Adaptive Filters

Hidayatullah Ahsan

Department of Electrical and Computer Engineering, Boise State University

April 12, 2010

Motivation

- d be a scalar-valued random variable (desired output signal)
  - $E[d] = 0$
  - $E\left[d^2\right] = \sigma_d^2$
  - With realization $\{d(i) : i = 0, 1, 2, \ldots\}$
- $u \in \mathbb{R}^M \left(\mathbb{C}^M\right)$ be a random vector (input signal)
  - $E[u] = 0$
  - $R_u = E[u^* u] > 0$
  - $R_{du} = E[du^*]$
  - With realization $\{u_i : i = 0, 1, 2, \ldots\}$

Problem

We want to solve

$$\min_{\omega} E\left[(d - u\omega)^2\right] \qquad (1)$$

where ω is the weights vector.

Solution

By the steepest-descent algorithm

$$\omega^o = R_u^{-1} R_{du}$$

which can be approximated by the following recursion with constant step-size $\mu > 0$

$$\omega_i = \omega_{i-1} + \mu \left[R_{du} - R_u \omega_{i-1} \right], \ \omega_{-1} = \text{initial guess.}$$

Remark
R_u and R_{du} should be **known**, and **fixed**.

Adaptive Filters

- "Smart Systems"
  - Learning: Learns the Statistics of the Signal
  - Tracking: Adjusts the Behavior to Signal Variations
- Practicle Reasons for Using Adaptive Filters
  - Lack of Statistical Information
    - Mean, Variance, Auto-correlation, Cross-correlation, etc
  - Variation in the Statistics of the Signal
    - Signal with Noise Randomly Moving in a Know/Unknown Bandwith with Time
- Types of Adaptive Filters
  - Least Mean Square (LMS) Filters
    - Normalized LMS Filters
    - Non-Canonical LMS Filters
  - Recursive Least Square (RLS) Filters
    - QR-RLS Filters

Least Mean Square (LMS) Filters

Development Using Instantaneous Approximation

- At time index i approximate
  - $R_u = E\left[u^* u\right]$ by $\widehat{R}_u = u_i^* u_i$
  - $R_{du} = E\left[du^*\right]$ by $\widehat{R}_{du} = d\left(i\right) u_i^*$
- Corresponding steepest-descent itteration

$$\omega_i = \omega_{i-1} + \mu u_i^* \left[d\left(i\right) - u_i \omega_{i-1}\right], \ \omega_{-1} = \text{initial guess}$$

where $\mu > 0$ is a constant stepsize.

- *Remarks*
  - Also known as the Widrow-Hoff algorithm.
  - Commonly used algorithm for simplicity.
  - μ is choosen to be 2^{-m} for $m \in \mathbb{N}$.
- *Computational Cost*
  - Complex-valued Signal: $8M + 2$ real multiplications, $8M$ real additions.
  - Real-values Signal: $2M + 1$ real multiplications, $2M$ real additions.

Least Mean Square (LMS) Filters

An Illustration

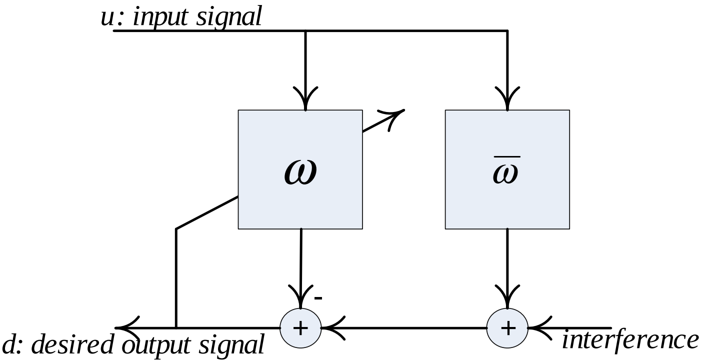
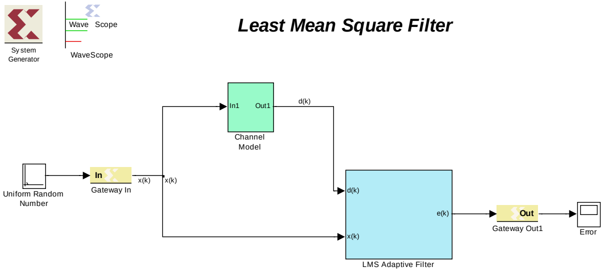
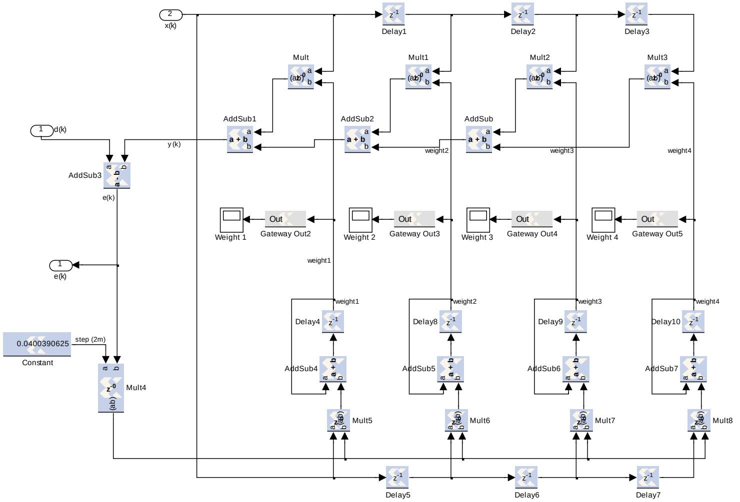


Figure: An Illustration for Least Mean Square Filter


Least Mean Square Filter
System Generator
Wave
Scope
WaveScope
In1
Out1
d(k)
Channel Model
Uniform Random Number
In
Gateway In
x(k)
x(k)
d(k)
x(k)
e(k)
LMS Adaptive Filter
Out
Gateway Out1
Error

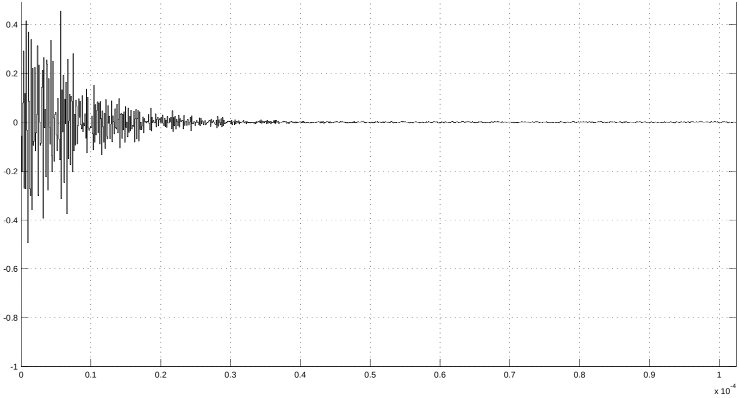
Least Mean Square (LMS) Filters

An Application (2/3)



Least Mean Square (LMS) Filters

An Application (Error)(3/3)



Time offset: 0

Normalized Least Mean Square (LMS) Filters

- Solution to (1) using regularized Newton Recursion

$$\omega_i = \omega_{i-1} + \mu(i)\left[\varepsilon(i) I - R_u\right]^{-1}\left[R_{du} - R_u \omega_{i-1}\right], \; \omega_{-1} = \text{initial guess.}$$

  where $\mu(i) > 0$ is the stepsize and $\varepsilon(i)$ is the regularization factor.
- With $\mu(i) = \mu > 0$ and $\varepsilon(i) = \varepsilon$ fixed for all i, using the instantaneous approximation

$$\begin{aligned} \omega_i &= \omega_{i-1} + \mu\left[\varepsilon I - u_i^* u_i\right]^{-1} u_i^*\left[d(i) - u_i \omega_{i-1}\right] \\ &= \cdots \\ &= \omega_{i-1} + \frac{\mu}{\varepsilon + \|u_i\|^2} u_i^*\left[d(i) - u_i \omega_{i-1}\right] \end{aligned}$$

- *Computational Cost*
  - Complex-valued Signal: $10M + 2$ real multiplications, $10M$ real additions and one real division.
  - Real-values Signal: $3M + 1$ real multiplications, $3M$ real additions and one real division.

Other LSM-Type Techniques

- Power Normalization
  - Replace $\frac{\mu}{\varepsilon + \|u_i\|^2}$ with $\frac{\mu / M}{\varepsilon / M + \|u_i\|^2 / M}$, where M is the order of the filter.

Definition

Non-Blind algorithms are so called since they employ a reference sequence $\{d(i) : i = 0, 1, 2, \ldots\}$.

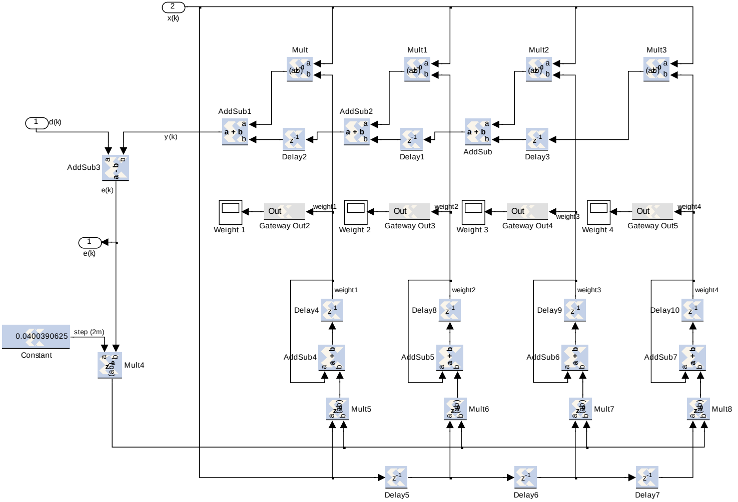
Non-Blind Algorithm

- Leaky LMS Algorithm
- LMF Algorithm
- LMMN Algorithm

Blind Algorithm

- CMA1-2, NCMA Algorithm
- CMA2-2 Algorithm
- RCA Algorithm
- MMA Algorithm

Non-Canonical Least Mean Square (LMS) Filters



Recursive Least Square (RLS) Filters

- Solution to (1) using regularized Newton Recursion

$$\omega_i = \omega_{i-1} + \mu(i)\left[\varepsilon(i) I - R_u\right]^{-1}\left[R_{du} - R_u\omega_{i-1}\right], \; \omega_{-1} = \text{initial guess.}$$

where $\mu(i) > 0$ is the stepsize and $\varepsilon(i)$ is the regularization factor.

- Approximate R_u by $\widehat{R}_u = \frac{1}{i+1}\sum_{j=0}^{i}\lambda^{i-j}u_j^* u_j$, i.e. by an exponential average of previous regressors.
  - If $\lambda = 1$ then all regressors have equal weight.
  - If $0 \ll \lambda < 1$ then recent regressors $(i-1, i-2, \ldots)$ are more relevant and remote regressors are forgotten.
  - Generally λ is choosen so that $0 \ll \lambda < 1$, therefore RLS has a memory or forgetting property.
- Assume $\mu(i) = \frac{1}{i+1}$ and $\varepsilon(i) = \frac{\lambda^{i+1}\varepsilon}{i+1}$ for all i. Then $\varepsilon(i) \to 0$ as $i \to \infty$, i.e. as time increases the regularization factor disappears.

Recursive Least Square (RLS) Filters

- Development using the instantaneous approximation

$$\omega_i = \omega_{i-1} + \left[\lambda^{i+1}\varepsilon I + \sum_{j=0}^{i} \lambda^{i-j} u_j^* u_j\right]^{-1} u_i^* \left[d\left(i\right) - u_i\omega_{i-1}\right]$$

- Define

$$\Phi_i = \lambda^{i+1}\varepsilon I + \sum_{j=0}^{i} \lambda^{i-j} u_j^* u_j$$

then

$$\Phi_i = \lambda\Phi_{i-1} + u_i^* u_i,\ \Phi_{-1} = \varepsilon I$$

- The matrix inversion formula for $P_i = \Phi_i^{-1}$ is given by

$$P_i = \lambda^{-1}\left[P_{i-1} - \frac{\lambda^{-1} P_{i-1} u_i^* u_i P_{i-1}}{1 + \lambda^{-1} u_i P_{i-1} u_i^*}\right],\ P_{-1} = \varepsilon^{-1} I$$

- With the simplification we obtain the RLS algorithm

$$\omega_i = \omega_{i-1} + P_i u_i^* \left[d\left(i\right) - u_i\omega_{i-1}\right],\ i = 0, 1, 2, \ldots$$

Least-Squares Problem

- Replace $E\left[|d-u\omega|^2\right]$ by $\frac{1}{N}\sum_{i=0}^{N-1}|d-u\omega|^2$, then problem (1) is modified to

$$\min_{\omega}\sum_{i=0}^{N-1}|d(i)-u_i\omega|^2=\min_{\omega}\|y-H\omega\|^2 \tag{2}$$

where

$$\begin{aligned} y &= \begin{bmatrix} d(0) & d(1) & \cdots & d(N-1) \end{bmatrix} \text{ and} \\ H &= \begin{bmatrix} u_0^T & u_1^T & \cdots & u_{N-1}^T \end{bmatrix}^T \end{aligned}$$

- Weighted Least-Squares
  - Let W be a weights matrix, then (2) can be modified to $\min_{\omega}(y-H\omega)^* W (y-H\omega)$.
- Regularized Least-Squares
  - Let $\Pi > 0$ be a regularization matrix, then (2) can be modified to $\min_{\omega}\left[\omega^*\Pi\omega+\|y-H\omega\|^2\right]$.

Not Presented

- Weighted, Regularized and Weighted and Regularized Least-Square Algorithms
- Array Methods for Adaptive Filters
- Given's Rotation
- CORDIC Cells
- QR-Recursive Least Square Algorithm

References

- Dr. Rafla's Notes for ECE 635
- Adaptive Filters by Ali H. Sayed