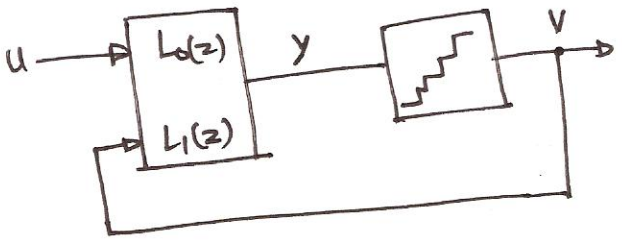
Higher-Order ΔΣ Modulators



$$\boxed{\begin{aligned} NTF(z) &= \frac{1}{1-L_1(z)} \\ STF(z) &= \frac{L_0(z)}{1-L_1(z)} \end{aligned}}$$

- $L_1(z)$ has high gain in the signal band
  ↳ reduces quantization noise in the signal band $\Rightarrow |NTF| \simeq \frac{1}{|L_1|}$
- $\Rightarrow L_0(z)$ must also be large in the signal band in order to keep STF close to 1 in the signal band. $\Rightarrow |STF| \simeq \frac{|L_0|}{|L_1|} \simeq 1$ in signal band
  ↳ both L_0 and L_1 should have their poles in the signal band.
  ↳ L_0 and L_1 have different zeros in general.
- $$NTF(z) = \frac{1}{1-L_1(z)} = \frac{1}{1-\frac{N_1(z)}{D(z)}} = \frac{D(z)}{D(z)-N(z)}$$ $$L_0(z) = \frac{N_0(z)}{D(z)}$$ $$L_1(z) = \frac{N_1(z)}{D(z)}$$
  $\Rightarrow$ poles of $L_1(z) \Longrightarrow$ zeros of $NTF(z)$ ← Important!
  - $NTF(z)$ and $STF(z)$ share the same poles
    ↳ i.e. the roots of $1-L_1(z)=0$
    ↳ zeros of $L_0(z)$ may cancel some/all poles in $D(z)$.
- A contrived example (not necessary a stable NTF):

$STF(z) = z^{-k}$ $NTF(z) = (1-z^{-1})^N$

$\Rightarrow |STF| = 1$

$$\Rightarrow L_0(z) = \frac{STF(z)}{NTF(z)} = z^{-k}(1-z^{-1})^{-N} = \frac{z^{N-k}}{(z-1)^N}$$

→ N-poles at $z=1$
$(N-k)$ zeros @ $z=0$
k zeros @ $z=\infty$.

$$L_1(z) = \frac{1}{NTF(z)} - 1 = (1-z^{-1})^{-N} - 1 = \frac{1-(1-z^{-1})^N}{(1-z^{-1})^N}$$

$$= \frac{z^N - (z^N-1)^N}{(z-1)^N}$$

⇒ N-poles at $z=1$

what about the zeros?

- for finding $L_1(z)$ zeros i.e. z_i

$$(1-z^{-1})^N = 1 = e^{j2\pi} \longleftarrow N^{th}\text{-roots of unity}$$

$$\Rightarrow (1-z_i^{-1}) = e^{j\frac{2\pi i}{N}}, \quad i=0 \text{ to } N-1$$

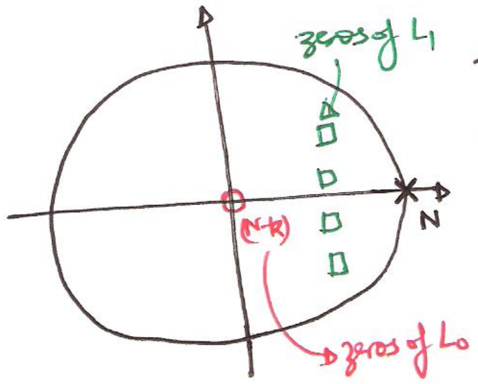
$$\Rightarrow z_i = \frac{1}{1-e^{j\frac{2\pi i}{N}}} = \frac{1}{2}\left[1 + j\cot\left(\frac{\pi i}{N}\right)\right]. \quad i=1,\dots,(N-1).$$

for $i=0 \Rightarrow z_0$ is at ∞.

⇒ 1 zero @ ∞

(N−1) zeros given by

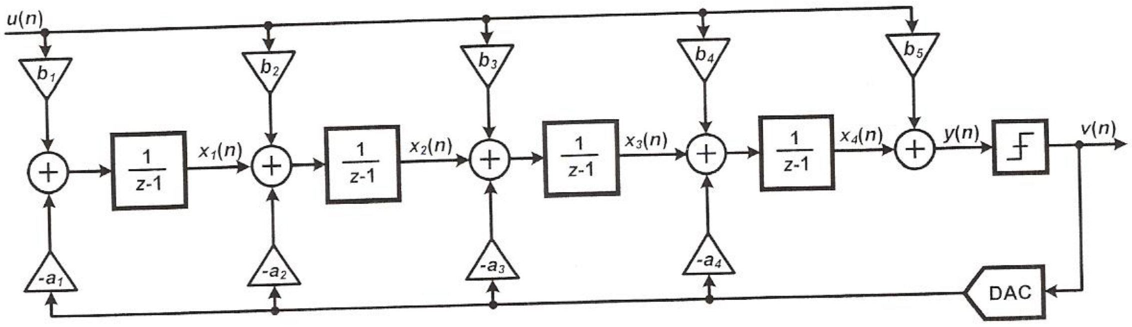
(MATLAB may show zeros at ∞ differently).



- * Both L_0 & L_1 have N-poles at $z=1$ (i.e DC)
- * $(N-k)$ zeros of L_0 lie on $z=0$, and other K zeros @ ∞.
- * zeros of L_1 are located within the unit circle at $z_i = \frac{1}{1-e^{j\frac{2\pi i}{N}}}$

Loop-Filter Architectures:

CIFB



CIFB → Cascaded Integrators with distributed feedBack.

- Cascade of N delaying integrators.
- Feedback from the quantizer to the input of each integrator.
- Multiple feed-in branches possible.

By inspection,

$$L_0(z) = \left(\left(\frac{b_1}{(z-1)} + b_2\right)\frac{1}{(z-1)} + b_3\right)\frac{1}{(z-1)} + \cdots \Big)\frac{1}{(z-1)} + b_{N+1}$$

$$= \sum_{i=1}^{N+1} \frac{b_i}{(z-1)^{N+1-i}}$$

$$= \frac{b_1 + b_2(z-1) + \cdots + b_{N+1}(z-1)^N}{(z-1)^N}$$

$$L_1(z) = -\left(\frac{a_1}{(z-1)^N} + \frac{a_2}{(z-1)^{N-1}} + \cdots + \frac{a_N}{(z-1)}\right)$$

$$= -\sum_{i=1}^{N} \frac{a_i}{(z-1)^{N+1-i}}$$

$$= -\frac{a_1 + a_2(z-1) + \cdots + a_N(z-1)^{N-1}}{(z-1)^N}, \quad a_i, b_i > 0.$$

$\Rightarrow$ $NTF(z) = \dfrac{1}{1-L_1(z)} = \dfrac{(z-1)^N}{D(z)}$, where

Note that the multiplier is 1 for $(z-1)^N$.

$D(z) = a_1 + a_2(z-1) + \cdots + a_N(z-1)^{N-1} + (z-1)^N$

$\Rightarrow$ All the NTF zeros lie at DC $\Rightarrow z=1$

↳ Since $NTF(\infty)=1$ $\Rightarrow$ the NTF is realizable.

↳ $\boxed{a_i\text{'s}}$ introduce finite non-zero poles into the NTF to control OBG. or $\|H\|_\infty$.

↳ also determine the zero of $L_1(z)$.

↳ found by comparing $D(z)$ with the denominator of the required $NTF(z)$. $a_i > 0 \ \forall i$

- $STF(z) = \dfrac{L_0(z)}{1-L_1(z)} = \dfrac{b_1 + b_2(z-1) + \cdots + b_{N+1}(z-1)^N}{D(z)}$

↳ $\boxed{b_i\text{'s}}$ determine the zeros of the STF and ais its poles.

↳ poles of NTF and STF are shared.

↳ STF zeros can be placed in such a way so as to cancel some of the poles, allowing STF to have a lower roll-off rate than set by $\dfrac{b_1}{D(z)}$.

- This NTF can be implemented using a Butterworth response as all the zeros are at DC.

  ↳ non-zero values of a_i's are required to realize the prescribed poles

  $\Rightarrow$ No flexibility in choosing a_i branches

- More flexibility in choosing the values of b_i's $\rightarrow$ $b(2:end)=0$ in MATLAB

  ↳ All b_i's except b_1 can be chosen to be zero

  $\Rightarrow$ only one feed-in branch

Ex. for $b(2:end)=0$

$$STF(z) = \frac{b_1}{D(z)} \Rightarrow \text{for } |STF|=1 \text{ at low frequencies } b_1 = D(0).$$

↳ All the zeros of STF lie at ∞.

- b's are obtained by coefficient matching of the numerator.
- when only b_1 is used, we need to make sure that $|D(e^{j\omega})|$ is maximally-flat in the signal band to make $|STF|$ constant in this region. ← Important!

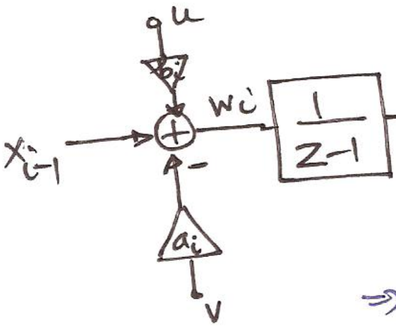
- **Low-Distortion CIFB case:**

when $b_i = a_i$ for $i \leq N$ and $b_{N+1} = 1$

↳ $STF = 1$ (for all frequencies)

$$\Rightarrow \boxed{V(z) = U(z) + NTF(z)\cdot E(z)}$$ ← important equation for Low-Dist Modulators

⇒ input to the i^{th} integrator for this case is:



$$\begin{aligned} w_i(z) &= X_{i-1}(z) - a_i V(z) + b_i U(z) \\ &= X_{i-1}(z) + a_i\left(U(z) - V(z)\right) \\ &= X_{i-1}(z) - a_i NTF(z)\cdot E(z) \end{aligned}$$

No signal here!

⇒ The input signal is not present in any ~~input~~ integrator input.

⇒ The loop-filter only processes the quantization noise e[n].

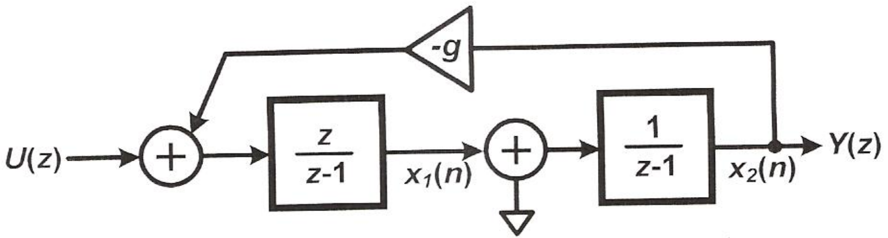
↳ reduced opamp output swings, especially for multibit quantizers.

⇒ integrator non-linearities will not introduce harmonic distortion into the output signal V(z).

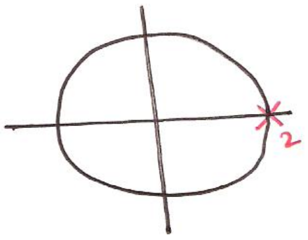
- This also results in lesser amount of dynamic range scaling (DRS).

↳ more convenient element (caps) values. ← Verify this point yourself.

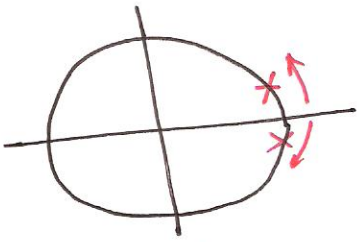
- How to implement complex zeros?
- Use feedback around pairs of integrators to form resonators.



- Non-delaying + delaying integrator pair
- Add a small amount of negative-feedback around the integrator pair to form a resonator.
  ↳ Move the open-loop poles away from the z=1 (DC) point along the unit circle.
  ↳ $L_1(z)$ poles eventually become NTF(z) zeros.



$$\frac{z}{(z-1)^2}$$



$$\frac{z}{z^2-(2-g)z+1}$$

$$U\frac{z}{(z-1)^2}-\frac{gz}{(z-1)^2}Y=Y$$

$$\Rightarrow R(z)=\frac{Y}{U}=\frac{z}{(z-1)^2+gz}=\frac{z}{z^2-(2-g)z+1}$$

$$=\frac{z}{z^2-(2\cos\alpha)z+1}$$

$$=\frac{z}{(z-e^{j\alpha})(z-e^{-j\alpha})}$$

$\Rightarrow$ $\cos\alpha = 1-\frac{g}{2}$

$\Rightarrow$ $\alpha = \cos^{-1}(1-\frac{g}{2})$

$\Rightarrow$ complex roots at $\omega = \pm\alpha = \pm\cos^{-1}(1-\frac{g}{2})$.

Now, consider,

$$\cos\alpha = 1-\frac{g}{2}$$

$$1-2\sin^2\frac{\alpha}{2} = 1-g/2$$

$\Rightarrow$ $\sin(\frac{\alpha}{2}) = \pm\frac{\sqrt{g}}{2}$, using for $\alpha << \pi$ $\sin(\alpha) \simeq \alpha$.

$\Rightarrow$ $\boxed{\alpha \simeq \pm\sqrt{g}}$

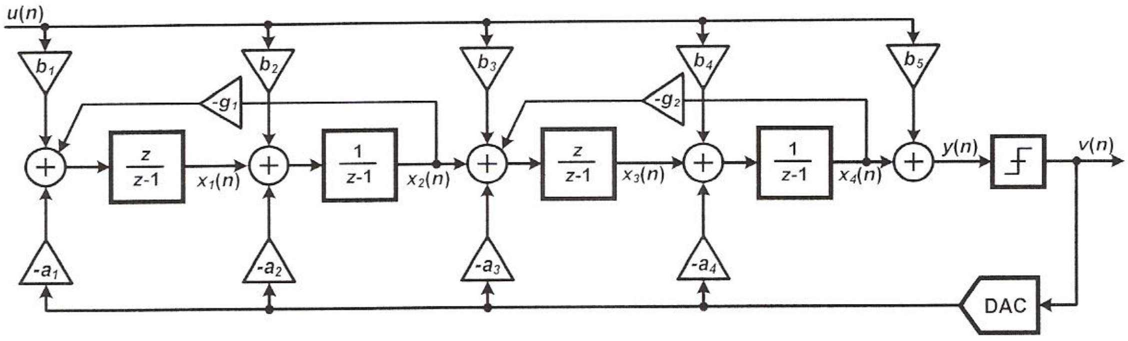
$\Rightarrow$ the roots are precisely on the unit-circle irrespective of the value of 'g'.

$\Rightarrow$ ∴ Resonator poles are located at

$$z = e^{\pm j\sqrt{g}}$$

- Combine the integrator pairs in the CIFB topology to form
- the CRFB (Cascade of Resonators with distributed feedBack)
  - ↳ Note that the resonator has one non-delaying integrator
- for odd-order, the first integrator is not paired-up.
  - ↳ reduces noise coupling from the 'g' branch.
- The value of g's can be directly obtained from the optimized zero locations.
- Now, we can implement either inverse-chebyshev or any other NTF ~~ke~~ response with complex zeros.

- Consider the 4th order CRFB topology



- Here 1st and 3rd integrators are delay-free to keep the $L_1(z)$ poles on the unit circle.

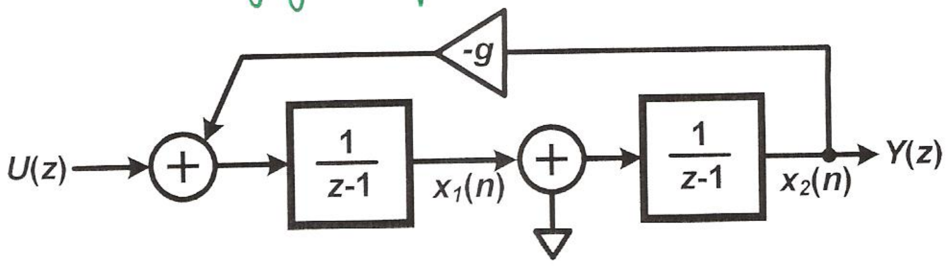
- In this topology, the transfer function from V to X_2 is

$$R_1(z) = \frac{X_2(z)}{V(z)}\bigg|_{U(z)=0} = \frac{-a_1 z + a_2(z-1)}{z^2 - (2-g_1 z) + 1}$$

- Now, for wideband ADC's realized using switched-capacitor integrators, it is advantageous to have a delay in each of the integrators.
  - ↳ relaxes speed requirements on the amplifiers used.

↳ Now, the resonator will need to accomodate two delaying integrators!

- Resonator with delaying integrators.



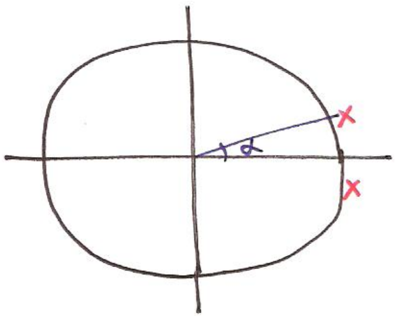
⇒ $R_1(z) = \frac{Y(z)}{U(z)} = \frac{1}{z^2 - 2z + (g+1)}$

⇒ the poles are at $z_i = 1 \pm j\sqrt{g}$

⇒ $|z_i| = \sqrt{1+g} > 1$ ⇒ outside the unit circle

- $\tan\alpha = \pm\sqrt{g_1}$

⇒ $\alpha = \pm\tan^{-1}(\sqrt{g_1}) \simeq \pm\sqrt{g_1}$ for $\omega_i << \pi$



This leads to the topology called "CIFB with Resonators" or simply CIFB in the ΔΣ toolbox.

- These resonators by themselves unstable, (∵ poles are outside the unit circle), but they are embedded in an overall stable feedback system which prevents oscillations.

↳ overall the loop is stable.

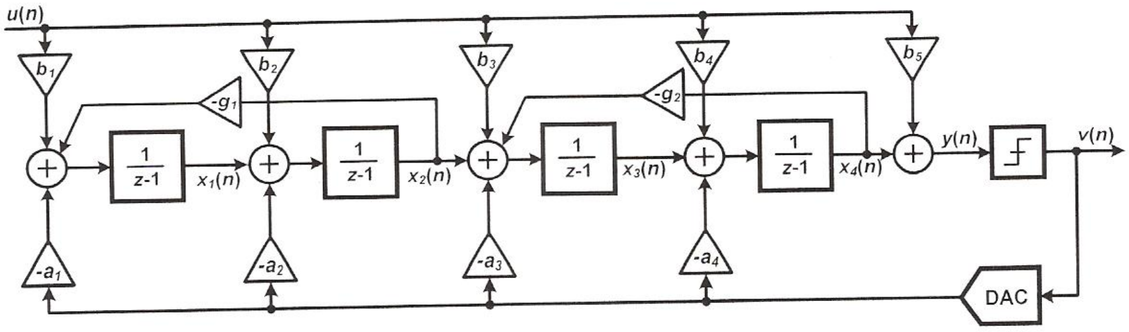
- The value of 'g_i' comes from the ω_i (or α).

↳ The rest of the parameters (a_i & b_i) are found by equating the polynomials of $L_0(z)$ and $L_1(z)$.

↳ Done using software tools (eg. ΔΣ toolbox)

- realizeNTF() function

CIFB topology with resonators.



Now to complete the analysis,

$$R_2(z) = -\frac{a_3 z + a_4(z-1)}{z^2-(2-g_2)z+1} = \frac{X_4(z)}{X_2(z)}\bigg|_{U(z)=0}$$

- $$\frac{X_4(z)}{V(z)}\bigg|_{U(z)=0} = \overset{V\to X_2}{R_1(z)} \cdot \overset{X_2\to X_4}{R_0(z)} + \overset{V\to X_4 \text{ directly}}{R_2(z)}$$ ← Think superposition here!

$$= R_1(z)\left(\frac{z}{z^2-(2-g_2)z+1}\right) + R_2(z)$$

⇒

$$L_1(z) = -\frac{(a_1 z + a_2(z-1))z + [z^2-(2-g_1)z+1]\cdot(a_3 z + a_4(z-1))}{(z^2-(2-g_1)z+1)(z^2-(2-g_2)z+1)}$$

$L_0(z)$ ⇒ negative of above, with b_i's replacing a_i in $R_1(z)$ and $R_2(z)$ and b_5 added as a constant term

$$\Rightarrow L_0(z) = \frac{(b_1 z + b_2(z-1))z + [z^2-(2-g_1)z+1](b_3 z + b_4(z-1)) + b_5}{(z^2-(2-g_1)z+1)(z^2-(2-g_2)z+1)}.$$

- Note the MATLAB warnings when 'CIFB' topology is used with optimized zeros.

ECE 697 Delta-Sigma Converters Design

Lecture#15 Slides

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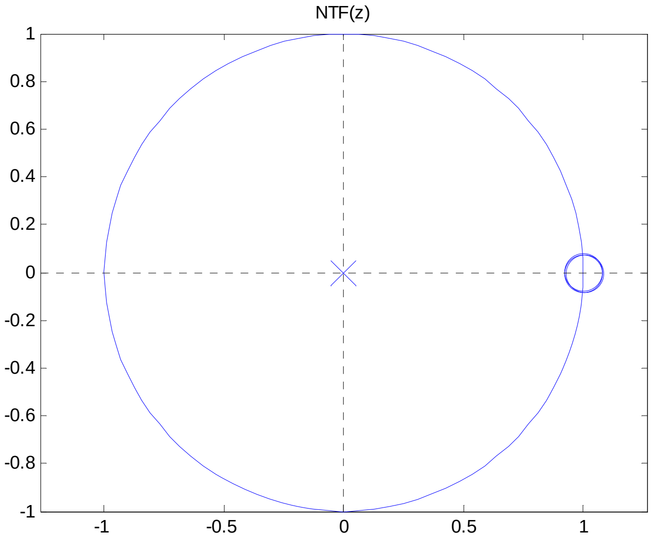
Fourth-Order Loop-Filter Example

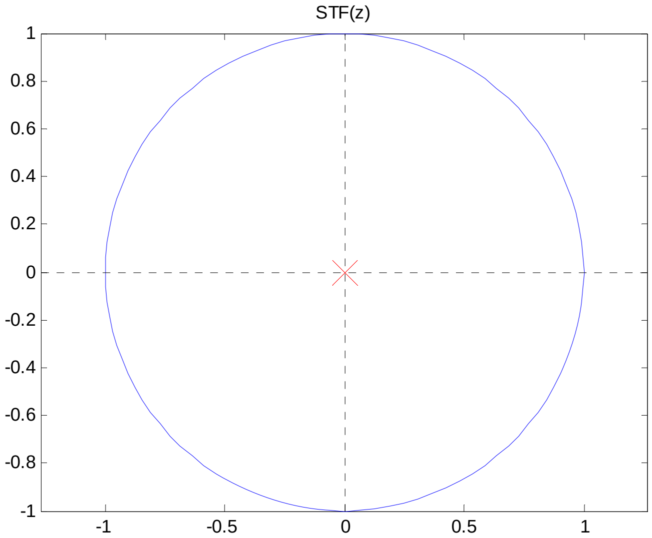
$$NTF(z) = (1 - z^{-1})^4$$

$$STF(z) = z^{-1}$$

$$L_0(z) = \frac{STF(z)}{NTF(z)} = \frac{z^{-1}}{(1 - z^{-1})^4}$$

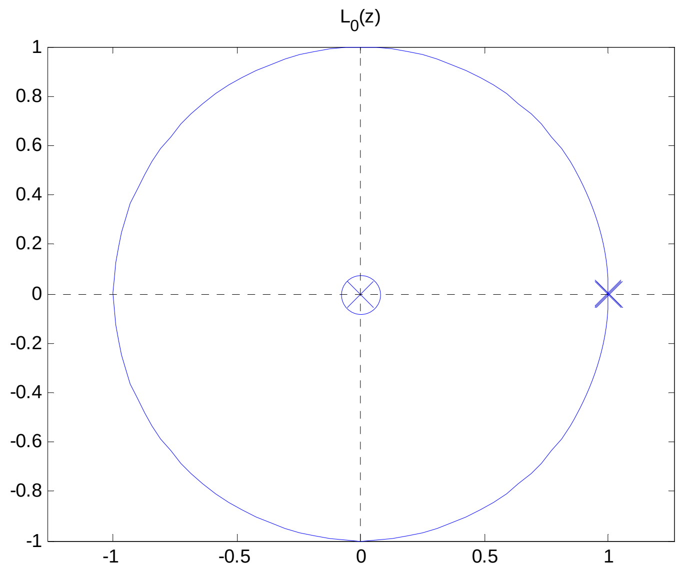
$$L_1(z) = \frac{1}{NTF(z)} - 1 = \frac{4z^{-1}\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}{(1 - z^{-1})^4}$$

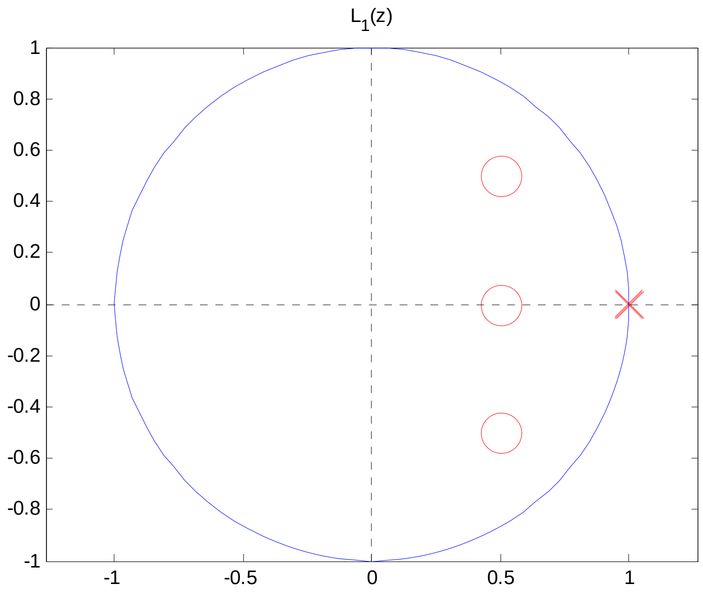




File: FourthOrderPolesZeros.m

Fourth-Order Loop-Filters: L_0 and L_1



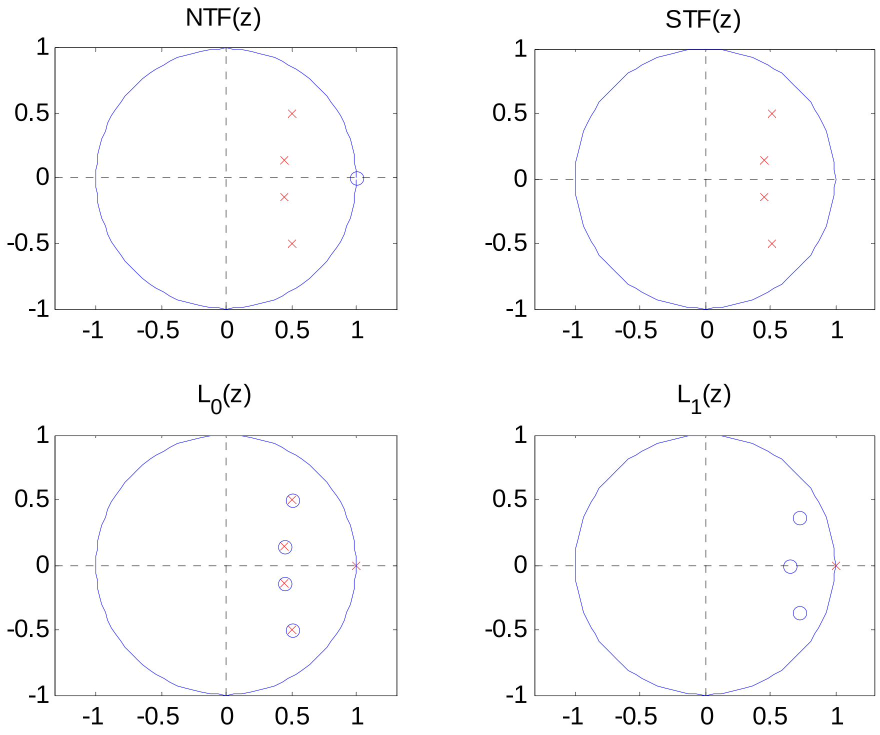


- L_0 and L_1 share same poles.
  - ✓ These poles become NTF zeros.
- L_1 zeros combined with the poles, give rise to NTF poles.
- L_0 zeros are different from those of L_1.
- Refer to the notes for details.

File: FourthOrderPolesZeros.m

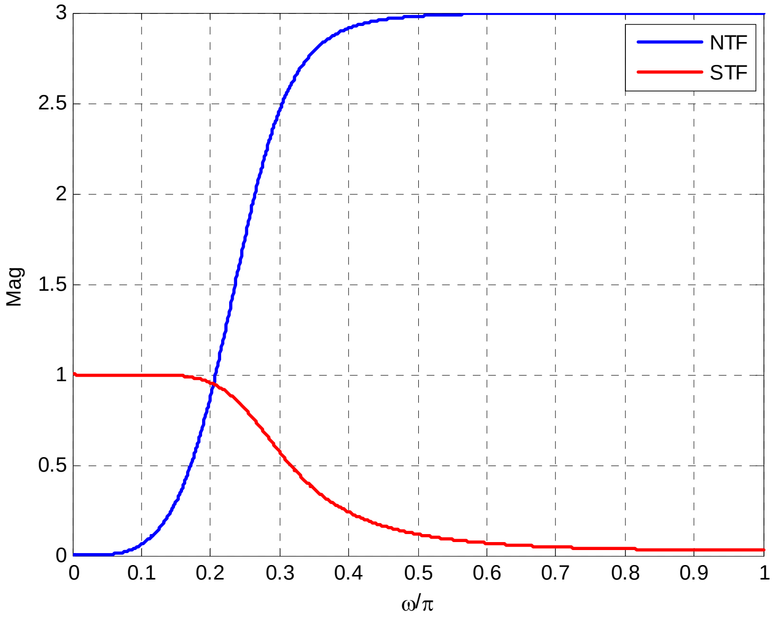
CIFB Example 1

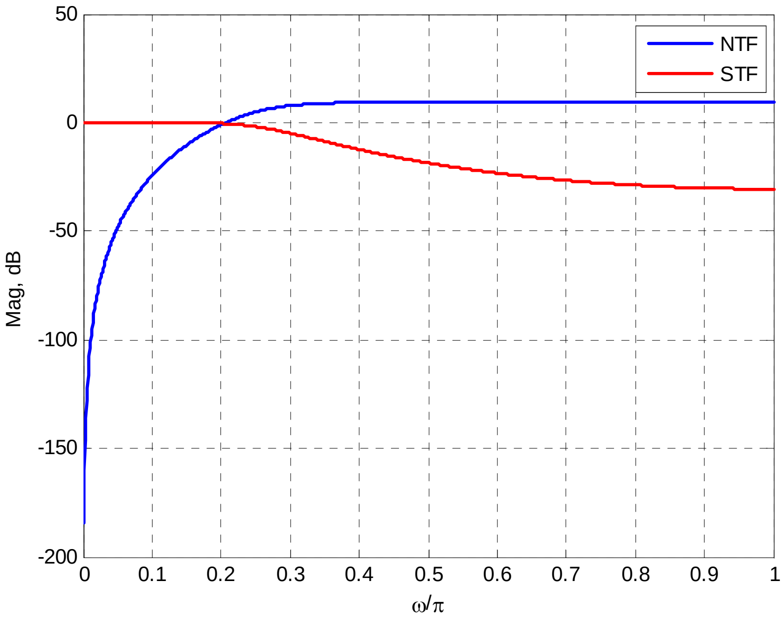
- CIFB, order = 4
- All NTF zeros at z=1, i.e. opt =0.
- OBG = 3, OSR = 16, nLev = 15.
- Only single input coupling is used
  - ✓ b(2:end) = 0
  - ✓ Maxflat poles in STF
- **a** = [0.16 0.86 1.9 2.1]
- **b** = [0.16 0 0 0]
- **c** = [1 1 1 1]
- **g** = [0 0]



File: CIFB_4^{th}_Order_1.m

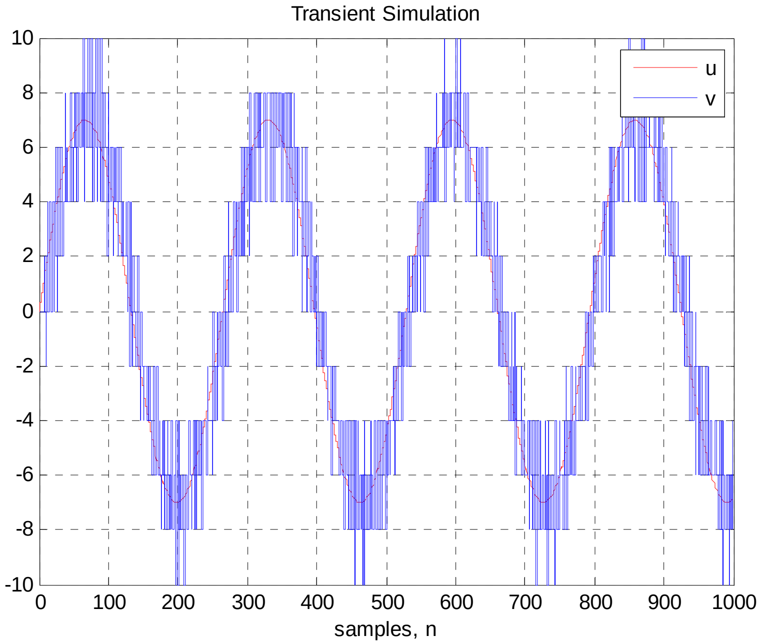
CIFB Example 1 contd. : NTF and STF

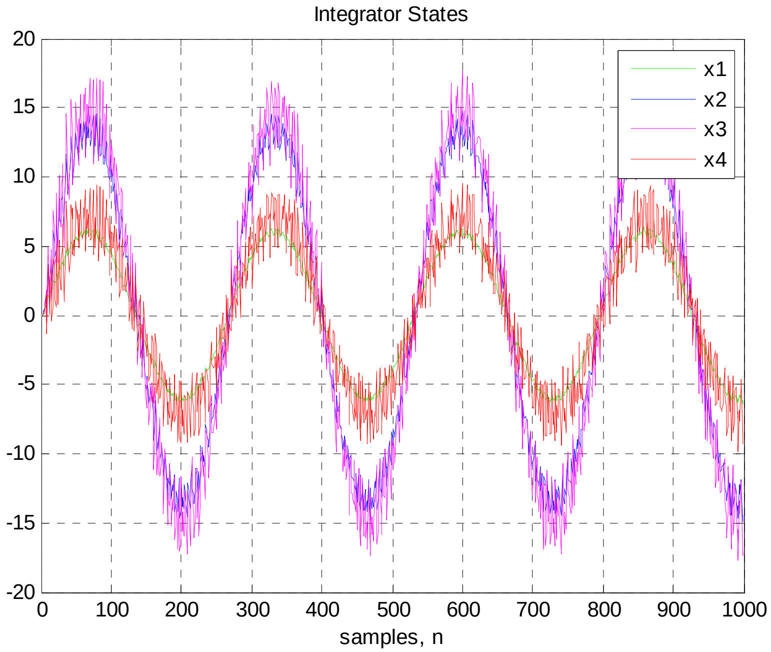




File: CIFB_4th_Order_1.m

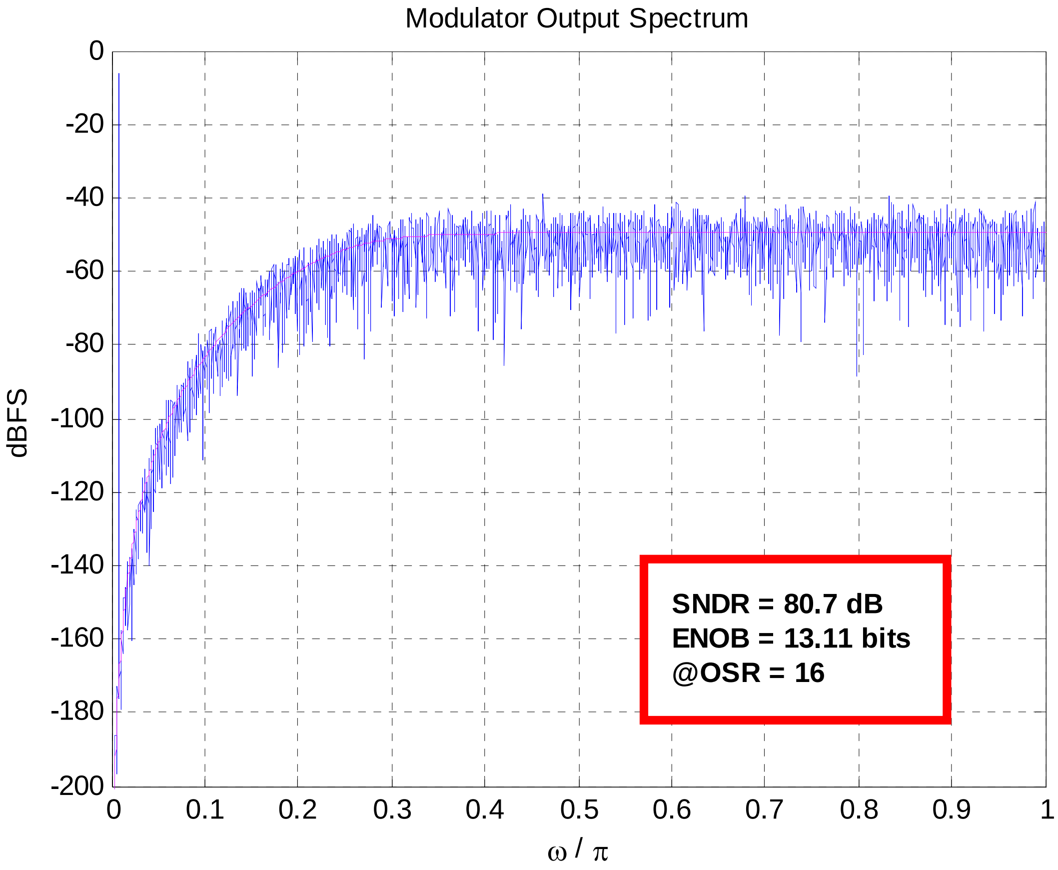
CIFB Example 1 contd. : Loop-Filter States





File: CIFB_4th_Order_1.m

CIFB Example 1 contd. : Simulated Spectrum

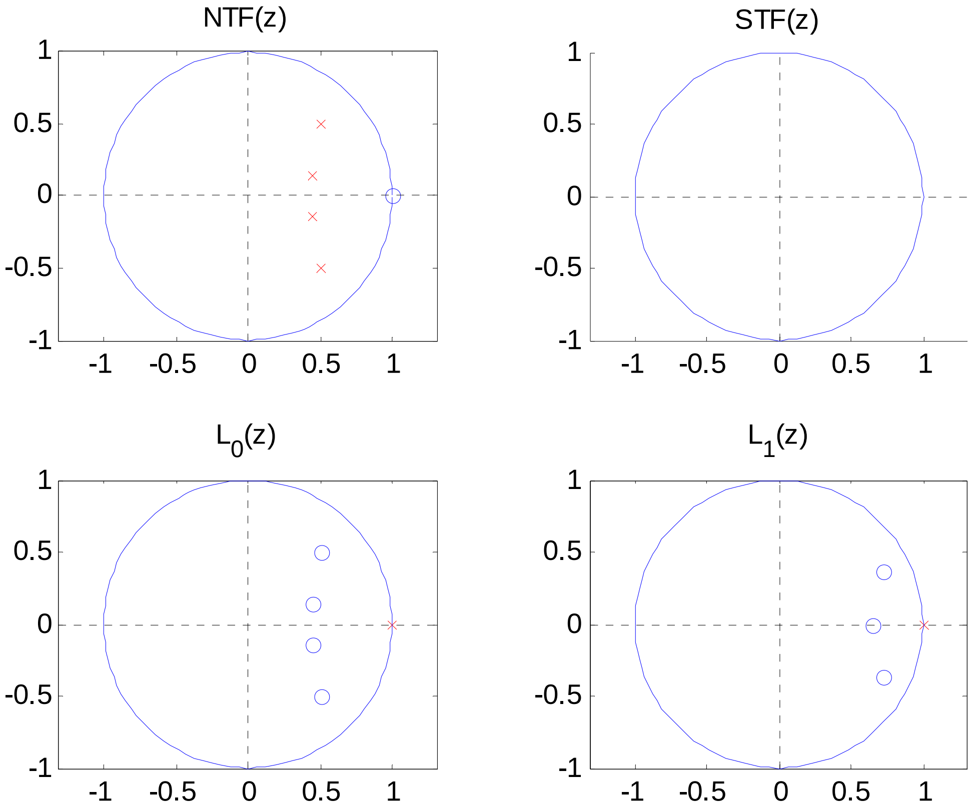


SNDR = 80.7 dB
ENOB = 13.11 bits
@OSR = 16

File: CIFB_4^{th}_Order_1.m

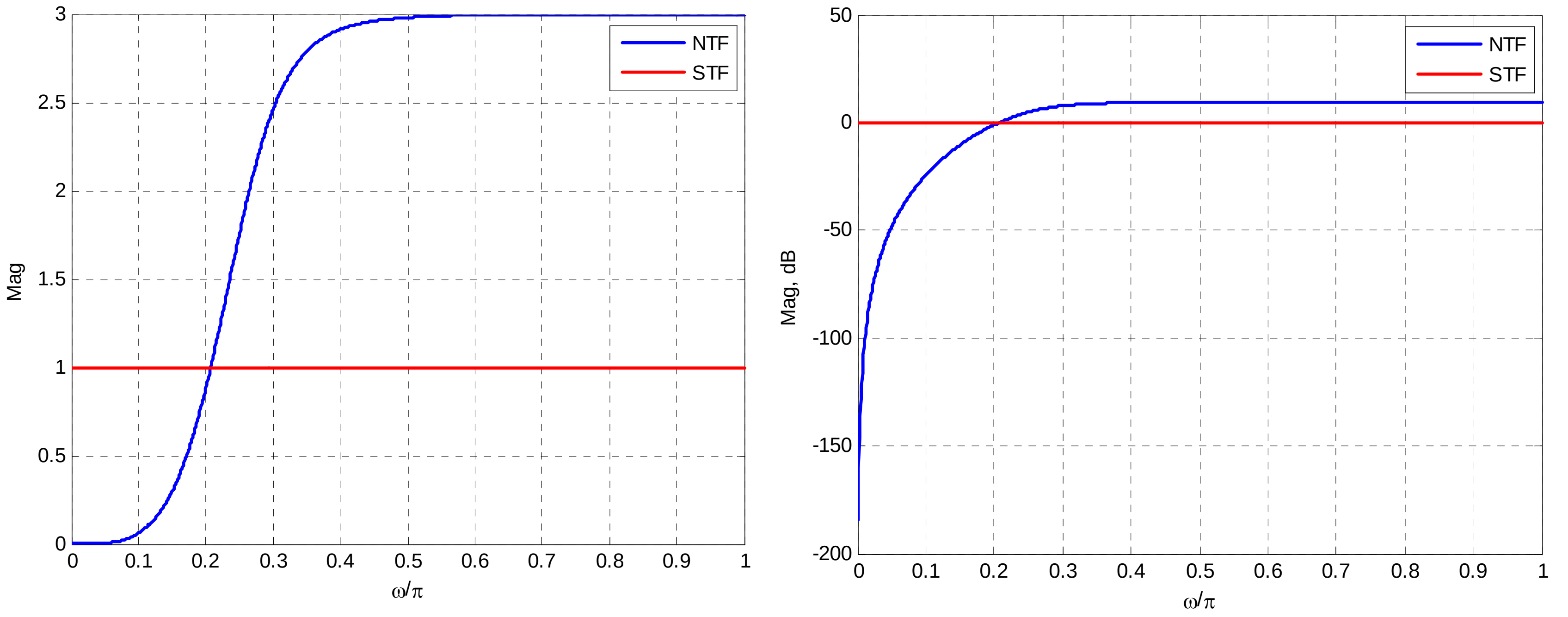
CIFB Example 2

- CIFB, order = 4
- All NTF zeros at z=1, i.e. opt =0.
- OBG = 3, OSR = 16, nLev = 15.
- Low-distortion topology
  - ✓ $b_i = a_i$
  - ✓ Maxflat poles in STF
- **a** = [0.16 0.86 1.9 2.1]
- **b** = [0.16 0.86 1.9 2.1]
- **c** = [1 1 1 1]
- **g** = [0 0]



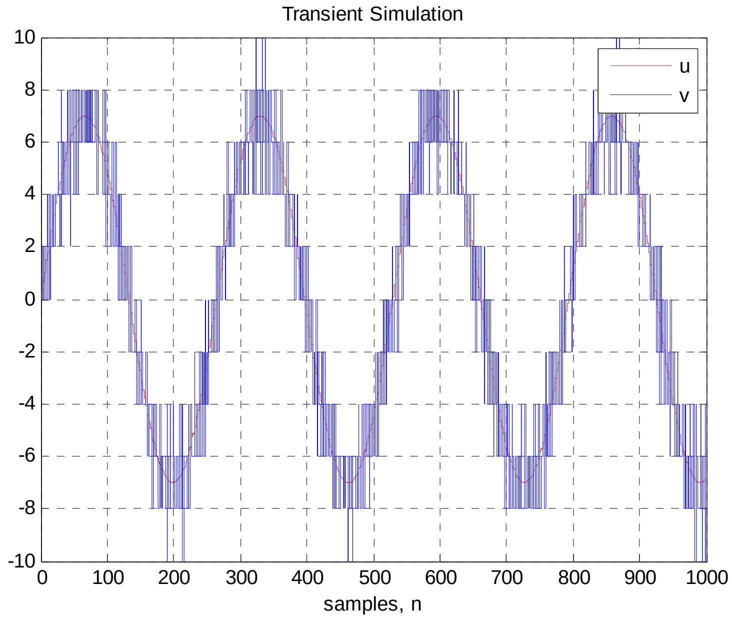
File: CIFB_4th_Order_2.m

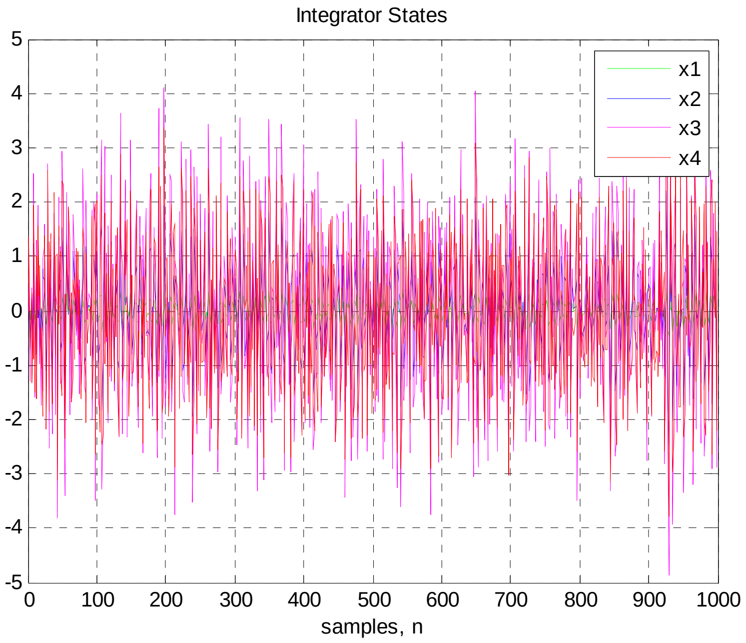
CIFB Example 2 contd. : NTF and STF



File: CIFB_4th_Order_2.m

CIFB Example 1 contd. : Loop-Filter States

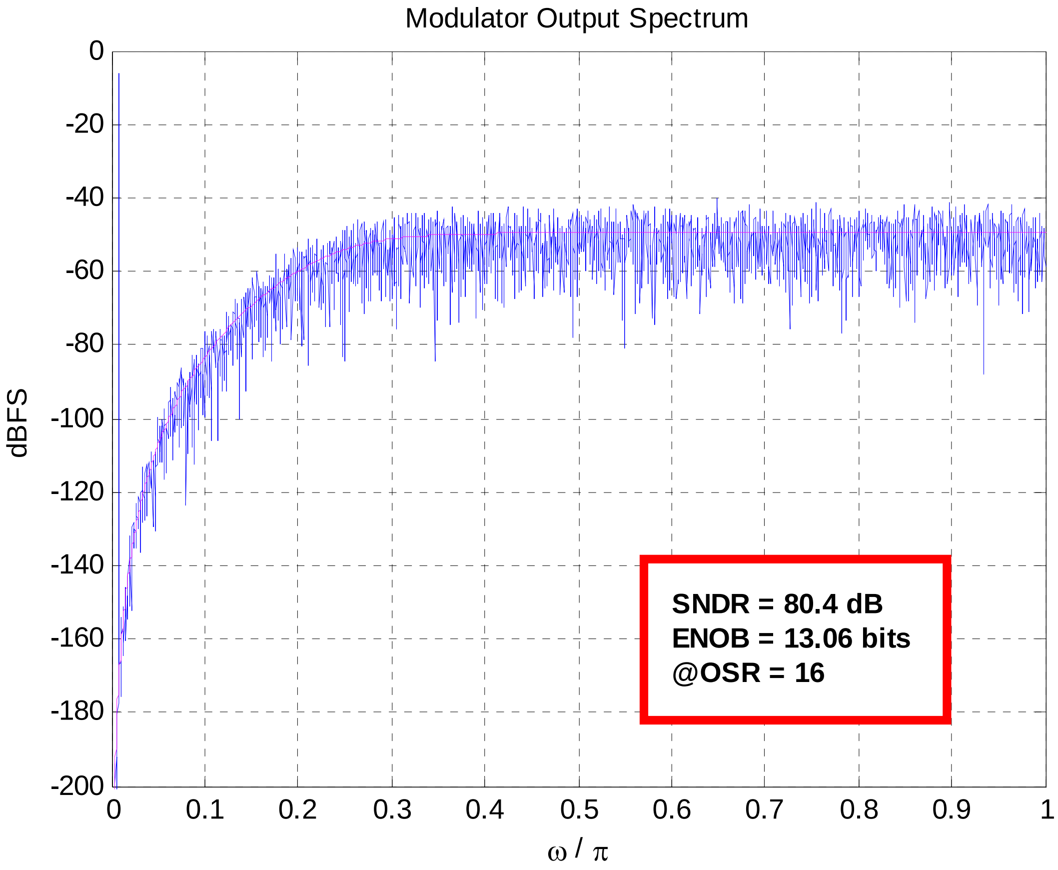




- ☐ Note that the integrator state excursions are drastically reduced.

File: CIFB_4th_Order_2.m

CIFB Example 2 contd. : Simulated Spectrum


Modulator Output Spectrum

SNDR = 80.4 dB
ENOB = 13.06 bits
@OSR = 16

File: CIFB_4th_Order_2.m

Other Examples of Feedback Topologies

- CRFB with single feed-in
  - ✓ CRFB_4th_Order_1.m
- Low-distortion CRFB topology
  - ✓ CRFB_4th_Order_2.m
- CIFB with single feed-in and optimized NTF zeros
  - ✓ CIFB_Opt_4th_Order_1.m
- Low-distortion CIFB topology with optimized NTF zeros
  - ✓ CIFB_Opt_4th_Order_2.m

File: CIFB_4th_Order_2.m