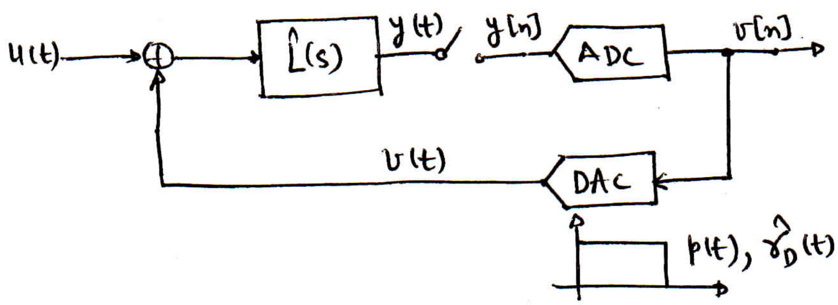
Synthesis of Higher-Order CT ΔΣ Modulators:



For the CT-DSM, we write the loop response as

$$l[n] = \hat{r}_D \circledast \hat{l}(t)\Big|_{t=nT_s} = \int_{-\infty}^{\infty} \hat{r}_D(\tau)\,\hat{l}(t-\tau)\,d\tau \Big|_{t=nT_s} \longrightarrow ①$$

This is also called the "Impulse-invariant transform" or "IIT".

In frequency domain IIT is written as

$$\mathcal{Z}^{-1}\{L(z)\} = \mathcal{L}^{-1}\{\hat{R}_D(s)\cdot\hat{L}(s)\}\Big|_{t=nT_s} \longrightarrow ②$$

Symbols and conventions (from Cherry's book):

- $l(t) \xrightarrow{\mathcal{L}} \hat{L}(s)$ is the CT loop filter (^) analog domain (CT).
- $\hat{r}_D(t)$, $\hat{r}_{DAC}(t)$ or simply $r_D(t)$ is the feedback DAC pulse-shape. $\hat{R}_D(s)$ is its $\mathcal{L}$-domain representation.

- Start with a DT modulator with an open-loop response (i.e. the loop-filter in DT) $L(z)$, that results in $NTF(z)$.
  - ↳ Map $L(z)$ to a CT ΔΣ modulator with a DAC pulse-shape $\hat{r}_D(t)$.
    - ↳ Use ① or ② to find the CT loop-filter $\hat{L}(s)$.

i.e.
$$\{L(z), \hat{r}_D(t)\} \xrightarrow{IIT} \hat{L}(s)$$

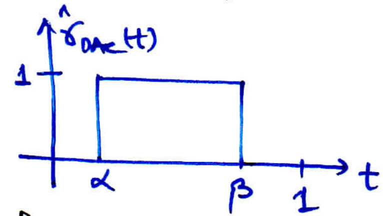
This IIT mapping is systematically done using the Tables derived in Cherry's book or papers. The steps are as follows:

(1) Write $L(z)$ as a partial fraction expansion

↳ choose a rectangular DAC pulse shape of magnitude 1 and timing $[\alpha, \beta]$. Note that to start with we design the CT DSM with a normalized frequency of $f_s = 1\,Hz \Rightarrow T_s = 1\,s$

DAC pulse-shape: $\hat{r}_{\alpha,\beta}(t) = \begin{cases} 1, & \alpha \le t \le \beta \\ 0, & \text{otherwise} \end{cases}$, $\quad 0 \le \alpha \le \beta \le 1$

$\Rightarrow \hat{R}_{\alpha,\beta}(s) = \frac{1}{s}\left[e^{-\alpha s} - e^{-\beta s}\right].$



(2) Use the table to convert each partial fraction pole from 'z'-domain to 's'-domain, and recombine to get $\hat{L}(s)$

- Using IIT, a z-domain pole of multiplicity 'l' at $z = z_k$ maps to a CT pole at $s = s_k$ with the same multiplicity, such that

$$s_k = \ln(z_k)$$

$$\Rightarrow \quad \frac{1}{(z - z_k)^l} \text{ maps to } \frac{y_0}{(s - s_k)^l}, \quad y_0 \in \mathbb{R}$$

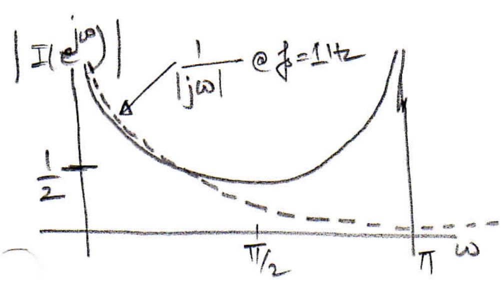
- A poor intuition:

Integrator $\quad I(z) = \frac{z^{-1}}{1 - z^{-1}} = \frac{1}{z-1} = \frac{1}{e^{sT_s} - 1}$

$= \frac{1}{e^s - 1}$ for $T_s = 1$

$= \frac{1}{\left(1 + s + \frac{s^2}{2!} + \cdots\right) - 1}$

$\simeq \frac{1}{1 + s - 1} = \frac{1}{s}$, $\quad |sT_s| \ll 1 \Rightarrow \omega \ll \pi$



$\Rightarrow \frac{1}{z-1}$ maps to $\frac{1}{s}$

DT integrator

CT integrator

s-Domain Equivalences for z-Domain Loop Filter Poles

s-DOMAIN EQUIVALENCES FOR z-DOMAIN LOOP FILTER POLES

z-domain pole	s-domain equivalent	Limit for $z_k = 1$
$\frac{1}{z-z_k}$	$\frac{r_0}{s-s_k} \times \frac{1}{z_k^{1-\alpha}-z_k^{1-\beta}}$ $r_0 = s_k$	$\frac{r_0}{s-s_k}$ $r_0 = \frac{1}{\beta-\alpha}$
$\frac{1}{(z-z_k)^2}$	$\frac{r_1 s+r_0}{(s-s_k)^2} \times \frac{1}{z_k(z_k^{1-\alpha}-z_k^{1-\beta})^2}$ $r_1 = q_1 s_k + q_0$ $r_0 = q_1 s_k^2$ $q_1 = z_k^{1-\beta}(1-\beta) - z_k^{1-\alpha}(1-\alpha)$ $q_0 = z_k^{1-\alpha} - z_k^{1-\beta}$	$\frac{r_1 s+r_0}{(s-s_k)^2}$ $r_1 = \frac{1}{2}\frac{\alpha+\beta-2}{\beta-\alpha}$ $r_0 = \frac{1}{\beta-\alpha}$
$\frac{1}{(z-z_k)^3}$	$\frac{r_2 s^2+r_1 s+r_0}{(s-s_k)^3} \times \frac{1}{z_k^2(z_k^{1-\alpha}-z_k^{1-\beta})^3}$ $r_2 = \frac{1}{2}q_2 s_k - q_1$ $r_1 = -q_2 s_k^2 + q_1 s_k + q_0$ $r_0 = \frac{1}{2}q_2 s_k^3$ $q_2 = (1-\beta)(2-\beta)(z_k^{1-\beta})^2$ $+ (1-\alpha)(2-\alpha)(z_k^{1-\alpha})^2$ $+ [\beta(\beta+3)+\alpha(\alpha+3)$ $- 4(1+\alpha\beta)]z_k^{1-\alpha}z_k^{1-\beta}$ $q_1 = (\frac{3}{2}-\beta)(z_k^{1-\beta})^2 + (\frac{3}{2}-\alpha)(z_k^{1-\alpha})^2$ $+ (\alpha+\beta-3)z_k^{1-\alpha}z_k^{1-\beta}$ $q_0 = (z_k^{1-\alpha} - z_k^{1-\beta})^2$	$\frac{r_2 s^2+r_1 s+r_0}{(s-s_k)^3}$ $r_2 = \frac{1}{12}\frac{1}{\beta-\alpha}[\beta(\beta-9)$ $+ \alpha(\alpha-9)+4\alpha\beta+12]$ $r_1 = \frac{1}{2}\frac{\alpha+\beta-3}{\beta-\alpha}$ $r_0 = \frac{1}{\beta-\alpha}$

z-Domain Equivalences for s-Domain Loop Filter Poles

z-DOMAIN EQUIVALENCES FOR s-DOMAIN LOOP FILTER POLES

s-domain pole	z-domain equivalent	Limit for $s_k = 0$
$\frac{1}{s-s_k}$	$\frac{y_0}{z-z_k} \times \frac{1}{s_k}$ $y_0 = z_k^{1-\alpha} - z_k^{1-\beta}$	$\frac{y_0}{z-z_k}$ $y_0 = \beta - \alpha$
$\frac{1}{(s-s_k)^2}$	$\frac{y_1 z + y_0}{(z-z_k)^2} \times \frac{1}{s_k^2}$ $y_1 = z_k^{1-\beta}[1 - s_k(1-\beta)] - z_k^{1-\alpha}[1 - s_k(1-\alpha)]$ $y_0 = z_k^{2-\alpha}(1 + s_k\alpha) - z_k^{2-\beta}(1 + s_k\beta)$	$\frac{y_1 z + y_0}{(z-z_k)^2}$ $y_1 = \frac{1}{2}[\beta(2-\beta) - \alpha(2-\alpha)]$ $y_0 = \frac{1}{2}(\beta^2 - \alpha^2)$
$\frac{1}{(s-s_k)^3}$	$\frac{y_2 z^2 + y_1 z + y_0}{(z-z_k)^3} \times \frac{1}{s_k^3}$ $y_2 = z_k^{1-\beta}[-1 + s_k(1-\beta) + \frac{s_k^2}{2}(1-\beta)^2] - z_k^{1-\alpha}[-1 + s_k(1-\alpha) + \frac{s_k^2}{2}(1-\alpha)^2]$ $y_1 = z_k^{2-\beta}[2 - s_k(1-2\beta) + \frac{s_k^2}{2}(-1 - 2\beta + 2\beta^2)] + z_k^{2-\alpha}[2 - s_k(1-2\alpha) + \frac{s_k^2}{2}(-1 - 2\alpha + 2\alpha^2)]$ $y_0 = z_k^{3-\alpha}(1 + s_k\alpha + \frac{s_k^2}{2}\alpha^2) - z_k^{3-\beta}(1 + s_k\beta + \frac{s_k^2}{2}\beta^2)$	$\frac{y_2 z^2 + y_1 z + y_0}{(z-z_k)^3}$ $y_2 = \frac{1}{6}(\beta^3 - \alpha^3) - \frac{1}{2}(\beta^2 - \alpha^2) + \frac{1}{2}(\beta - \alpha)$ $y_1 = \frac{1}{3}(\beta^3 - \alpha^3) - \frac{1}{2}(\beta^2 - \alpha^2) - \frac{1}{2}(\beta - \alpha)$ $y_0 = \frac{1}{6}(\beta^3 - \alpha^3)$

- Poles at DC (i.e. $z_k=1$) end up giving $\frac{0}{0}$ terms, as the numerator of the z-domain equivalent in CT.
  - ↳ Apply L'Hospital's rule → results in Column 3 of the table
  - ⇒ Use column '3' for ideal integrators which have poles at $z_k=1$.

Example 1: 2nd order CT-DSM

NRZ DAC i.e. $(\alpha, \beta) = (0, 1)$

$$NTF(z) = (1-z^{-1})^2$$

$$\Rightarrow L(z) = \frac{NTF(z)-1}{NTF(z)} = \frac{(1-z^{-1})^2 - 1}{(1-z^{-1})^2}$$

$$= \frac{-2z^{-1}+z^{-2}}{(1-z^{-1})^2}$$

$$\Rightarrow \boxed{L(z) = \frac{-2z+1}{(z-1)^2}}$$

← always write in $\frac{1}{(z-z_k)^l}$ form when using the tables.

- MATLAB can be used to find the partial fraction

[r, p, k] = residue(b,a);

$$\frac{b(z)}{a(z)} = \frac{b_1 z^m + b_2 z^{m-1} + \cdots + b_{m+1}}{a_1 z^n + a_2 z^{n-1} + \cdots + a_{n+1}}$$

the residue command gives

$$\frac{b(z)}{a(z)} = \frac{r_1}{z-p_1} + \frac{r_2}{(z-p_2)} + \cdots + \frac{r_n}{(z-p_n)} + k(z)$$

direct term

in case of repeated poles, it implies

$$= \frac{r_1}{(z-p_1)} + \frac{r_2}{(z-p_1)^2} + \cdots \text{ other terms}$$

See documentation for further details.

- Now we express $L(z)$ as the partial fraction expansion

$$L(z) = \frac{-2}{(z-1)} + \frac{-1}{(z-1)^2}$$

$$\Rightarrow z_k = 1 \longrightarrow s_k = 0.$$

- Converting each of the partial fraction into its CT equivalent by applying column 3 of the table.

  - $$\frac{1}{z-z_k} \overset{\text{IIT}}{\longleftrightarrow} \frac{\gamma_0}{s-s_k}, \quad \gamma_0 = \frac{1}{(\beta-\alpha)}$$

$$\Rightarrow \frac{1}{z-1} \longleftrightarrow \frac{\gamma_0}{s} \longrightarrow ①$$

  - $$\frac{1}{(z-1)^2} \longleftrightarrow \frac{\gamma_1 s + \gamma_0}{s^2}, \quad \gamma_1 = \frac{1}{2}\left(\frac{\alpha+\beta-2}{\beta-\alpha}\right), \quad \gamma_0 = \frac{1}{\beta-\alpha} \longrightarrow ②$$

with $(\alpha, \beta) = (0, 1)$, we get

$$\Rightarrow \frac{1}{(z-1)} \longrightarrow \frac{1}{s}$$

$$\frac{1}{(z-1)^2} \longrightarrow \frac{(1-0.5s)}{s^2}$$

$$\Rightarrow \because L(z) = \frac{-2}{z-1} + \frac{-1}{(z-1)^2}$$

$$\Rightarrow \hat{L}(s) = \frac{-2}{s} + \frac{-1+0.5s}{s^2}$$

$$\Rightarrow \boxed{\hat{L}(s) = -\frac{1+1.5s}{s^2}}$$

$\leftarrow$ The $-$ve sign depends upon the feedback sign convention used in the block diagram (Think!)

$\rightarrow$ first derived by Candy in 1985

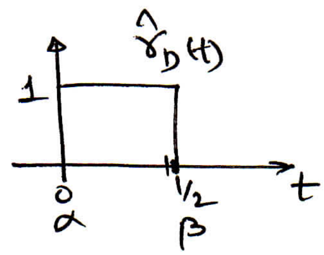
CT equivalent of the double-integration modulator L_0 (Boser-Wooley topology)

- for $T_s \neq 1$, we simply replace 's' by 'sT_s' to denormalize the loop-filter

$$\Rightarrow \hat{L}(s) = -\frac{1+1.5(sT_s)}{(sT_s)^2} = -\frac{1+(1.5T_s)s}{s^2(T_s)^2}$$

Example 2: RZ DAC

$\Rightarrow (\beta-\alpha)=\frac{1}{2}$



$$\Rightarrow \frac{1}{(z-1)} \longrightarrow \frac{2}{s}$$

and $\frac{1}{(z-1)^2} \longrightarrow \frac{2-1.5s}{s^2}$

$$\because L(z) = -\frac{2}{z-1} + \frac{-1}{(z-1)^2}$$

$$\Rightarrow \hat{L}(s) = \frac{-4}{s} + \frac{-2+1.5s}{s^2}$$

$$= \frac{-4s-2+1.5s}{s^2}$$

$$\boxed{\hat{L}(s)_{RZ} = -\left(\frac{2+2.5s}{s^2}\right)}$$

Recall that: $\hat{L}(s)_{NRZ} = -\frac{(1+1.5s)}{s^2}$

$\Rightarrow$ The numerator coefficients are higher in magnitude
↳ In order to get same quantizer input voltage ($y[n]$) with a shorter DAC pulse, we require larger integrator gains.

Using MATLAB

$[a, b, c, d] = \text{tf2ss}([\underbrace{0\ -2\ 1}_{num}], [\underbrace{1\ -2\ 1}_{den}], \underbrace{1}_{T_s=1})$

```
sysd = ss(a, b, c, d, 1); % state space, Ts=1
sysc = d2c(sysd); % convert to CT system from DT
```

For rectangular DAC pulses (α, β), only the $\hat{B}$ matrix changes.

See [Norsworthy ch4]

$$\hat{B}_{(\alpha,\beta)} = \left[e^{\hat{A}(1-\alpha)} - e^{\hat{A}(1-\beta)}\right]^{-1} (A - I)\, \hat{B}_{(0,1)}$$

→ sometimes it may not work when $\hat{A}$ is singular

e.g. $\hat{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

- d2c command takes care of $\hat{A}$ singularity.

The code:

```
ac = sysc.a; % get Ac matrix
bc = (inv(expm(ac*(1-alpha)) - expm(ac*(1-beta)))*(sysd.a-eye(2))*sysc.b
sysc.b = bc; % reset bc
tf(sysc); % print tf
```

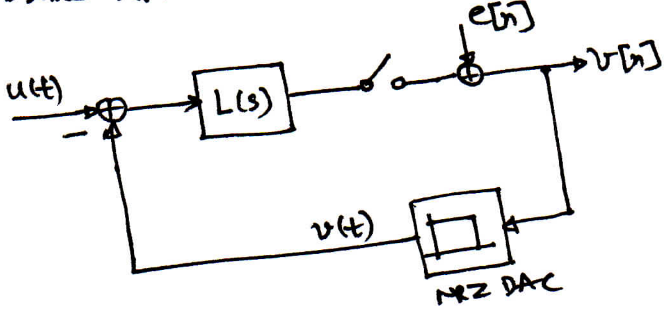
returns $\dfrac{(-2.5s - 2)}{s^2}$ as the resulting transfer function. $= \hat{L}(s)$

PS: The ΔΣ toolbox function realizeNTF_ct can also be used to determine $\hat{L}(s)$, using the numerical fitting approach, covered later.

Time-domain intuition of CT loop filter design:

Ex. $NTF(z) = (1-z^{-1})^2$

Assume NRZ DAC.



- Note that we drop the superscript ^ in $\hat{L}(s)$ for analog quantities.
- Also note that the −ve sign in the feedback in the block diagram may or may not be used.
  - ↳ using +ve sign in the feedback will result in −ve terms in $L(s)$.

$NTF(z) = \dfrac{1}{1+L(z)}$ ← Discrete-time equivalent of the CT loop response (closed-loop)

$\Rightarrow L(z) = \dfrac{1}{NTF(z)} - 1 = \dfrac{1}{(1-z^{-1})^2} - 1$

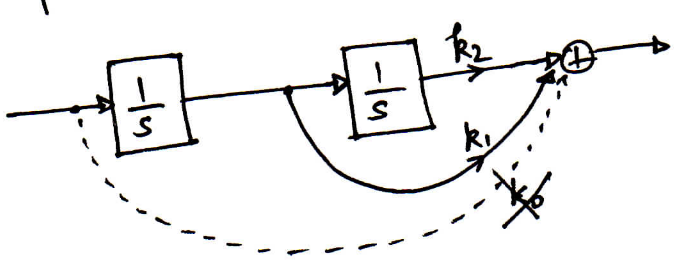
$= \dfrac{z^{-1}(2-z^{-1})}{(1-z^{-1})^2}$

$= \dfrac{z^{-1}}{(1-z^{-1})} + \dfrac{z^{-1}}{(1-z^{-1})^2}$ → partial fraction expansion in the form of delaying integrators.

$$\boxed{\begin{array}{l} NTF(\infty) = \dfrac{1}{1+L(\infty)} = 1 \\ \Rightarrow L(\infty) = 0 \\ \Rightarrow l[0] = 0. \\ \Rightarrow \text{No zero-delay loop!} \end{array}}$$

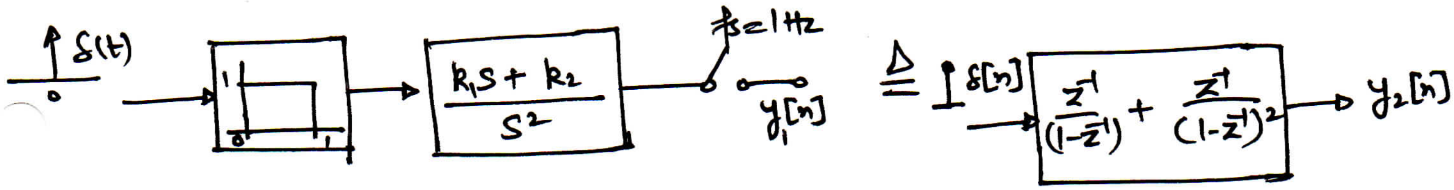
- We have two integrators in the loop-filter
  $\Rightarrow \dfrac{1}{(1-z^{-1})^2} \Rightarrow \dfrac{1}{\omega^2} \Rightarrow \dfrac{1}{s^2}$ ⇒ 2 CT integrators are needed to implement the loop-filter.

Ex. feed-forward (CIFF/FF) implementation:



$k_0 = 0 \Rightarrow$ No delay-free loop.

$\Rightarrow L(s) = \cancel{k_0}^{\,0} + \dfrac{k_1}{s} + \dfrac{k_2}{s^2}$



$\Rightarrow$ We want $y_1[n] = y_2[n]$ for IIT.

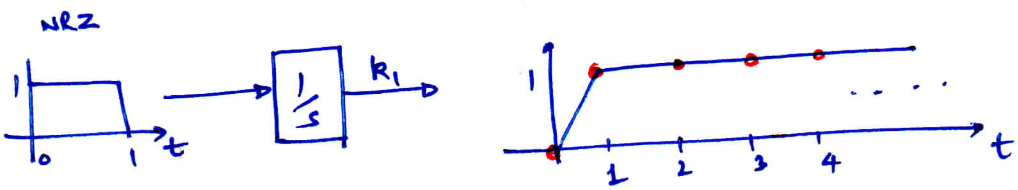
$\Rightarrow$ $\Delta\Sigma$ loop will not be able to differentiate the difference between the DT and CT loop.

- Now, we have

DT:
$$y_2[n] = \mathcal{Z}^{-1}\left\{\frac{z^{-1}}{1-z^{-1}}\right\} \longrightarrow \{0, 1, 1, \dots$$
$$+ \mathcal{Z}^{-1}\left\{\frac{z^{-1}}{(1-z^{-1})^2}\right\} \longrightarrow \{0, 1, 2, 3, \dots$$

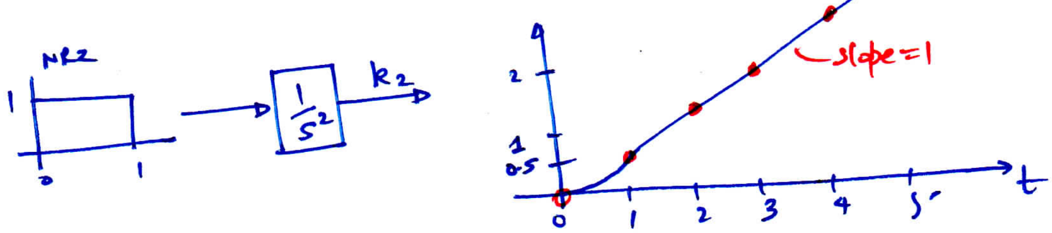
$$y_2[n] \longrightarrow \{0, 2, 3, 4, 5, \dots\}$$

CT:



$$\Rightarrow k_1 \cdot \mathcal{L}^{-1}\left\{ NRZ \circledast \frac{1}{s} \right\} \xrightarrow{1\,Hz} k_1\{0, 1, 1, \dots\}$$

Also:



$$\Rightarrow k_2 \cdot \mathcal{L}^{-1}\left\{ NRZ \circledast \frac{1}{s^2} \right\} \xrightarrow{1\,Hz} k_2\{0, 0.5, 1.5, 2.5, \dots\}$$

$\Rightarrow$ for $y_1[n] = y_2[n]$, we have

$$k_1\{0, 1, 1, \dots\} + k_2\{0, 0.5, 1.5, 2.5, \dots\} = \{0, 2, 3, 4, 5, \dots\} \longrightarrow (1)$$

from ①, we have two consistent set of equations:

$$k_1 + \frac{1}{2}k_2 = 2 \longrightarrow \text{(i)}$$

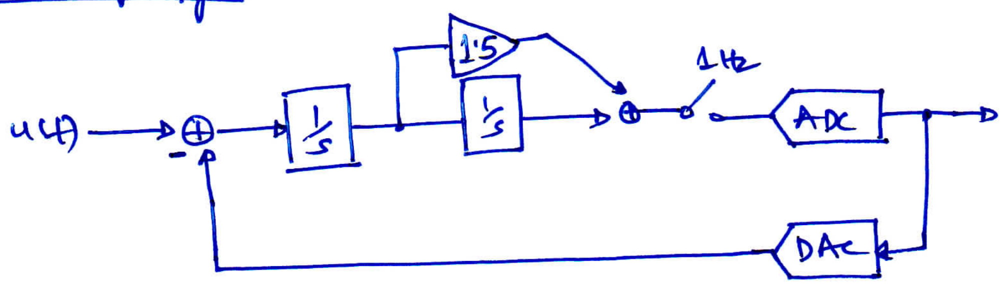
$$k_1 + \frac{3}{2}\cdot k_2 = 3 \longrightarrow \text{(ii)}$$

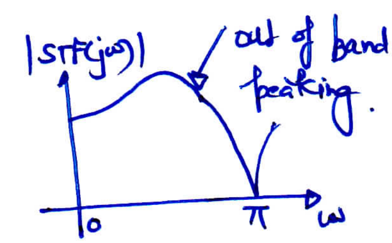
which have the solution $\{k_1, k_2\} = \{1.5, 1\}$

$$\Rightarrow L(s) = \frac{1.5}{s} + \frac{1}{s^2} = \frac{1.5s + 1}{s^2}$$

- which is same as the sol^n from the table based method.
- This method is called the numerical fitting method
  - ↳ Least-square fit the CT-ΔΣ loop response to that of the DT-ΣΔ by finding appropriate coefficients $\{k_1, k_2, \ldots\}$

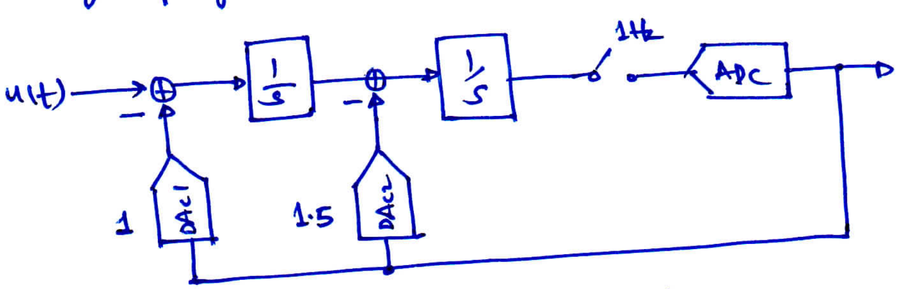
CIFF topology:

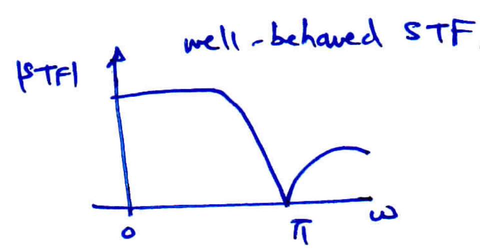




CIFB topology:

Many topologies are possible:





Need two DAC's

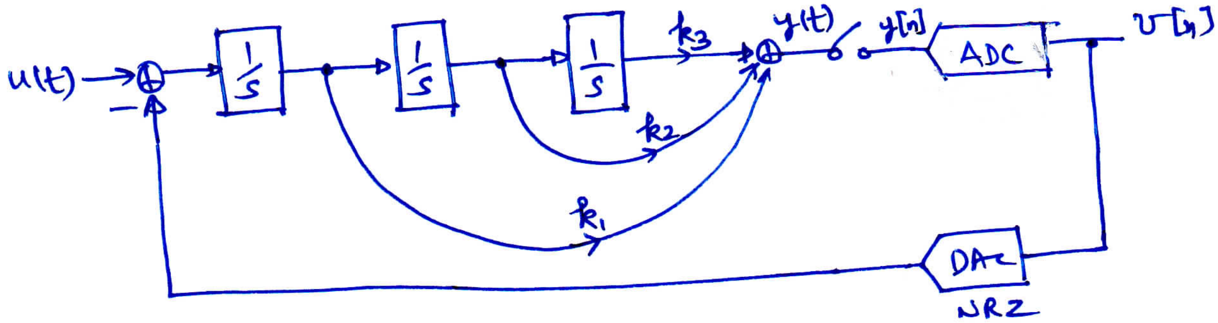
* NTFs of both the loops are the same but the STFs are different.

→ High frequency rejection of this 'simple' FF topology is worse than the FB case.

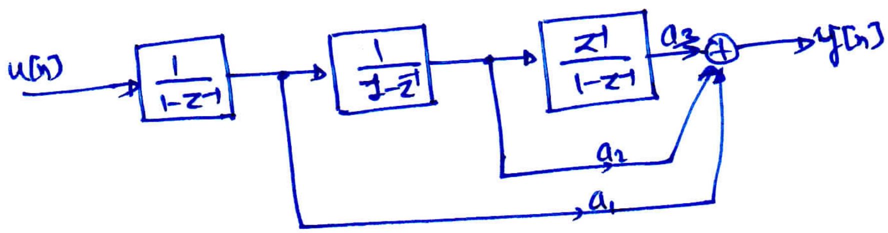
3rd order Butterworth NTF : Numerical Fitting Method :

* All NTF zeros at $z=1$.

* Start with a CT loop-filter implementation



. Find a corresponding DT loop-filter of this form: (CRFF here)



. Find $L(z)$ and expand into partial fractions

$$= \frac{1}{(1-z^{-1})} + \frac{(\cdot)}{(1-z^{-1})^2} + \frac{(\cdot)}{(1-z^{-1})^3}$$

loop impulse response $l[n]$ = DAC pulse shape passing through the loop filter, and then sampled at f_s rate.

$$\begin{array}{c} \downarrow \frac{1}{s} \quad \downarrow \frac{1}{s^2} \quad \downarrow \frac{1}{s^3} \end{array} \qquad \xleftarrow{z^{-1}} L(z) \leftarrow \frac{1}{1+L(z)} = NTF(z)$$

$$\underbrace{\begin{bmatrix} c_1 & c_2 & c_3 \\ & & \\ & & \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}}_{\mathbf{k}} = \underbrace{\begin{bmatrix} l_0 \\ l_1 \\ l_2 \\ \vdots \end{bmatrix}}_{\mathbf{l}}$$

$c_1 \Rightarrow$ DAC pulse through $1/s$ and then sampled

$c_2 \Rightarrow$ " " " $1/s^2$ " " "

$c_3 \Rightarrow$ " " " $1/s^3$ " " "

- many equations $\Rightarrow$ unique solution when using ideal integrators, and when the mapping is possible
- when using non-ideal components do a matrix 'least-square fit' to obtain $\{k_1, k_2, k_3\}$

$$C k = \ell$$

$$\Rightarrow (C^T C) k = C^T \ell$$

$$\Rightarrow \boxed{k^* = (C^T C)^{-1} (C^T \ell)}$$

- suitably choose the length of impulse response used for fitting.