Cylindrical Radiation and Scattering

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ELEG 648—Radiation and Scattering in Cylindrical Coordinates
Outline

1. Cylindrical Radiation
   - Sources of Cylindrical Radiation
   - Green’s Function and Far Field
   - Wave Transformations
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2. Scattering
   - Scattering from a Circular Cylinder
   - Scattering from a Wedge
   - 2.5-D Problems
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Two-Dimensional Sources

Two-dimensional radiation is created by sources independent of $z$.

The simplest such source (akin to a Hertzian Dipole) is an infinite filament.

Such a current radiates $\text{TM}_z$.

Consider a filamentary current $I$ on the $z$-axis. Since its radiation is

1. Independent of $\phi$,
2. Independent of $z$, and
3. Outwardly traveling,

the magnetic vector potential must be of the form

$$A_z = \mu CH_0^{(2)}(k\rho)$$
Radiation from a Filament

Now, we must have

\[
\lim_{\rho \to 0} \int_{0}^{2\pi} H_\phi \rho \, d\phi = I
\]

Now

\[
H = \frac{1}{\mu} \nabla \times A
\]

implies

\[
H_\phi = -C \frac{\partial}{\partial \rho} \left[ H_0^{(2)}(k\rho) \right] = kCH_1^{(2)}(k\rho).
\]

For small \( \rho \), we have

\[
H_1^{(2)}(k\rho) \to \frac{k\rho}{2} + \frac{j}{\pi} \frac{2}{k\rho}
\]
Substituting this into our equation,

\[ kC \lim_{\rho \to 0} \int_0^{2\pi} \left( \frac{k\rho}{2} + \frac{j}{\pi} \frac{2}{k\rho} \right) \rho \, d\phi = I \]

\[ \frac{2jC}{\pi} \frac{2\pi}{2\pi} = I \]

so finally

\[ C = \frac{1}{4j} \]

and we have the

**Magnetic Vector Potential**

\[ A_z = \frac{\mu I}{4j} H_0^{(2)}(k\rho) \]
Fields of a Filament

Given $A_z$ we can easily compute

**Filamentary Fields**

\[
E_z = -\frac{k^2 I}{4\omega \epsilon} H_0^{(2)}(k\rho)
\]

\[
H_\phi = \frac{kI}{4j} H_1^{(2)}(k\rho)
\]

Using large argument approximations, we can find

**Far Fields**

\[
E_z = -\eta kl \sqrt{\frac{j}{8\pi k\rho}} e^{-jk\rho}
\]

\[
H_\phi = kl \sqrt{\frac{j}{8\pi k\rho}} e^{-jk\rho}
\]
Near the source, $E$ and $H$ are not in phase and have a complex relationship.

Far from the source, $E$ and $H$
- Are in phase,
- Have ratio $\eta$, and
- Decrease as $\rho^{-\frac{1}{2}}$.

The total radiation (per unit length) must be independent of radius (which can be proven directly). Therefore, we can use the far field expression to compute the power:

$$P = - \int_{0}^{2\pi} E_z H_{\phi}^* \rho d\phi = - \int_{0}^{2\pi} \left( -\eta kl \sqrt{\frac{j}{8\pi k\rho}} e^{-jk\rho} \right) \left( k l^* \sqrt{\frac{-j}{8\pi k\rho}} e^{jk\rho} \right) \rho d\phi$$

$$= \frac{\eta k^2 |l|^2}{8\pi k\rho} 2\pi \rho = \frac{\eta k}{4} |l|^2$$
To proceed further, it is useful to define

\[ \rho = xu_x + yu_y \]
\[ \rho' = x'u_x + y'u_y \]

The distance between these points is

\[ |\rho - \rho'| = \sqrt{(\rho - \rho') \cdot (\rho - \rho')} \]
\[ = \sqrt{\rho^2 + (\rho')^2 - 2\rho \cdot \rho' \cos(\phi - \phi')} \]
Radiation Due to a Filament

The radiation due to a current $I$ at the origin we have seen is

$$A_z = \frac{\mu I}{4j} H_0^{(2)}(k\rho).$$

Therefore, if the current is located at $\rho'$, we have

**Filamentary Radiation**

$$A_z(\rho) = \frac{\mu I}{4j} H_0^{(2)}(k|\rho - \rho'|)$$

Similarly, a filamentary magnetic current $K$ at $\rho'$ radiates

$$F_z(\rho) = \frac{\epsilon K}{4j} H_0^{(2)}(k|\rho - \rho'|)$$
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Green’s Function

If a current density $J_z(\rho)$ is independent of $z$, we can think of it as a bundle of filaments with current $J_z(\rho)dS$. Then we have the

**General Formulas for Two-Dimensional Radiation**

$$A_z(\rho) = \frac{\mu}{4j} \iint J_z(\rho') H_0^{(2)}(k|\rho - \rho'|)dS'$$

$$F_z(\rho) = \frac{\epsilon}{4j} \iint M_z(\rho') H_0^{(2)}(k|\rho - \rho'|)dS'$$

From here, the fields can be computed in the usual way.
Far Fields

In the far field, we can make the standard approximation

$$|\rho - \rho'| \to \rho - \rho' \cos(\phi - \phi')$$

Similarly, for $x \to \infty$,

$$H_0^{(2)}(x) \to \sqrt{\frac{2j}{\pi x}} e^{-jx}.$$

Combining these (and remembering to use a simpler approximation in the denominator) we have

The Far Field

$$A_z(\rho) = \mu \frac{e^{-jk\rho}}{\sqrt{8j\pi k\rho}} \int\int J_z(\rho') e^{jk\rho' \cos(\phi - \phi')} dS'$$

$$F_z(\rho) = \epsilon \frac{e^{-jk\rho}}{\sqrt{8j\pi k\rho}} \int\int M_z(\rho') e^{jk\rho' \cos(\phi - \phi')} dS'$$
Far Fields

- We can compute the far fields using the definitions of $A$ and $F$.
- The results are similar to three-dimensional results.

**Relationship Between $E$ and $H$**

\[
E_\phi = \eta H_z \quad E_z = -\eta H_\phi
\]

**Far Electric Fields**

\[
E_\phi = -j\omega \eta F_z \\
E_z = -j\omega A_z
\]
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We often want to express plane waves in cylindrical wave functions. The result must be

- Finite at the origin, and
- $2\pi$-periodic.

Thus

$$e^{-jx} = e^{-j\rho \cos \phi} = \sum_{n=-\infty}^{\infty} a_n J_n(\rho) e^{in\phi}.$$

To find the $a_n$, use orthogonality

$$\int_{0}^{2\pi} e^{-j\rho \cos \phi} e^{-jm\phi} d\phi = \sum_{n=-\infty}^{\infty} a_n J_n(\rho) \int_{0}^{2\pi} e^{i(n-m)\phi} d\phi.$$
Plane Wave Expansion

Using orthogonality

\[ \int_{0}^{2\pi} e^{-j\rho\cos\phi} e^{-jm\phi} d\phi = 2\pi a_m J_m(\rho) \]

The integral on the right-hand side is well-known

\[ J_n(x) = \frac{j^n}{2\pi} \int_{0}^{2\pi} e^{-jx\cos\phi} e^{-jm\phi} d\phi \]
Substituting this in, we find

\[ a_m = j^{-m} \]

and finally

The Plane Wave Expansion of Cylindrical Waves

\[ e^{-jx} = e^{-j \rho \cos \phi} = \sum_{n=-\infty}^{\infty} j^{-n} J_n(\rho) e^{jn\phi} \]

How might this formula be modified for waves travelling in other directions?
The Addition Theorem

- We are also interested in expanding the field of a filament with respect to a different center.
- We have seen that a filamentary current \( I \) located at \( \rho = \rho' \) radiates a field with \( A_z = \mu \psi \) with
  \[
  \psi(\rho, \phi) = \frac{I}{4\pi} H_0^{(2)}(k|\rho - \rho'|).
  \]
- We can think of the current generating this field as a current sheet, confined to the \( \rho = \rho' \) cylinder, with
  \[
  J_z(\phi) = \frac{I \delta(\phi - \phi')}{\rho}
  \]
  where the denominator ensures that
  \[
  \int_0^{2\pi} J_z(\phi) \rho d\phi = I
  \]
The Addition Theorem

We can expand the field inside and outside the tube of current:

\[ \psi(\rho, \phi) = \begin{cases} 
\sum_{n=-\infty}^{\infty} a_n^- J_n(k\rho)e^{jn\phi} & \text{for } \rho < \rho' \\
\sum_{n=-\infty}^{\infty} a_n^+ H_n^{(2)}(k\rho)e^{jn\phi} & \text{for } \rho > \rho'
\end{cases} \]

Since \( E_z \propto \psi \), \( \psi \) must be continuous at \( \rho' \). Thus

\[ a_n^- J_n(k\rho') = a_n^+ H_n^{(2)}(k\rho'). \]

This is solved if we let

\[ a_n^- = H_n^{(2)}(k\rho')a_n \]
\[ a_n^+ = J_n(k\rho')a_n \]
The Addition Theorem

We now have

\[ \psi(\rho, \phi) = \begin{cases} 
\sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(k\rho') J_n(k\rho) e^{jn\phi} & \text{for } \rho < \rho' \\
\sum_{n=-\infty}^{\infty} a_n J_n(k\rho') H_n^{(2)}(k\rho) e^{jn\phi} & \text{for } \rho > \rho' 
\end{cases} \]

Now

\[ H_\phi = -\frac{\partial \psi}{\partial \rho} = \begin{cases} 
-k \sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(k\rho') J'_n(k\rho) e^{jn\phi} & \text{for } \rho < \rho' \\
-k \sum_{n=-\infty}^{\infty} a_n J_n(k\rho') H_n^{(2)'}(k\rho) e^{jn\phi} & \text{for } \rho > \rho' 
\end{cases} \]
The Addition Theorem

From boundary conditions, we have

\[ H_\phi(\rho = \rho'^+) - H_\phi(\rho = \rho'^-) = J_z \]

We thus have

\[ -k \sum_{n=-\infty}^{\infty} a_n [J_n(k\rho')H_n^{(2)'}(k\rho') - H_n^{(2)}(k\rho')J_n'(k\rho')] e^{in\phi} = J_z \]

The standard Wronskian formula gives

\[ J_n(k\rho')H_n^{(2)'}(k\rho') - H_n^{(2)}(k\rho')J_n'(k\rho') = \frac{-2j}{\pi k\rho} \]
The Addition Theorem

Therefore, plugging in,

\[ \frac{2j}{\pi \rho} \sum_{n=-\infty}^{\infty} a_n e^{jn\phi} = \frac{l \delta(\phi - \phi')}{\rho} \]

We can now find the \( a_n \) using orthogonality:

\[ \frac{j}{\pi} \sum_{n=-\infty}^{\infty} a_n \int_{0}^{2\pi} e^{in\phi} e^{-jm\phi} d\phi = l \int_{0}^{2\pi} \delta(\phi - \phi') e^{-jm\phi} d\phi \]

\[ 4ja_m = le^{-jm\phi'} \]

\[ a_m = \frac{l}{4j} e^{-jm\phi'} \]
The Addition Theorem

We thus find that the field of a filament can be expanded as

\[ A_z = \mu \psi \text{ with } \]

\[ \psi(\rho, \phi) = \begin{cases} 
\frac{i}{4j} \sum_{n=-\infty}^{\infty} H_n^{(2)}(k\rho')J_n(k\rho)e^{in(\phi'-\phi)} & \text{for } \rho < \rho' \\
\frac{i}{4j} \sum_{n=-\infty}^{\infty} J_n(k\rho')H_n^{(2)}(k\rho)e^{in(\phi'-\phi)} & \text{for } \rho > \rho' 
\end{cases} \]

Equating our earlier expression, this gives

The Addition Theorem

\[ H_0^{(2)}(k|\rho - \rho'|) = \begin{cases} 
\sum_{n=-\infty}^{\infty} J_n(k\rho)H_n^{(2)}(k\rho')e^{in(\phi'-\phi')} & \text{for } \rho < \rho' \\
\sum_{n=-\infty}^{\infty} J_n(k\rho')H_n^{(2)}(k\rho)e^{in(\phi'-\phi')} & \text{for } \rho > \rho' 
\end{cases} \]
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Consider a PEC cylinder of radius $a$.
Let it be excited by a $z$-polarized incident wave

$$E_z^i = E_0 e^{-jkx} = E_0 e^{-j\rho\cos\phi}$$

Using our plane wave expansion, we may write

$$E_z^i = E_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n(k \rho) e^{in\phi}$$

The total field is of course

$$E_z = E_z^i + E_z^s;$$

it must vanish on the surface of the cylinder.
We may expand the scattered field as
We may expand the scattered field as

$$E_s^z = E_0 \sum_{n=-\infty}^{\infty} j^{-n} a_n H_n^{(2)}(k\rho)e^{jn\phi}$$

The total field on the surface of the cylinder is thus

$$E_z = E_0 \sum_{n=-\infty}^{\infty} j^{-n} \left[ J_n(ka) + a_n H_n^{(2)}(ka) \right] e^{jn\phi} = 0$$

Therefore

$$a_n = -\frac{J_n(ka)}{H_n^{(2)}(ka)}$$
From here we can find any information about the scattering we want. For instance

\[ J_z = H_\phi \bigg|_{\rho=a} = \frac{1}{j\omega \mu} \frac{\partial E_z}{\partial \rho} \bigg|_{\rho=a} \]

Now

\[
\frac{1}{j\omega \mu} \left. \frac{\partial E_z}{\partial \rho} \right|_{\rho=a} = \frac{E_0}{j\omega \mu} \sum_{n=-\infty}^{\infty} j^{-n} \left[ J'_n(ka) + a_n H_n^{(2)'}(ka) \right] e^{jn\phi} \\
= \frac{-E_0}{j\omega \mu} \sum_{n=-\infty}^{\infty} j^{-n} \left[ H_n^{(2)}(ka)J'_n(ka) + J_n(ka)H_n^{(2)'}(ka) \right] \frac{e^{jn\phi}}{H_n^{(2)}(ka)}
\]
Using the Wronskian relation, we find

The Surface Current

\[ J_s(\phi) = \frac{-2E_0}{\omega \mu \pi a} \sum_{n=-\infty}^{\infty} \frac{j^{-n}e^{jn\phi}}{H_n^{(2)}(ka)} \]

For a thin wire, the first term dominates and we can even write

\[ I = 2\pi \frac{E_0}{j\omega \mu \log ka} \]
Of course, the exact scattered field is given above:

\[ E_{Z}^{s} = E_{0} \sum_{n=-\infty}^{\infty} j^{-n} a_{n} H_{n}^{(2)}(k\rho)e^{jn\phi} \]

Using the far field formula for Hankel Functions, we find the Scattered Far Field

\[ E_{Z}^{s} = E_{0}e^{-jk\rho} \sqrt{\frac{2}{\pi k\rho}} \sum_{n=-\infty}^{\infty} a_{n}e^{jn\phi} \]

We may treat the other polarization in the same way.
We can also consider the scattering due to a current $I$ located at $\rho'$. In this case, the incident field is

$$E^i_z = \frac{-k^2 I}{4\omega \varepsilon} H_0^{(2)}(k|\rho - \rho'|) = \frac{-k^2 I}{4\omega \varepsilon} \sum_{n=-\infty}^{\infty} H_n^{(2)}(k\rho')J_n(k\rho)e^{jn(\phi - \phi')} \quad \text{for } \rho < \rho'$$

We can write the scattered field in the form

$$E^s_z = \frac{-k^2 I}{4\omega \varepsilon} \sum_{n=-\infty}^{\infty} c_n H_n^{(2)}(k\rho')H_n^{(2)}(k\rho)e^{jn(\phi - \phi')}$$
Scattering due to Filamentary Excitation

From the preceding, it is obvious that

\[ c_n = -\frac{J_n(ka)}{H_n^{(2)}(ka)} \]

The final solution is thus the

Total Field from Filamentary Scattering

\[ E_z = \frac{k^2 I}{4\omega\epsilon} \sum_{n=-\infty}^{\infty} H_n^{(2)}(k\rho^\ast) \left[ J_n(k\rho) + c_n H_n^{(2)}(k\rho) \right] e^{jn(\phi - \phi')} \]

Here

\[ \rho^\ast = \min(\rho, \rho') \quad \rho = \max(\rho, \rho') \]
Our answer is symmetrical with respect to the substitution \( \rho \leftrightarrow \rho' \) and \( \phi \leftrightarrow \phi' \). Why?
Observations

- Our answer is symmetrical with respect to the substitution $\rho \leftrightarrow \rho'$ and $\phi \leftrightarrow \phi'$. Why?

- The coefficients $c_n$ are the same as the $a_n$ from the last problem. Why?
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- The coefficients $c_n$ are the same as the $a_n$ from the last problem. Why? How can we assure they are the same for every problem?
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It is often said this problem is a generalization of the plane wave scattering problem. Why?
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Observations

- Our answer is symmetrical with respect to the substitution $\rho \leftrightarrow \rho'$ and $\phi \leftrightarrow \phi'$. Why?
- The coefficients $c_n$ are the same as the $a_n$ from the last problem. Why? How can we assure they are the same for every problem?
- It is often said this problem is a generalization of the plane wave scattering problem. Why? How can we recover the plane wave solution from this one?
- How else might we approach the solution to this problem?
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Now we look at filamentary scattering from a wedge.

The wedge is composed of two half planes $\phi = \alpha$ and $\phi = 2\pi - \alpha$.

The filament carries current $I$ and is located at $(\rho', \phi')$. 
We can write the

Total Electric Field

\[ E_z = \sum_{\nu} a_{\nu} J_{\nu}(k\rho_\leq) H_{\nu}^{(2)}(k\rho_\geq) \sin[\nu(\phi' - \alpha)] \sin[\nu(\phi - \alpha)] \]

- The dependence on \( \rho \) and \( \phi \) is clear enough.
- How can I just write a dependence on \( \rho' \) and \( \phi' \)?
- Since \( E_z(\alpha) = E_z(2\pi - \alpha) = 0 \),

\[ \nu = \nu_m = \frac{m\pi}{2(\pi - \alpha)} \]
Expansion of the Current

The current can be thought of as a cylindrical sheet current (A/m) with an impulsive distribution:

\[ J_z = I \frac{\delta(\phi - \phi')}{\rho'} \]

This current can be expanded in a Fourier series in the usual way:

\[ J_z = \frac{I}{(\pi - \alpha)\rho'} \sum_{m=1}^{\infty} \sin \nu_m (\phi' - \alpha) \sin \nu_m (\phi - \alpha) \]

From boundary conditions, this current is related to the field by

\[ J_z = H_\phi (\rho'^+) - H_\phi (\rho'^-) \]
Scattering from a Wedge

From the usual equations we can find

\[
H_\phi = \begin{cases} 
\frac{k}{j\omega \mu} \sum_{m=1}^{\infty} a_{\nu m} H^{(2)}_{\nu m}(k \rho') J'_{\nu m}(k \rho) \sin \nu_m (\phi' - \alpha) \sin \nu_m (\phi - \alpha) & \rho < \rho' \\
\frac{k}{j\omega \mu} \sum_{m=1}^{\infty} a_{\nu m} H^{(2)*}_{\nu m}(k \rho') J_{\nu m}(k \rho) \sin \nu_m (\phi' - \alpha) \sin \nu_m (\phi - \alpha) & \rho > \rho'
\end{cases}
\]

Using the Wronskian, we can write

\[
J_z = -\frac{2}{\omega \mu \pi \rho'} \sum_{m=1}^{\infty} a_{\nu m} \sin \nu_m (\phi' - \alpha) \sin \nu_m (\phi - \alpha)
\]

Equating the two expressions for \(J_z\) we find

\[
a_{\nu m} = -\frac{\omega \mu \pi l}{2(\pi - \alpha)}
\]
We can generalize our solution to plane wave scattering by taking the limit as the limit as the source recedes. The original incident field was

$$E^i_z = \frac{-k^2 l}{4\omega \epsilon} H_0^{(2)}(k|\rho - \rho'|)$$

Using the large argument approximation and the far field approximation of $|\rho - \rho'|$ we write

$$E^i_z = -\frac{\omega \mu l}{4} \sqrt{\frac{2j}{\pi k \rho'}} e^{-jk\rho'} e^{jk\rho \cos(\phi - \phi')}$$
This can be written as

\[ E_z^i = E_0 e^{jk\rho \cos(\phi-\phi')} \]

where

\[ E_0 = \frac{-\omega \mu l}{4} \sqrt{\frac{2j}{\pi k\rho'}} e^{-jk\rho'} \]

Why is \( E_0 \) dependent on \( \rho \)?
This can be written as

\[ E_z^i = E_0 e^{jk\rho \cos(\phi - \phi')} \]

where

\[ E_0 = -\frac{\omega \mu l}{4} \sqrt{\frac{2j}{\pi k\rho'}} e^{-jk\rho'} \]

Why is \( E_0 \) dependent on \( \rho' \)? For large \( \rho' \) our solution is

\[ E_z = \sqrt{\frac{2j}{\pi k\rho'}} \sum_{m=1}^{\infty} a_n j^\nu_m J_\nu_m(k\rho) \sin \nu_m(\phi' - \alpha) \sin \nu_m(\phi - \alpha) \]
Plugging in for $l$ and $a_n$ we find

The Total Field

$$E_z = \frac{2\pi E_0}{\pi - \alpha} \sum_{m=1}^{\infty} j^{\nu_m} J_{\nu_m}(k\rho) \sin \nu_m(\phi' - \alpha) \sin \nu_m(\phi - \alpha)$$
Cylindrical Radiation

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2.5 Dimensions?

- 2.5 dimensions is a “term of art,” not an actual physical description.
- The geometry of the problem is assumed two-dimensional.
- The source, however, may be three dimensional.
- The problems are attacked by superposition (i.e. Fourier Methods).

Suppose

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \psi = 0
\]
Since the boundaries are independent of $z$, we can define $\psi$ in terms of

$$
\psi(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(x, y, k_z) e^{jk_zz} dk_z
$$

Substituting into the Helmholtz equation, we find that

$$
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (k^2 - k_{z}^2) \right] \tilde{\psi} = 0
$$

We define (as usual) $k_{\rho}^2 = k^2 - k_{z}^2$. 
A Filament

- Suppose we have a filament of current \( I(z) \).
- We of course have a TM\(_z\) field, \( A_z = \mu \psi \).
- The wave function can be written in the form

\[
\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k_z) H_0^{(2)}(k_\rho \rho) dk_z.
\]

- Here, then, the transform of the function is

\[
\tilde{\psi} = f(k_z) H_0^{(2)}(k_\rho \rho).
\]
A Filament

The azimuthal magnetic field is

$$\tilde{H}_\phi = -\frac{\partial \tilde{\psi}}{\partial \rho} = -k_\rho f(k_z) H_0^{(2)'}(k_\rho \rho).$$

Now, by Ampère’s law,

$$\lim_{\rho \to 0} \oint \tilde{H}_\phi d\ell = \tilde{l}$$

For small $\rho$,

$$\tilde{H}_\phi = k_\rho f(k_z) \frac{d}{dx} \left( \frac{2j}{\pi} \ln \frac{\gamma x}{2} \right)_{x=k_\rho} = \frac{2j}{\pi \rho} f(k_z)$$
A Filament

Thus

\[ \lim_{\rho \to 0} \int \tilde{H}_\phi \, d\ell = 2\pi \rho \frac{2j}{\pi \rho} f(k_z) = \tilde{l} \]

or

\[ f(k_z) = \frac{\tilde{l}}{4j} \]

Finally, we find the

2.5-D Filament Solution

\[ \psi = \frac{1}{8\pi j} \int_{-\infty}^{\infty} \tilde{l}(k_z) H_0^{(2)} \left( \rho \sqrt{k^2 - k_z^2} \right) e^{jk_z z} \, dk_z \]
Another Way

We can always make a line current out of dipoles. We thus have

**The Spatial Approach**

\[
\psi = \int_{-\infty}^{\infty} I(z') \frac{e^{-jk\sqrt{\rho^2 + (z-z')^2}}}{4\pi \sqrt{\rho^2 + (z - z')^2}} \, dz'
\]

Now, suppose we choose \( I(z') = I\ell \delta(z') \). Then

\[
\psi = I\ell \frac{e^{-jkr}}{4\pi r}.
\]

Also, we have \( \tilde{I}(k_z) = I\ell \), since the Fourier Transform of the delta function is unity.
A New Identity

Plugging this transform into the 2.5-D solution,

\[ \psi = \frac{I\ell}{8\pi j} \int_{-\infty}^{\infty} H_0^{(2)}(\rho \sqrt{k^2 - k_z^2}) e^{jk_z z} dk_z \]

Setting these two expressions to each other, we find

\[ \frac{e^{ikr}}{r} = \frac{1}{2j} \int_{-\infty}^{\infty} H_0^{(2)}(\rho \sqrt{k^2 - k_z^2}) e^{jk_z z} dk_z \]