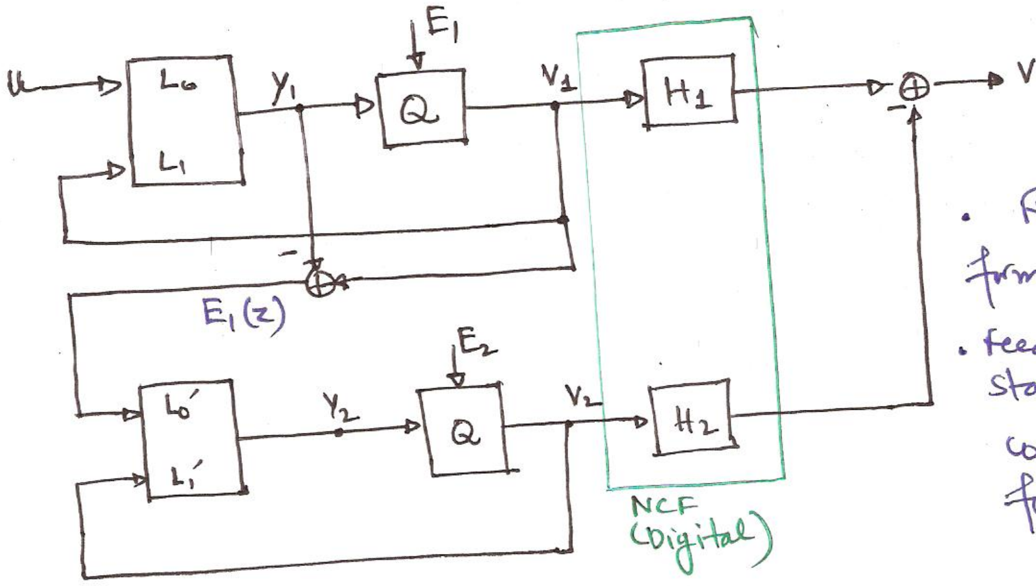
Cascaded (MASH) Delta-Sigma Modulators

Multi Stage Noise Shaping (MASH)

- Second stage in the cascade is another $\Delta\Sigma$ modulator.



- Find $e_1[n]$ in analog form by subtracting $e_1 = V_1 - y_1$
- feed $e_1[n]$ to the second stage of the modulator and convert it to its digital form.

1st stage: $V_1(z) = STF_1(z)\, U(z) + NTF_1(z) \cdot E_1(z)$

2nd stage: $V_2(z) = STF_2(z)\, E_1(z) + NTF_2(z) \cdot E_2(z)$

Final output:

$$V(z) = H_1 V_1 - H_2 V_2$$

$$= H_1 (STF_1 \cdot U + NTF_1 \cdot E_1) - H_2 (STF_2 \cdot E_1 + NTF_2 \cdot E_2)$$

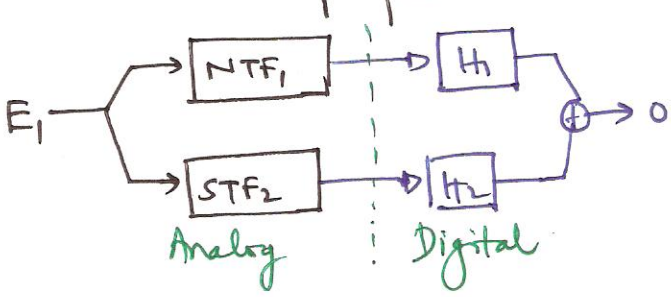
$$= (H_1 STF_1\, U - H_2 \cdot NTF_2 \cdot E_2) + \underbrace{H_1 NTF_1 \cdot E_1 - H_2 \cdot STF_2 \cdot E_1}_{\text{unwanted terms}}$$

$$= (H_1 STF_1\, U - H_2 \cdot NTF_2 \cdot E_2) + \cancel{(H_1 \cdot NTF_1 - H_2 \cdot STF_2)}^{\,0} E_2$$

- for cancelling E_1, we MUST satisfy

$$\boxed{H_1 \cdot NTF_1 = H_2 \cdot STF_2}$$

$\Rightarrow$ The two paths seen by 'E_1' must be identical for perfect cancellation of noise $e_1[n]$.



- The perfect noise cancellation never really occurs due to the mismatch between analog and digital circuitry, and due to the 'tonal' nature of the quantization noise $e_1(n)$.

- The simplest and most practical choice for H_1 and H_2 are

$$H_1 NTF_1 = H_2 STF_2$$

$$\Rightarrow \quad \left.\begin{array}{l} H_1 = STF_{2,d} \\ H_2 = NTF_{1,d} \end{array}\right\} \text{All digital filters}$$

- Since STF_2 is often just a delay $\Rightarrow$ H_1 is easy to realize

The overall output, then, is given by

$$V = H_1 V_1 - H_2 V_2 = STF_1 \cdot STF_2 \cdot U - NTF_1 \cdot NTF_2 \cdot E_2$$

$$\Rightarrow \boxed{\begin{array}{l} NTF = \prod_i NTF_i \\ STF = \prod_i STF_i \end{array}}$$

Example: 2-2 MASH, aka (SOSO) ← second-order - second-order cascade

$$\Rightarrow \quad STF_1 = STF_2 = z^{-2}, \quad NTF_1 = NTF_2 = (1-z^{-1})^2$$

$$\Rightarrow \quad V(z) = z^{-4} U(z) - \underbrace{(1-z^{-1})^4 E_2(z)}$$

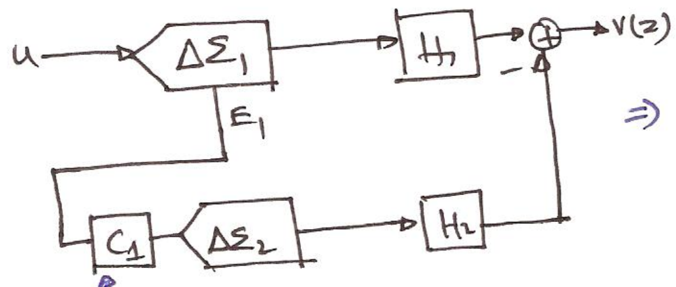
→ fourth order noise shaping

↳ Stability restrictions of only a 2^{nd}-order loop!

Important:

- E_1 input to the second modulator stage needs to be scaled to fit it within the stable range (MSA) of the second $\Delta\Sigma$ modulator.
  - ↳ For a 2^{nd} order, single-bit first-stage $\Delta\Sigma$ output, the usual scaling factor is $1/4$.
  - ↳ If multi-bit quantization is used in the first-stage, the scaling factor can be greater than 1.
  - ↳ The inverse of this scaling factor (or coupling factor) needs to be included in H_2 to cancel $E_1(z)$.

Coupling of Stages:



interstage coupling coefficient

$\Rightarrow H_1 = STF_{2,d}$

$H_2 = \frac{1}{C_1} \cdot NTF_{1,d}$

$$Y(z) = STF_1 \cdot STF_{2,d}\, U(z) + \frac{1}{C_1} \cdot NTF_{1,d} \cdot NTF_2\, E(z).$$

$$\Rightarrow NTF_{casc} = \frac{(1-z^{-1})^{N_{casc}}}{\prod_{i=1}^{M} c_i}, \qquad N_{casc} = \sum_{i=1}^{M} N_i$$

$M \Rightarrow$ stages

$N_{cas} \Rightarrow$ total order of the MASH ΔΣ modulator

- Due to the interstage coupling coefficients, the overall in-band noise is given as

$$IBN = \frac{\sigma_e^2}{\pi} \frac{\int_0^{\pi/OSR} |NTF_1(e^{j\omega})|^2 |NTF_2(e^{j\omega})|^2\, d\omega}{\prod_i c_i^2} = \frac{IBN_0}{\prod_i c_i^2}$$

c_i's are usually >1 for single-bit MASH first stage.

$\Rightarrow$ if $c_i < 1 \Rightarrow$ IBN is increased

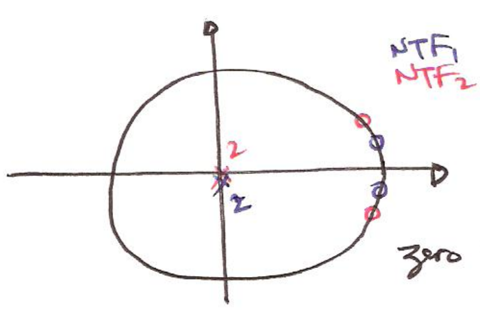
$\Rightarrow$ MASH performance is below the ideal value

- For NTF zero optimization, the complex NTF zeros from the synthesis can be paired-up into individual stage NTFs

$\Rightarrow$ $NTF_1(z) \cdot NTF_2(z) = NTF(z)$ with zero optimization

↳ need to think about the pole placement in individual loops.

↳ 2^{nd} order loops are generally stable with a reasonable MSA, even when the poles are at 0.



zero pairing in 2-2 MASH.

Noise leakage in MASH:

- If $H_1 \cdot NTF_1 \neq H_2 \cdot NTF_2$ due to imperfections in the realization of the analog transfer function with the digital ~~logic~~ filters H_1 & H_2, then
  - ↳ E_1 will appear at the output multiplied by the leakage transfer function $(STF_2 \cdot NTF_{1a} - NTF_1 \cdot STF_{2a})$, where 'a' denotes the actual value of the analog transfer function.
  - ↳ This is known as Noise leakage in MASH ADC.
  - ↳ results in serious deterioration in the noise performance of the ADC.

- It is advantageous for MASH system to use a low-distortion loop-filter structure in all stages, especially the one where the quantization noise is isolated by subtraction.
  - ↳ Easy to obtain $e_1[n]$ without any subtraction, i.e. the output of the last integrator. (see Silva-Steensgaard structure)

- other advantages:

① $e_2[n]$ is generated by quantizing $e_1[n]$, which itself is "noise-like".
  - ↳ $e_2[n]$ is very close to white noise (uncorrelated with the signal and "itself")
    - $R_x(e_2[n]) = \delta[n] \cdot \sigma_{e_2}^2$
  - ↳ $e_2[n]$ behaves like white noise even if $e_1[n]$ is 'tonal'.

  see book plot 4.25.

  ⇒ MASH modulator is less likely to use dithering than a single-loop modulator.

2. Allows the use of a multi-bit quantizer in the second stage of the MASH without any DEM correction for the feedback DAC non-linearity.

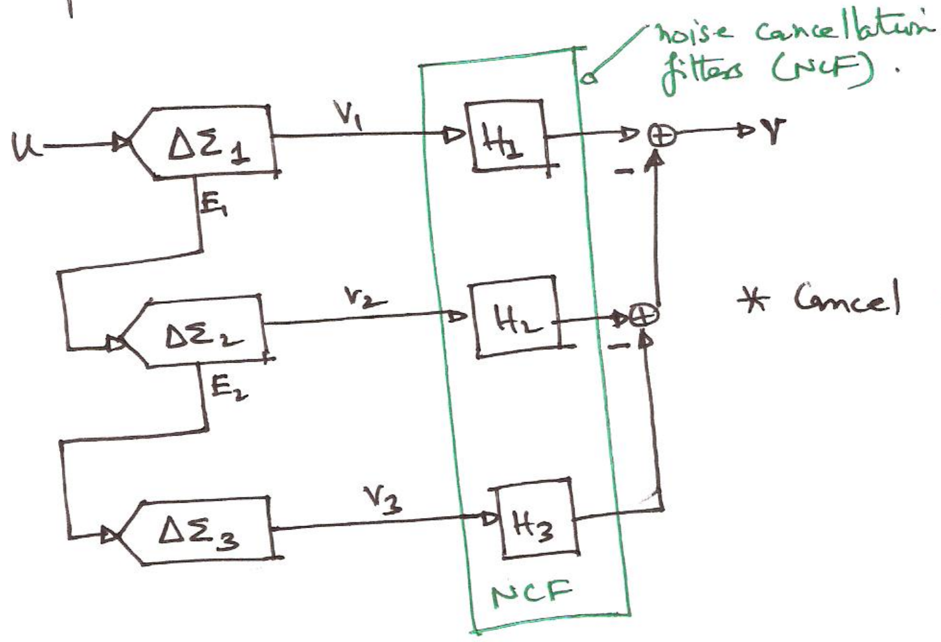
↳ as $u_2 = e_1[n]$ is a "noise-like" input and produces much less amount of distortion tones due to the non-linearity in the feedback.

↳ Also, the non-linearity error of the second-stage feedback DAC (in V_2) is multiplied by $-H_2(z)$ before being added in the output signal V. Since, $H_2(z) = NTF_1(z)$ the DAC_2 non-linearity is suppressed in the baseband.

↳ remnant tones are suppressed, not shaped out, so not as good as DWA, but works well as u_2 is a noise-like signal.

↳ "DAC_2 distortion noise/tones is small and tolerable"!

- Can further extend MASH to more number of stages (>2).



* Cancel E_1 & E_2 in the NCF.

The cancellation conditions for the 3-stage MASH are:

$$\left.\begin{aligned} H_1\cdot NTF_1 - H_2\cdot STF_2 &= 0 \\ H_2\cdot NTF_2 - H_3\cdot STF_3 &= 0 \end{aligned}\right\} \rightarrow (1)$$

If E_1 & E_2 are perfectly cancelled:

$$V = (STF_1\cdot U + NTF_1\cdot E_1)H_1 - (STF_2\cdot E_1 + NTF_2\cdot E_2)H_2 + (STF_3\cdot E_2 + NTF_3\cdot E_3)H_3$$

$$= STF_1\cdot H_1\cdot U + NTF_3\cdot H_3\cdot E_3$$

$$\Rightarrow \quad V = STF_1\cdot H_1\cdot U + \left(\frac{H_1\cdot NTF_1 - NTF_2\cdot NTF_3}{STF_2\cdot STF_3}\right)E_3$$

- Example NCF design:

  $\Rightarrow H_1 = STF_{2d}, \quad H_2 = NTF_{1d}$

  $\Rightarrow H_3 = \dfrac{NTF_{1d}\cdot NTF_2}{STF_3}$, choose appropriate $STF_3 = z^{-k}$

  for the overall MASH DSM:

  $$\Rightarrow \begin{cases} STF = STF_1\cdot H_1 = STF_1\cdot STF_{2d} \\ NTF = \dfrac{NTF_1\cdot NTF_2\cdot NTF_3}{STF_3} \end{cases}$$

  ↳ Also need to include the interstage coupling coefficients c_i later.

⇒ H_1 and other STF usually contain simple delays

↳ flat gain in the signal band

⇒ NTF's provide 'triple' noise suppression in the signal band.

↳ Ideal the quantization errors of fist and second stages (ie E_1 & E_2) are cancelled.

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Example:

$\Rightarrow$ 2-2-2 $\Rightarrow$ 6^{th} order noise-shaping with the stability requirements of only second-order stages!

↳ performance is limited by noise leakage of E_1.

↳ ~~and~~ Imperfect matching between analog $NTF_{1,2,3}$, $STF_{1,2,3}$ and digital $H_{1,2,3}$.

Analysis of Noise-leakage in MASH ADCs:

- In the single-loop ΔΣ modulators, the issues are (assume SC implementation)
  - ↳ imperfect matching of Capacitors (C's)
  - ↳ finite-gain of opamps.
  - ↳ incomplete-settling and slewing in opamps.

↳ These anamolies change the NTF and STF, but will not usually affect the SQNR significantly as long as $L(z)$ is large in the signal band.

$$\because |NTF| \approx \frac{1}{|L_1|} << 1$$

eg. for opamp gain as low as $\frac{OSR}{\pi}$, the SQNR decreases only by a few dB's.

- In 2-stage MASH:

↳ large SQNR is achieved by accurate cancellation of $E_1(z)$, which is shaped by a low-order $H_2 = NTF_{1d}$.

↳ requires accurate matching between the analog and digital ~~components~~. transfer function combinations.

$H_1 \cdot NTF_1$ and $H_2 \cdot STF_2$

↳ Designer needs to be aware of the matching between the circuit blocks to keep E_1-leakage low.

- For a 3-stage MASH, leakage of $E_2(z)$ should also be analyzed.
  ↳ equations describing leakage become complex.
  ↳ Need accurate behavioral simulations for characterization.

Simple Analysis:

The 'leakage transfer-functions' for E_1 & E_2 to the overall MASH output V are:

$$H_{\ell_1} = H_1 \cdot NTF_1 - H_2 \cdot STF_2$$
$$H_{\ell_2} = H_2 \cdot NTF_2 - H_3 \cdot STF_3$$

- Ideally both of these leakage TFs are zero,
  But due to imperfect analog components (circuit blocks), the NTF & STF will be inaccurate.
  ⇒ H_{ℓ_1} & H_{ℓ_2} will be non-zero, allowing E_1 & E_2 to leak into V.

$$H_{\ell_1} = \underbrace{H_1}_{z^{-2}} \cdot \underbrace{NTF_1}_{(1-z^{-1})^2} - \underbrace{H_2}_{(1-z^{-1})^2} \cdot \underbrace{STF_2}_{z^{-2}}$$ for 2-2 MASH

Simplifying assumptions:

① The leakage of E_2 is less important than that of E_1
Since H_{ℓ_2} represents higher-order noise shaping than H_{ℓ_1}.

Ex. in a 2-2-1 MASH ⇒ H_{ℓ_1} noise_shaping is 2^{nd} order
H_{ℓ_2} noise-shaping is 4^{th} order
↳ also if multi-bit quantizer is used in the 2^{nd}-stage
⇒ $|E_2| << |E_1|$.

② In H_{ℓ_1}, the effect of imperfect NTF_1 dominates that of imperfect STF_2, even though the gain-error is same for both.

∵ Since $H_{2d} = NTF_1 \triangleq (1-z^{-1})^2$, the errors in STF_2 are inherently noise-shaped. However the errors in NTF_1 are not noise-shaped as $H_1 = STF_{2d} \Rightarrow$ flat in the signal band

3, Using 2 $STF_2 = H_1 = 1$ in the signal band, then

$|H_{e_1}| \simeq |NTF_1 - H_2| = |NTF_{1a} - NTF_{1i}|$

↳ 'i' ⇒ ideal model

↳ 'a' ⇒ analog implementation

4 ∵ $NTF_1 = \frac{1}{1-L_1}$,

Assuming small errors we have, $|L_1| >> 1$ in the signal band

$\Rightarrow \boxed{|H_{e_1}| \simeq \left|\frac{1}{L_{1i}} - \frac{1}{L_{1a}}\right|} \longrightarrow$ 2

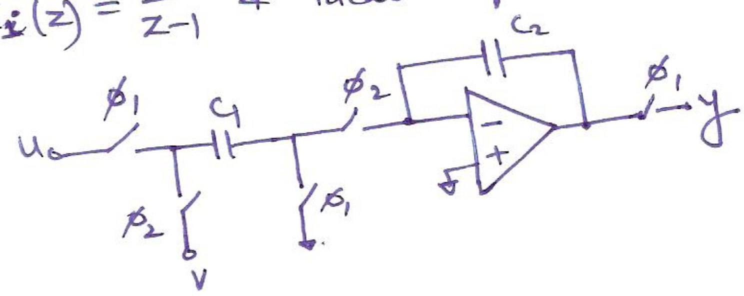
↳ much simpler to evaluate than the original $|H_{e_1}|$.

Example:

A 1-1 or 1-1-1 MASH modulator.

integrator gain $= \frac{C_1}{C_2}$

· loop-filter in the 1st-stage, $I_i(z) = \frac{a}{z-1}$ ← ideal integrator TF.



errors: · Capacitor ratio error

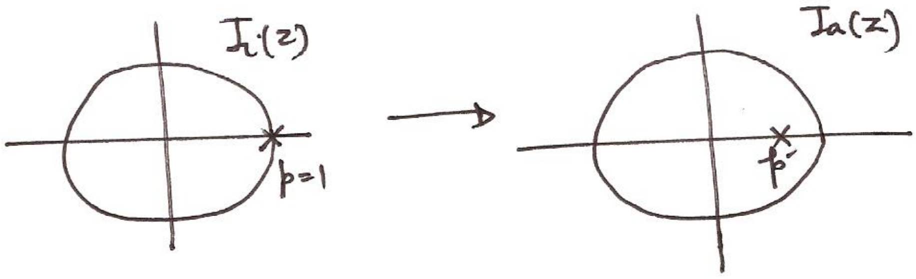
$\frac{C_1}{C_2} = (1+D) \rightarrow$ changes 'a'

· Finite op-amp DC gain $A \rightarrow$ changes both 'a' and the pole location.

· Finite op-amp $f_{un} \Rightarrow$ changes 'a' due to incomplete settling

↳ effects of slewing ignored for now.

$\Rightarrow \boxed{I_a(z) = \frac{a'}{z-p'}}$ ← model for imperfect integrator.



for $D \ll 1$, & $\frac{a}{A} \ll 1$ we can write

$$\begin{cases} a' \triangleq a\left[1 - D - \frac{(1+a)}{A}\right]. \\ p' = 1 - \frac{a}{A} \end{cases}$$

Here, $L_1(z) = -I(z)$

$$\Rightarrow |H_{e,1}| = \left| \frac{(z-1)}{a} - \frac{(z-p')}{a'} \right|$$

$$\simeq \left|\frac{1}{a'}\right| \cdot \left| \frac{a}{A} + (z-1)\left(D + \frac{1+a}{A}\right)\right|$$

$$\Rightarrow \boxed{|H_{e,1}| \simeq \underset{\text{direct noise feedthrough}}{\frac{1}{A}} + \underbrace{(z-1)\left[\frac{D}{a} + \left(1 + \frac{1}{a}\right)\frac{1}{A}\right]}_{1^{st}\text{-order filtered}}}.$$

↳ Unfiltered E_1-leakage component $\simeq \boxed{\frac{E_1}{A}}$

and a first-order filtered component approximately equal to $\boxed{(z-1)\left[\frac{D}{a} + \left(1+\frac{1}{a}\right)\frac{1}{A}\right] \cdot E_1}$

⇒ for high SQNR, → very high opamp gain with excellent settling required to reduce the unfiltered leakage to sufficiently low-level.

⇒ for low-OSR, the second component will also become significant, ⇒ ~~[illegible]~~ $(z-1)\frac{E_1}{A}$ becomes larger in signal band.

↳ matching accuracy of the caps should be better now. So that $D \ll 1$.

- for a second-order first-stage in the MASH, the leakage of E_1 can be reduced.
  - ↳ complicated analysis.

- Assume 2-0 MASH
  - ↳ first-stage is a low-distortion 2^{nd}-order modulator
    - ↳ two cascaded integrators

Now, $I_{a_k}(z) = \frac{a_k'}{(z-p_k')}$, for both the integrators, $k=1,2$..

Using Taylor series expansion around $z=1$, we have

$$H_{e_1}(z) = A_0 + A_1(1-z^{-1}) + A_2(1-z^{-1})^2 + \dots$$

. Assume $A \gg 1$ & $D \ll 1$

$\Rightarrow$ for the series coefficient, we get

$A_0 = \frac{1}{A^2}$ $\longrightarrow$ unfiltered leakage $\propto \frac{1}{A^2}$ $\Rightarrow$ very small

$A_1 = \left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{1}{A}$ $\longrightarrow$ linearly filtered leakage

$A_2 = \frac{1}{a_1 a_2} - 1 + 2\left[1 - \frac{1}{a_1 a_2} - \frac{1}{a_2^2}\right]\cdot\frac{1}{A} + \frac{2D}{a_1 a_2}$ $\longrightarrow$ $(1-z^{-1})^2$ filtered leakage.

For $OSR \gg 1$, A_1 & A_2 terms dominate $|H_{e_1}|$.

- The derivation above ignored leakage due to the coupling ~~errors~~ branch (c_1) and the errors in the second stage.
  - ↳ these terms only contribute to $A_2, A_3, \dots$ as H_2 is a 2^{nd}-order HPF.

$\Rightarrow$ A second-order first stage is beneficial in a MASH to achieve higher (near-ideal) SQNR with reasonable opamp gains (A).

Ref: Textbook pgs 132-136.
Also see the yellow-book of $\Delta\Sigma$.

PS: CT-MASHes to be covered later
↳ much more complicated!