Lecture 27

A1

NRZ DAC

$$e_j[n] = \left(y[n]-y[n-1]\right)\cdot\frac{\Delta T_s(n)}{T}$$

clocking uncertainty.
↳ iid
↳ white process.

variance $\sigma^2_{e_j} = \sigma^2_{dy}\cdot\frac{\sigma^2_{\Delta T_s}}{T^2}$

$\sigma^2_{e_j}$ is depended on input signal through y

$dy[n] = y[n] - y[n-1]$.

$\Rightarrow$

$$\sigma^2_{dy} \simeq \frac{\sigma^2_{LSB}}{\pi}\int_0^{\pi}\left|(1-e^{-j\omega})\,NTF(e^{j\omega})\right|^2 d\omega.$$

$$\sigma^2_{LSB} = \frac{\Delta^2}{12}$$

Assumption ↳ modulator is linear
↳ quantization noise is additive } works well with a multibit quantizer.

$\Rightarrow$ **idle channel jitter noise**
in-band noise due to jitter

$$J = \frac{\sigma^2_{\Delta T_s}}{T^2}\cdot\frac{\sigma^2_{LSB}}{\pi\, OSR}\int_0^{\pi}\left|(1-e^{-j\omega})NTF(e^{j\omega})\right|^2 d\omega \longrightarrow \left(\frac{\sigma^2_{\Delta T_s}}{T^2}\cdot\sigma^2_{dy}\cdot\frac{1}{OSR}\right)$$

$\Rightarrow$

J depends on the area A_J under the curve

$$\left|(1-e^{-j\omega})NTF(e^{j\omega})\right|^2 d\omega.$$

$$\Rightarrow \quad A_J = \int_0^{\pi}\left|(1-e^{-j\omega})NTF(e^{j\omega})\right|^2 d\omega$$

In band Quantization Noise

$$Q = \frac{\sigma^2_{LSB}}{\pi}\int_0^{\pi/OSR}\left|NTF(e^{j\omega})\right|^2 d\omega$$

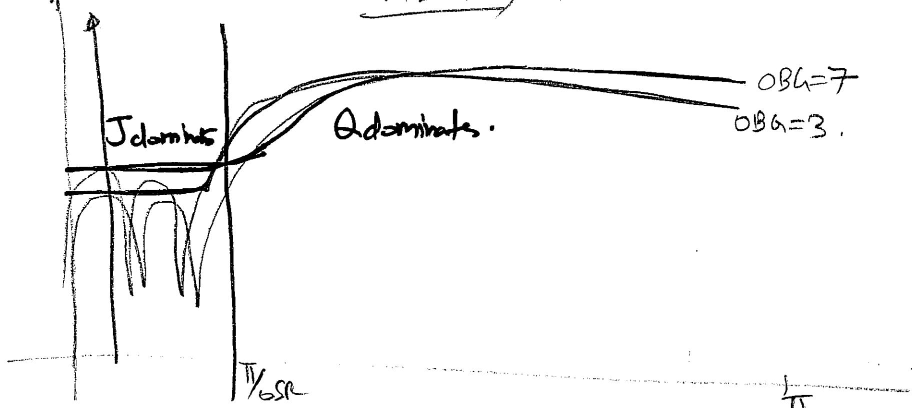
⇒ J for α mostly depends upon the out-of band behaviour of the NTF. as $|NTF(e^{j\omega})|$ is small within the signal band and also it is HPF'd by $|1-e^{-j\omega}|$.

→ Q depends upon the NTF within the signal bandwidth.

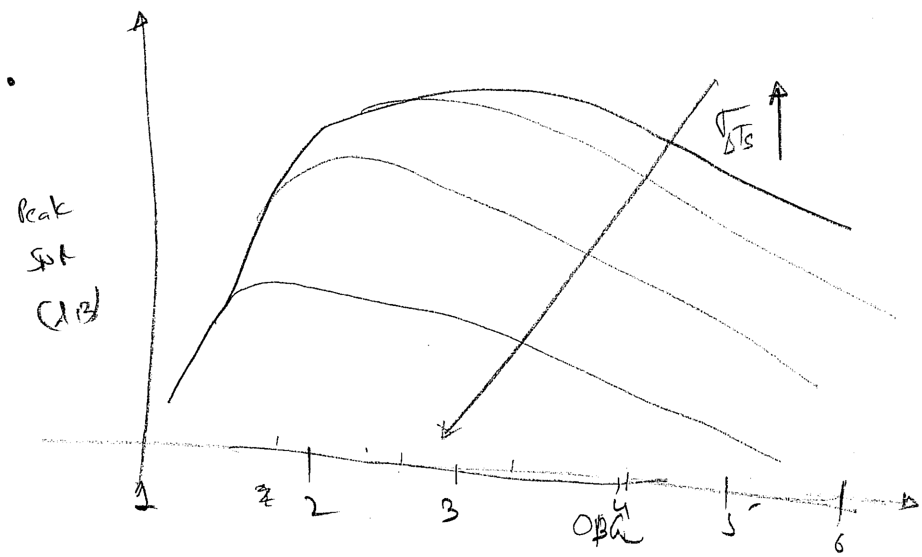
⇒ for a fixed order and maximally flat NTF.

OBG ↑ ⇒ Q ↓ and J ↑

N=4, 4-bit.



⇒ Modulator with a lower OBG → larger in-band noise dominated by the jitter.



Small OBG ⇒ Q is large ⇒ SNR is low

OBG ↑ ⇒ SNR ↑

after certain OBG,

J ↑ ⇒ swamps in-band ↑ SNR ↓

MSA ↓ ⇒ SNR ↓

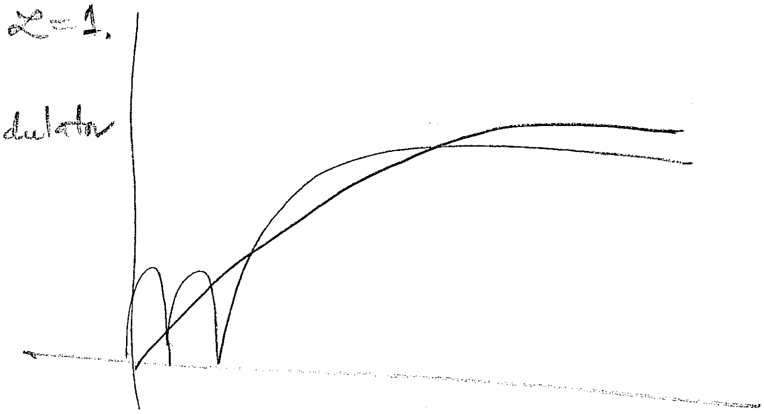
optimum OBG* where the SNR is maximum

OBG* decreases $\sigma_{\Delta T}$ ↑.

(A3)

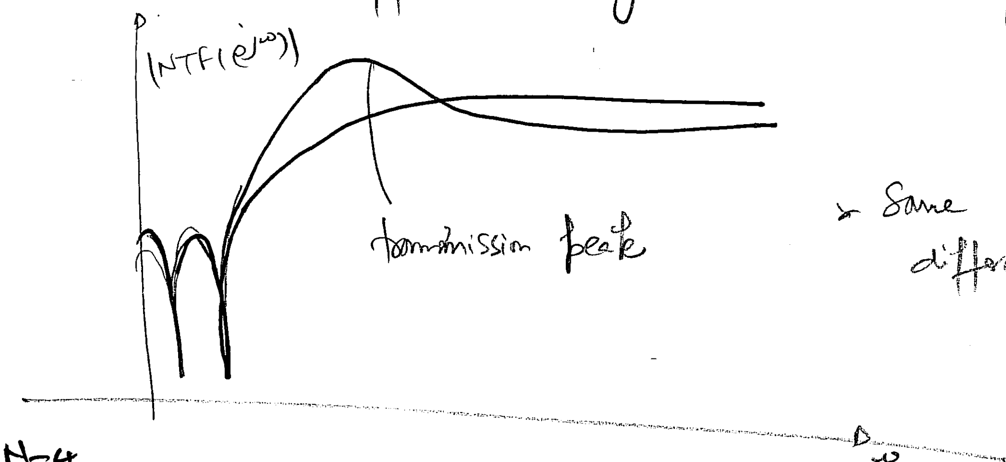
- For a given Q and NTF order, the OBG for a modulator with optimally spread passband zeros, is smaller than that of an NTF with all zeros at ~~z~~ $z = 1$.

  ⇒ J is lower for a modulator with optimal NTF zeros.



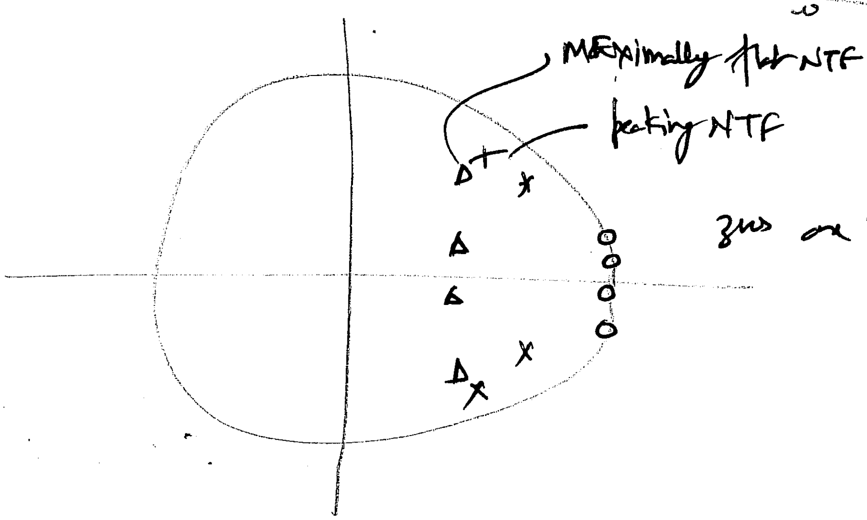
Now lets remove the constraint that the NTF need not be maximally flat.

⇒ Many NTFs with the same in-band characteristics, but different out-of band behavior possible.



- Same NTF zeros but different pole locations

N=4



zeros are the same.

$\Rightarrow$ both have the same Q but J can be different as the Out of band characteristics are not the same

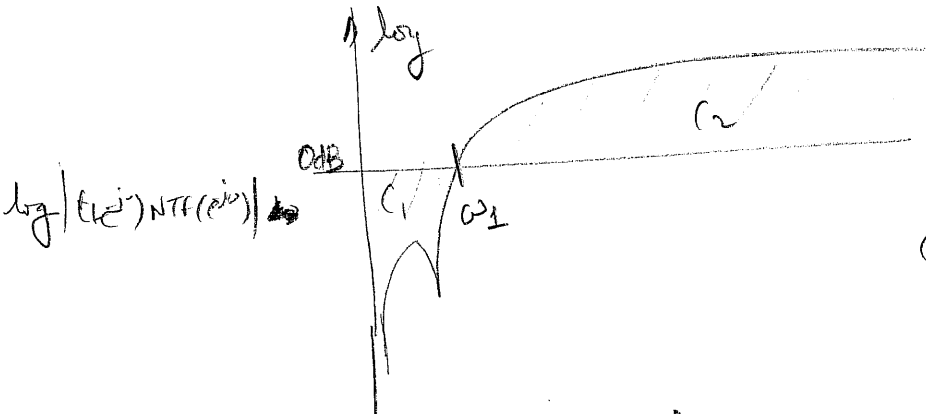
↳ Better mathematical understanding.

↳ Bode's Sensitivity Integral

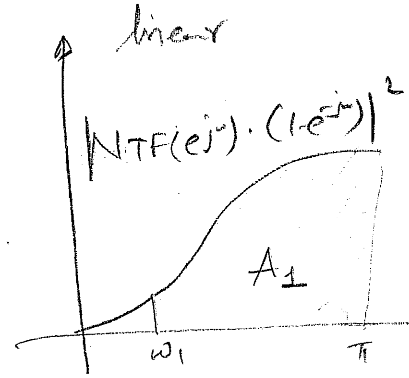
we had:
$$\int_0^{\pi} \log|NTF(e^{j\omega})|\,d\omega = 0$$
for a stable NTF with all zeros inside or on the unit circle.

$$\Rightarrow \int_0^{\pi} \log|NTF(e^{j\omega})(1-e^{-j\omega})|\,d\omega = 0. \longrightarrow (1)$$

$\Rightarrow$ in dB $\Rightarrow$ $20\log|\,|$ plot the area above the 0dB line is equal to the area below 0dB line.



$C_1 = C_2$



$\omega_1 \Rightarrow$ cross over point.

$$\Rightarrow A_J > \int_{\omega_1}^{\pi} |(1-e^{-j\omega})NTF(e^{j\omega})|^2\,d\omega \leftarrow \text{the shaded area} \longrightarrow (2)$$

want to relate (1) and (2) somehow.

(1) — $\log|\,|$

(2) — ~~linear with~~ $|\,|^2$

Using the AM ≥ GM integral inequality. (A6)

No (A5)

$$\int_a^b |f(x)|^2 dx \geq (b-a) \exp\left(\frac{2}{(b-a)}\int_a^b \log|f(x)|\, dx\right) \longrightarrow (3)$$

equality when $|f(x)|$ is constant in $[a,b]$

from (3) & (2) we have $f(x) = (1-e^{-j\omega})\, NTF(e^{j\omega})$

$$\int_{\omega_1}^{\pi} |(1-e^{-j\omega})NTF(e^{j\omega})|^2 d\omega \geq (\pi-\omega_1)\exp\left(\frac{2}{\pi-\omega_1}\underbrace{\int_{\omega_1}^{\pi} \log|(1-e^{-j\omega})NTF(e^{j\omega})|\, d\omega}_{C}\right)$$

~~equality~~

Let C be the area above/below the 0dB line of $\log|NTF(e^{j\omega})(1-e^{-j\omega})|$.

$\Rightarrow \quad J > J_{min}$

and

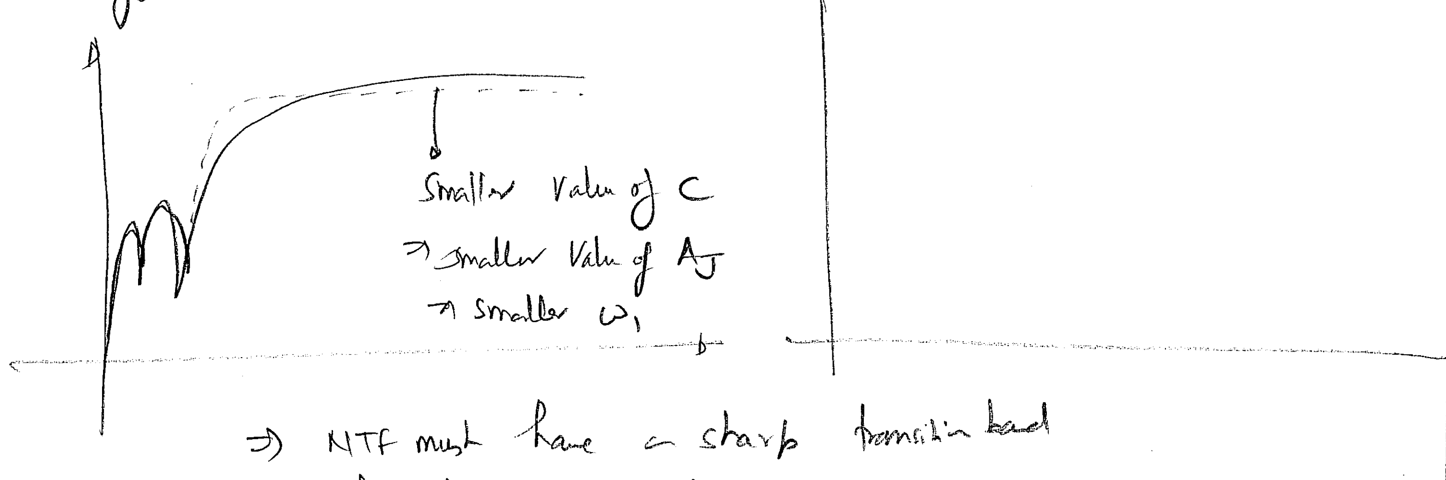
$$J_{min} = \frac{\sigma_{\Delta T_s}^2}{T^2}\cdot\frac{\sigma_{LSB}^2}{\pi\, OSR}(\pi-\omega_1)\exp\left(\frac{2C}{\pi-\omega_1}\right)$$

lower bound on the in-band jitter noise, for a given NTF and clock jitter

$\Rightarrow J > J_{min} \propto \exp\left(\frac{C}{\pi-\omega_1}\right) \Rightarrow C\downarrow,\ \omega_1\downarrow$

→ exponentially proportional to the area above/below the 0dB line.

Now go back to the two NTFs.



$\Rightarrow$ NTF must have a sharp transition band for lower values of ω_1 and C

Observations on the jitter bound derived above:

- J_{min} is a tight bound.
  - ↳ almost an equality.
  - ↳ useful in design
- J_{min} is related to the two parameters of the NTF → C and ω_1
  - ↳ more on this in a bit.

· NTF design for reduced jitter sensitivity.

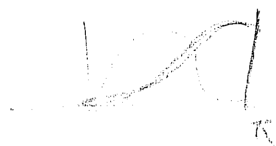
⇒ 'C' depends upon Q.

⇒ By varying the locations of the NTF poles, ~~change~~ ~~C~~ ~~and~~ ~~evenly~~ distribute C in such a way so as to minimize A_1

$$A_1 = \int_{\omega_1}^{\pi} \left| NTF(e^{j\omega})(1-e^{-j\omega}) \right|^2 d\omega$$

⇒ The equality in this equation occurs when $|f(x)|$ is a constant.

⇒ $NTF(e^{j\omega})\cdot(1-e^{-j\omega}) = \text{constant}$ for $\omega_1 < \omega < \pi$

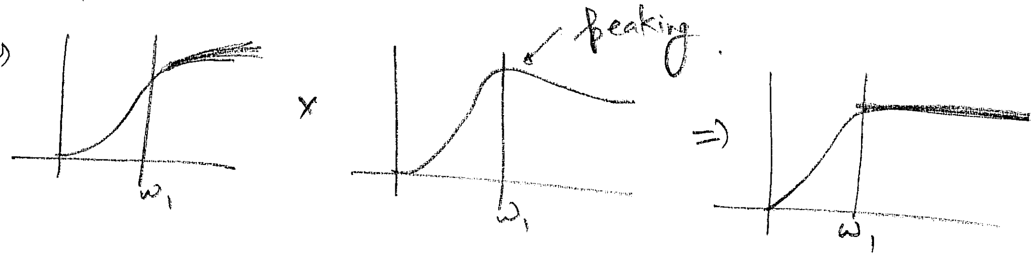


- Since $|1-e^{-j\omega}|$ is monotonically increasing.

⇒ $J = J_{min}$ when $|NTF(e^{j\omega})|$ has peaking.

$|1-e^{-j\omega}| \times |NTF(e^{j\omega})| = \text{const}$

⇒



- It turns out that a maximally flat NTF with a properly chosen OBG, does a good job of getting very close to the optimal jitter value.

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① Maximally flat NTF's

$$|NTF(e^{j\omega})| = \frac{\omega^{l}\prod_{i=1}^{N-l}(\omega-\omega_i)}{|D(e^{j\omega})|}$$

$\omega_i \Rightarrow$ zeros of transmission in the signal band.

$\omega_i << \omega_1$

$|D(e^{j\omega})| \simeq$ constant in ~~signal band~~ the range $0 < \omega < \omega_1$

Let $k = \dfrac{1}{|D(e^{j\omega})|}$

$\Rightarrow$

$$|NTF(e^{j\omega})| \simeq k\,\omega^{l}\prod_{i=1}^{N-l}(\omega-\omega_i) \quad \text{for} \quad \omega \in [0, \omega_1]$$

$\omega_1 = ?$

$$10\log\Big|\underbrace{(1-e^{-j\omega_1})}_{\simeq\omega_1} k\,\omega_1^{l}\prod_{i=1}^{N-l}\underbrace{(\omega_1-\omega_i)}_{\omega_1}\Big| = 0$$

$\Rightarrow \Rightarrow \log|k\omega_1^{N+1}| \simeq 0$

$k\omega_1^{N+1} \simeq 1$

$\Rightarrow$ $\boxed{\omega_1 \simeq \dfrac{1}{k^{\frac{1}{N+1}}}}$ $\Rightarrow k \simeq \dfrac{1}{\omega_1^{N+1}}$

$\Rightarrow$

$$C \simeq -\int_0^{\omega_1}\log|k\omega^{N+1}|\,d\omega = -\int_0^{\omega_1}\log\left|\left(\frac{\omega}{\omega_1}\right)^{N+1}\right|d\omega$$

$$= -(N+1)\int_0^{\omega_1}\log\left|\frac{\omega}{\omega_1}\right|d\omega = -(N+1)\,\omega_1\int_0^{1}\log|y|\,dy \Rightarrow -1$$

$$= (N+1)\,\omega_1$$

$\Rightarrow$ $\boxed{C \simeq (N+1)\,\omega_1}$

A9

$\Rightarrow$

$$J \geq \frac{\sigma_{DT_s}^2}{T^2} \cdot \frac{\sigma_{LSB}^2}{\pi} \cdot \frac{(\pi - \omega_1)}{OSR} \exp\left(\frac{2(N+1)\omega_1}{\pi - \omega_1} \right)$$

$\Rightarrow$ only ω_1 is an indep parameter

$c = f(\omega_1)$ is eliminated

Optimization:

The total inband noise can be reduced by minimizing

$(J + \alpha Q)$

$\alpha = 3$

$(\alpha - 1)Q \Rightarrow$ Thermal noise

$\Rightarrow$ Total noise $\Rightarrow J + Q + \underbrace{(\alpha-1)Q}_{\text{Thermal}}$

$(\alpha - 1)$ is chosen such that Thermal noise is several times larger than Q

↳ reduce idle tones by dithering

↳ perf of the ADC is only limited by the thermal noise not Q.

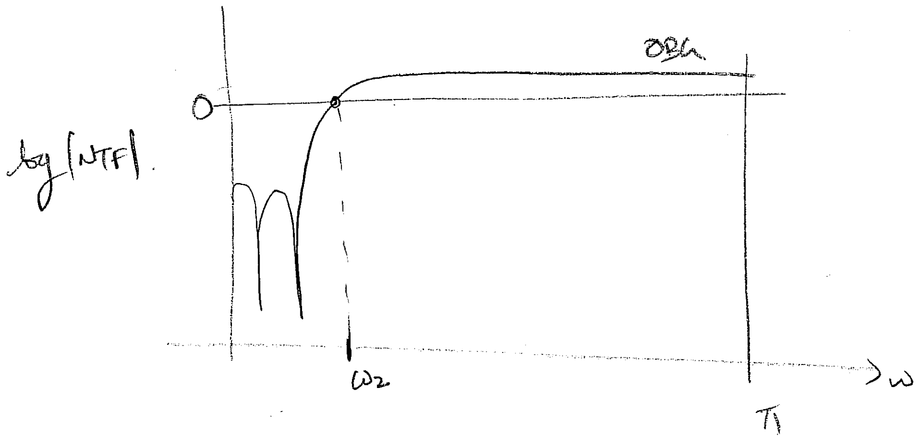
$\Rightarrow$

$$Q \simeq C_y \cdot \frac{\sigma_{LSB}^2}{\pi} \int_0^{\pi/OSR} \left(\frac{\omega}{\omega_1}\right)^{2N} d\omega.$$

$C_z, C_y \rightarrow$ terms depending upon the location of ω_i within the sig. band (a correction factor)

$$= \frac{\sigma_{LSB}^2}{\pi} \cdot \frac{\pi^{2N+1}}{(2N+1)\,\omega_1^{2N+1}\, OSR^{2N+1}} \cdot C_z.$$

$\Rightarrow$ both J & Q now depend upon a single parameter ω_1



Let $\omega_2 \to$ freq where $|NTF|$ goes to 0dB

recall $\omega_1 \to$ freq where $|(1-e^{-j\omega})NTF|$ goes to 0dB.

we see that
$$\omega_2 = \omega_1^{\frac{N+1}{N}}$$

Also The area of $\log|NTF|$ below 0dB line $= N\omega_2$

Appx area above 0dB line $= (\pi - \omega_2)\log(OBG)$.

$\Rightarrow$ $C_1 = C_2$

$\Rightarrow$ $\boxed{OBG^* \simeq \exp\left(\dfrac{N\omega_1^{\frac{N+1}{N}}}{\pi - \omega_1^{\frac{N+1}{N}}}\right)}$ is the optimal OBG for a maximally flat NTF.

this gives
$$J^*_{max} = \frac{\sigma^2_{\Delta T_s}}{T^2} \cdot \frac{\sigma^2_{LSB}}{OSR} \cdot \frac{8B_n^2\left(\tan\left(\frac{\omega_c}{2}\right)\right)}{\tan\left(\frac{\omega_c}{2}\right)} \left|(C_1 + C_2)\right|.$$

$\omega_c \Rightarrow$ 3dB HP corner of the NTF.

$B_n \Rightarrow$ n^{th} order Butterworth polynomial

$C_1, C_2 \sim f(\omega_c)$ see the paper.

Multibit Modulators using RZ DACs :

$$e_j(n) = 2y(n)\left(\overset{\text{rising edge}}{\frac{\Delta T_{S_1}(n)}{T}} + \overset{\text{falling edge}}{\frac{\Delta T_{S_2}(n)}{T}}\right)$$

$$\sigma_{e_j}^2 = 4\sigma_y^2 \cdot \frac{1}{T^2} \cdot \left(\sigma_{\Delta T_{S_1}}^2 + \sigma_{\Delta T_{S_2}}^2\right) \quad \text{using } \sigma_{\Delta T_{S_1}}^2 = \sigma_{\Delta T_{S_2}}^2$$

the idle channel jitter noise

$\rightarrow$
$$J_{RZ} = 8\,\frac{\sigma_{\Delta T_S}^2}{T^2} \cdot \frac{\sigma_{LSB}^2}{\pi\, OSR} \cdot \int_0^{\pi} |NTF(e^{j\omega})|^2 d\omega .$$

$\rightarrow$
$$\frac{J_{RZ}}{J_{NRZ}} = 8 \cdot \frac{\int_0^{\pi} |NTF(e^{j\omega})|^2 d\omega}{\int_0^{\pi} |(1-e^{-j\omega})|NTF(e^{j\omega})|^2 d\omega} .$$

$\because$
$$|\underbrace{(1-e^{-j\omega})}_{<2} NTF(e^{j\omega})|^2 < 4|NTF(e^{j\omega})|^2$$

$\Rightarrow$ idle channel noise with RZ DAC is at least 3dB higher than with the NRZ DAC.

from simulation. $\frac{J_{RZ}}{J_{NRZ}} \sim 4-5$ dB on average

$\hookrightarrow$ becomes worse in the presence of input signal

$\hookrightarrow$ ~10dB for large signals.

$\rightarrow$ RZ DAC in a multibit modulator leads to 'severe' sensitivity when compared to NRZ DACs

Final Discussion:

① J and Q tradeoff

↳ use complex, optimized zero

↳ (local) OBG

② NTF shape:

Usually maximally flat (MF) NTFs are used.

- An NTF with a "gentle" peak can reduce J when compared to a MF-NTF for the same Q.

  ↳ an optimum OBG ~~combo~~ with MF-NTF can get very close to the optimal value.

- minimize $J + \alpha Q$ to get OBG^*

- peaking NTF with FF topology will give ~~p~~ peaking STF (nominally).

  ↳ reduce ~~fo~~ STF peaking by using 'b's

③ Effect of excess loop delay:

ELD leads to latency

↳ causes NTF peaking, if not compensated.

↳ small amount of ELD is beneficial as it reduces J →

↳ use it strategically.