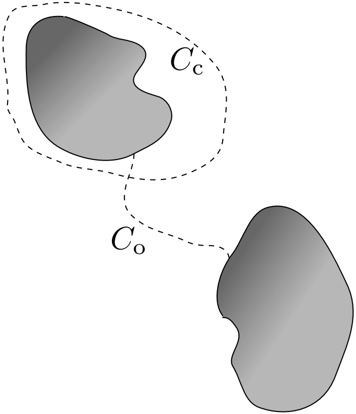

Definitions

$$V = -\int_{C_o} \mathbf{E} \cdot d\mathbf{l} \qquad I = \oint_{C_c} \mathbf{H} \cdot d\mathbf{l}$$

Given the relationship between the transverse fields

$$\begin{aligned} I &= \frac{1}{\eta} \oint_{C_c} \mathbf{u}_z \times \mathbf{E} \cdot d\mathbf{l} \\ &= \frac{1}{\eta} \oint_{C_c} d\mathbf{l} \times \mathbf{u}_z \cdot \mathbf{E} \end{aligned}$$

Relationship to Maxwell's Equations

Shrinking C_c to the surface of the conductor, we can write

$$I = \frac{1}{\eta} \oint_{C_\text{c}} E_\text{n} \, \mathrm{d}l$$

Now, the charge per unit length on the center conductor is

$$q = \epsilon \oint_{C_\text{c}} E_\text{n} \, \mathrm{d}l$$

from boundary conditions. Therefore, since $C = q/V$

$$V = \frac{\epsilon}{C} \oint_{C_\text{c}} E_\text{n} \, \mathrm{d}l$$

Thus

$$Z_0 = \frac{V}{I} = \eta \frac{\epsilon}{C}$$

Relationship to Maxwell's Equations

It may similarly be shown that

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2. In a transverse plane, **E** and **H** are static in nature,
3. By computing C or L using a static solver, the entire transmission line can be characterized.
4. None of this is strictly true if the dielectric is inhomogeneous (even piecewise inhomogeneous).

Outline

- This theory applies to transmission lines with lossy dielectrics perfectly.

Lossy Transmission Lines

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In any case, even with lossy conductors, we can assume the fields are pretty much unchanged and just use the above equations with

$$\gamma = \alpha + j\beta$$

We still have

$$V = V_0 e^{-(\alpha + j\beta)z} \qquad I = \frac{V}{Z_0}$$

The time average power flow is

$$\overline{P_\mathrm{f}} = \frac{1}{2}\mathrm{Re}\left\{VI^*\right\} = \frac{|V_0|^2}{2\mathrm{Re}\left\{Z_0\right\}}e^{-2\alpha z}$$

Now, a power dissipated per unit length, $\overline{P_\mathrm{d}}$ must be responsible for the decrease:

$$\overline{P_\mathrm{d}} = -\frac{\mathrm{d}\overline{P_\mathrm{f}}}{\mathrm{d}z} = 2\alpha\overline{P_\mathrm{f}}$$

Perturbation Theory

The time average power flow is

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$$\overline{P_\text{d}} = -\frac{\text{d}\overline{P_\text{f}}}{\text{d}z} = 2\alpha\overline{P_\text{f}}$$

Therefore, if we can approximate $\overline{P_\text{d}}$ (by, say assuming the current distribution is unchanged by the introduction of loss), we find the

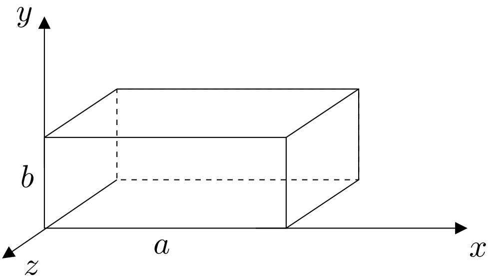
Perturbation Formula for the Dissipation Rate

$$\alpha = \frac{\overline{P_\text{d}}}{2\overline{P_\text{f}}}$$

Outline

The Basics

- Transmission lines are two conductor systems that carry TEM waves.
- Other systems are possible. Anything that guides waves is called a waveguide.
- For now, we study the simplest of these, the rectangular waveguide.



Solution by Superposition

Consider two waves traveling at angles $\pm\phi$ with respect to the y–z plane. Assuming a y-polarization for the **E**-field, we can write

$$\begin{aligned} E_y &= A\left(e^{-jkx\sin\phi} + e^{jkx\sin\phi}\right)e^{-jkz\cos\phi} \\ &= -2jA\sin\left(kx\sin\phi\right)e^{-jkz\cos\phi} \end{aligned}$$

Define

$$\begin{aligned} E_0 &= 2jA \\ k_c &= k\sin\phi \\ \gamma &= jk\cos\phi \end{aligned}$$

Of course $\gamma^2 = k_c^2 - k^2$ since $\sin^2\phi + \cos^2\phi = 1$.

Solution by Superposition

Our superposition is of the form

Modal Electric Field

$$E_y = E_0 \sin\left(k_c x\right) e^{-\gamma z}$$

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The first condition is satisfied automatically; for the second we find

The Cutoff Wavenumber

$$k_c = \frac{n\pi}{a} \qquad n = 1, 2, 3 \ldots$$

Modes

- Each choice of n is called a mode. It represents a solution that can exist without a source (i.e. an eigenfunction).
- This mode is called transverse electric TE because the electric field is normal to the direction of propapagation.
- There are transverse magnetic (TM) modes, in which $H_z = 0$.
- The particular modes we have found here are called the TE_{n0} modes. The reason for this nomenclature will be clear when we solve the problem in general.

Outline

Propagation Constant

We have seen that $\gamma^2 = k_c^2 - k^2$. This means we have (for $k \in \mathbb{R}$)

The Propagation Constant

$$\gamma = \begin{cases} j\beta = j\sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2} & \text{for } k \geq \frac{n\pi}{a} \\ \alpha = \sqrt{\left(\frac{n\pi}{a}\right)^2 - k^2} & \text{for } k \leq \frac{n\pi}{a} \end{cases}$$

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When $\gamma \in j\mathbb{R}$, the wave propagates in the z-direction.
When $\gamma \in \mathbb{R}$, the wave evanesces in the z-direction.
The transition from evanescence to propagation occurs at cutoff, when

$$k = k_c = \frac{n\pi}{a}$$

Cutoff

Associated with the cutoff wavenumber, there is a

Cutoff Frequency

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} = \frac{n}{2\pi\sqrt{\mu\epsilon}}$$

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Below the cutoff frequency, waves evanesce. Above it, they propagate.

Alternative Formula for Cutoff

Note that

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Note that

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Thus, we can write

$$\sqrt{k^2 - k_c^2} = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

and

The Propagation Constant

$$\gamma = \begin{cases} j\beta = jk\sqrt{1 - \left(\frac{f_c}{f}\right)^2} & \text{for } f \geq f_c \\ \alpha = k_c\sqrt{1 - \left(\frac{f}{f_c}\right)^2} & \text{for } f \leq f_c \end{cases}$$

Guide Wavelength and Phase Velocity

The guide wavelength λ_g is the distance over which the phase of propagation in the z-direction changes by 2π. Thus

Guide Wavelength

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{k\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

By the same token, the guided phase velocity is the speed at which a constant phase plane moves down the guide, i.e.

Guided Phase Velocity

$$v_g = f\lambda_g = \frac{f\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{v_p}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

A Word About Phase Velocity

Note that the guided phase velocity is faster than the speed of light! Is this a violation of Einstein's theory of relativity?

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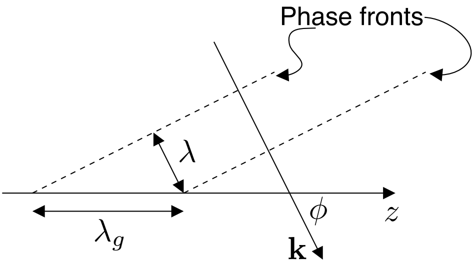
Note that the guided phase velocity is faster than the speed of light! Is this a violation of Einstein's theory of relativity?
Can we use this observation to make a faster-than-light communication system?
Recall our derivation....



A Word About Phase Velocity

From the figure,

$$\lambda_g = \frac{\lambda}{\cos\phi}$$

This demonstrates that the wave is merely traveling obliquely, and that the larger wavelength is merely a geometrical artifact. It also demonstrates that the artifact is controlled by a

Geometrical Factor

$$\cos\phi = \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Magnetic Field

We can compute the magnetic field from Maxwell's Equation

$$\mathbf{H} = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E}$$

This immediately gives

The Magnetic Field

$$\begin{aligned} H_x &= -\frac{\gamma}{j\omega\mu} E_0 \sin\left(k_c x\right) e^{-\gamma z} \\ H_z &= \frac{k_c}{j\omega\mu} E_0 \cos\left(k_c x\right) e^{-\gamma z} \end{aligned}$$

Mode Impedance

Definition of Mode Impedance

$$Z_0 = Z_z = -\frac{E_y}{H_x} = \frac{j\omega\mu}{\gamma}$$

Why is this the relevant definition? Why the minus sign?

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Why is this the relevant definition? Why the minus sign?

We call it Z_0 by analogy with transmission lines. Plugging in for γ we have the frequently more useful expression for

Modal Impedance

$$Z_0 = \begin{cases} \dfrac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} & \text{for } f \geq f_c \\ \dfrac{j\eta}{\sqrt{\left(\frac{f}{f_c}\right)^2-1}} & \text{for } f \leq f_c \end{cases}$$

The Dominant Mode

The mode with the lowest cutoff frequency in a particular waveguide is called the dominant mode.

- For two-conductor, homogeneously-filled waveguiding systems, this is the TEM mode.
- For the rectangular waveguide, it is the TE_{10} mode.

The waveguide should be used between the cutoff of its dominant mode and that of its first higher-order mode. Why?

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The waveguide should be used between the cutoff of its dominant mode and that of its first higher-order mode. Why? Will other modes exist in the guide in this operating bandwidth? Where?

Power Transmission

The transverse fields can be written

$$\begin{aligned} E_y &= E_0 \sin(k_c x)\, e^{-\gamma z} \\ H_x &= -\frac{E_0}{Z_z} \sin(k_c x)\, e^{-\gamma z} \end{aligned}$$

Above cutoff, we can compute

Modal Power

$$\begin{aligned} \overline{P_\text{f}} &= -\frac{1}{2}\int_0^a \int_0^b E_y H_x^* \, \mathrm{d}y \, \mathrm{d}x \\ &= \frac{|E_0|^2}{2Z_0} \int_0^a \int_0^b \sin^2\left(\frac{\pi x}{a}\right) \mathrm{d}y\mathrm{d}x = \end{aligned}$$

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$$= \frac{|E_0|^2}{2Z_0}\int_0^a \int_0^b \sin^2\left(\frac{\pi x}{a}\right) \mathrm{d}y\mathrm{d}x = \frac{|E_0|^2}{4Z_0} ab$$

How does this change below cutoff?

Outline

Dielectric Losses

Dielectric loss essentially changes nothing above:

- γ is complex (rather than pure real or imaginary) since $k \notin \mathbb{R}$
- Z_0 is complex (rather than pure real or imaginary)
- Either one can be simply computed from the formulas given above.

The only issue is that there is no real cutoff in a guide with losses. Why?

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The only issue is that there is no real cutoff in a guide with losses. Why?

If the losses are small, we can ignore the issue and define a real f_c. What happens around this frequency?

Conductive Losses

If there are losses in the conductor, our solution is not strictly correct since the boundary condition has changed. Nonetheless, we can use our current solution with the perturbation theory we discussed for transmission lines.

Perturbation Formula for the Dissipation Rate

$$\alpha = \frac{\overline{P_\mathrm{d}}}{2\overline{P_\mathrm{f}}}$$

We have already computed $\overline{P_\mathrm{f}}$. How can we compute $\overline{P_\mathrm{d}}$?

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We have already computed $\overline{P_f}$. How can we compute $\overline{P_d}$?

$\mathbf{J} = \mathbf{u}_n \times \mathbf{H}$ and we know the surface resistance!

The Left Wall

On the $x = 0$ wall we have

$$
\begin{aligned}
\overline{P_\mathrm{d}}\Big|_{x=0} &= \frac{\mathcal{R}}{2}\int_0^b |H_z|^2\,\mathrm{d}y \\
&= \frac{\mathcal{R}}{2}\left|\frac{k_c E_0}{j\omega\mu}\right|^2 \int_0^b \mathrm{d}y \\
&= \frac{b\mathcal{R}}{2}\left(\frac{k_c}{\eta k}\right)^2 |E_0|^2 \\
&= \frac{b\mathcal{R}}{2}\left(\frac{f_c}{\eta f}\right)^2 |E_0|^2
\end{aligned}
$$

The same amount of power is dissipated on the $x = b$ wall.

The Bottom Wall

On the $y = 0$ wall we have

$$
\begin{aligned}
\overline{P_\mathrm{d}}\Big|_{y=0} &= \frac{\mathcal{R}}{2}\int_0^a |H_z|^2 + |H_x|^2 \,\mathrm{d}x \\
&= \frac{\mathcal{R}}{2}|E_0|^2\int_0^b \left[\frac{\sin^2(\pi x/a)}{Z_0^2} + \left(\frac{f_c}{\eta f}\right)^2 \cos^2\left(\frac{\pi x}{a}\right)\right]\mathrm{d}x \\
&= \frac{\mathcal{R}a}{4}|E_0|^2\left[\frac{1}{Z_0^2} + \left(\frac{f_c}{\eta f}\right)^2\right]
\end{aligned}
$$

The same expression is obtained for $y = b$.

The Dissipation Rate

Adding it all together, we have

$$\overline{P_\text{d}} = \frac{\mathcal{R}\left|E_0\right|^2}{2}\left[\frac{a}{Z_0^2} + \left(\frac{f_c}{\eta f}\right)^2 (2b + a)\right]$$

Therefore, we have the

Dissipation Constant

$$\alpha = \frac{\mathcal{R} Z_0}{ab}\left[\frac{a}{Z_0^2} + \left(\frac{f_c}{\eta f}\right)^2 (2b + a)\right]$$

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What are the units of α?

Resonators

- A resonator is a box with conducting walls, that is, a microwave cavity.
- It works because if the walls are made of perfect conductor, only discrete frequencies can exist in the resonator.
- We can use the same superposition approach to find the modes in a cavity.
- To do this, we will assume that the cavity is simply our waveguide with metal shorting plates at $z = 0$ and $z = c$ (not to be confused with the speed of light!)
- For simplicity, we work with the fundamental mode.

Finding the Resonant Frequency

Superposing waves traveling in the $\pm z$ directions we get

$$
\begin{aligned}
E_y = E_y^+ + e_y^- &= A \sin\left(\frac{\pi x}{a}\right)\left(e^{-j\beta z} - e^{j\beta z}\right) \\
&= E_0 \sin\left(\frac{\pi x}{a}\right) \sin \beta z
\end{aligned}
$$

For this to vanish at $z = c$, we need $\beta c = \pi$. This gives

$$\pi = \frac{2\pi f}{\sqrt{\mu\epsilon}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

Solving this gives the

Resonant Frequency

$$f_r = \frac{1}{2ac}\sqrt{\frac{a^2 + c^2}{\epsilon\mu}}$$

Cavity Fields

This is the fundamental mode of the cavity, the TE_{101} mode. Its magnetic field can be computed from Faraday's Law giving

$$
\begin{aligned}
E_y &= E_0 \sin\frac{\pi x}{a}\sin\frac{\pi z}{c} \\
H_x = \frac{1}{j\omega\mu}\frac{\partial E_y}{\partial z} &= \frac{\pi E_0/c}{jk\eta}\sin\frac{\pi x}{a}\cos\frac{\pi z}{c} \\
&= \frac{aE_0}{j\eta\sqrt{a^2+c^2}}\sin\frac{\pi x}{a}\cos\frac{\pi z}{c} \\
H_z = -\frac{1}{j\omega\mu}\frac{\partial E_y}{\partial x} &= \frac{\pi E_0/a}{jk\eta}\cos\frac{\pi x}{a}\sin\frac{\pi z}{c} \\
&= \frac{cE_0}{j\eta\sqrt{a^2+c^2}} cos \frac{\pi x}{a}\sin\frac{\pi z}{c}
\end{aligned}
$$

Energy Storage

Poynting's Theorem in simple media is

$$\overline{P_{\text{s}}} = \overline{P_{\text{f}}} + \overline{P_{\text{d}}} + 2j\omega(\overline{W_{\text{m}}} - \overline{W_{\text{e}}})$$

In an empty lossless cavity, there is no

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3. power dissipation.

Therefore, one way to think of resonance is

Equality of Electric and Magnetic Energy Storage

$$\overline{W_m} = \overline{W_e}$$

Energy Storage

This energy is the

Average Field Energy in the Cavity

$$\overline{W_\mathrm{m}} = \overline{W_\mathrm{e}} = \frac{\epsilon}{4}\int_0^a\int_0^b\int_0^c |E_0|^2 \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi z}{c}\right)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \frac{\epsilon}{16}|E_0|^2 abc$$

- Because energy is conserved, at any time, the sum of the electric and magnetic energy must be constant.
- Thus, when the magnetic energy goes to zero, the electric energy must be all of the energy.
- Since the whole process is sinusoidal, the maximum must be twice the average.

Total Energy Stored

$$W = 2\overline{W_\mathrm{e}} = \frac{\epsilon}{8}|E_0|^2 abc$$

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The quality factor of a resonant system is defined as

Quality Factor

$$Q = \frac{\omega W}{\overline{P_d}}$$

It is shown in any sophomore circuit theory book that

- the quality factor is the ratio of the bandwidth to the center frequency
- the response of a resonant system is proportional to Q.

Quality Factors

The quality factor due to a lossy dielectric is easily computed:

Dielectric Q

$$Q_d = \frac{\omega \epsilon'' \iiint |E_0|^2 \, dv}{\omega \epsilon' \iiint |E_0|^2 \, dv} = \frac{\epsilon''}{\epsilon'}$$

Conductor Q is tougher, but can be computed using perturbation. For our mode it is

Conductor Q

$$Q_c = \frac{\pi \eta}{2\mathcal{R}} \frac{b\left(a^2 + c^2\right)^{\frac{3}{2}}}{ac\left(a^2 + c^2\right) + 2b\left(a^3 + c^3\right)}$$

Total Quality Factor

Since

$$\frac{1}{Q} = \frac{\overline{P_{\mathrm{d}}}}{\omega W},$$

and since the dissipation is due to both the conductor and the dielectric, we have

Total Quality Factor

$$\frac{1}{Q} = \frac{1}{Q_c} + \frac{1}{Q_d}$$