④ Clipping of the output waveform occurs more frequently $\Rightarrow$ 'k_1' drops further

$\Rightarrow$ poles move faster towards unit circle

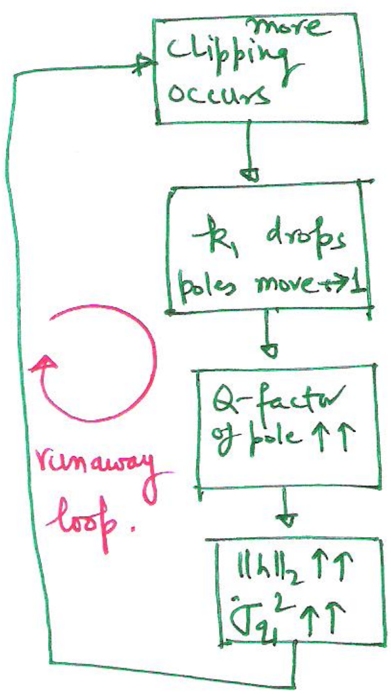
runaway loop

$\Rightarrow$ noise variance increases further

⑤ System becomes unstable

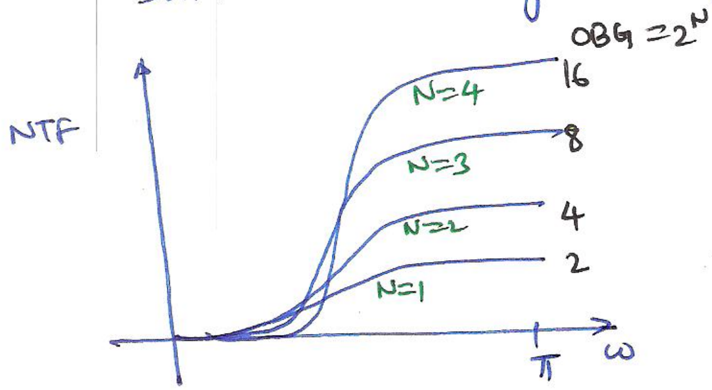
↳ State variable become unbounded.

↳ SNR drops down



• Very rapid runaway to instability.

$\Rightarrow$ Modulator can not be stable for the signal level upto the Saturation level of the quantizer. (MSA $\Rightarrow$ maximum stable amplitude)



As OBG increases, the variance of the noise is higher.

$\Rightarrow$ Quantizer overloads more often

$\Rightarrow$ MSA starts decreasing rapidly.

• OBG $= H(-1) = \|h\|_1$

* Variance of the quantization noise at the quantizer input

$$= \frac{\Delta^2}{12}\left(\sum_n h^2(n) - 1\right) = \begin{cases} 5\frac{\Delta^2}{12} & \text{for } N=2 \\ 19\frac{\Delta^2}{12} & \text{for } N=3 \\ 69\frac{\Delta^2}{12} & \text{for } N=4. \end{cases}$$

* Maximum stable amplitude (MSA): → aka 'U_{max}'
  - MSA < full scale range of the quantizer.
  - As the NTF order, $N \uparrow \Rightarrow$ MSA $\downarrow\downarrow$

Multi-bit Modulators:

- If the number of levels is increased:
  - ⇒ LSB size decreases $\left(\Delta = \frac{FS}{2^n}\right)$
  - ⇒ amplitude and variance of the quantizer noise at 'y' goes down
  - ⇒ Can allow larger input signal amplitude without overloading the quantizer.
  - ⇒ MSA increases! (better stability for same order)

- "A binary (single-bit) ΔΣ modulator with NTF = H(z) is likely to be stable if $\max_\omega |H(e^{j\omega})| < 1.5$ ← Lee criterion
  - ⇒ $OBG = \max_\omega |NTF(e^{j\omega})| < 1.5$ for single-bit modulators
    - ↳ criterion is neither sufficient nor necessary.
    - ↳ has no solid theoretical foundation
      - ↳ need simulations to verify

- $\max_\omega |H(e^{j\omega})|$ is also called the infinity-norm of H, denoted by $\|H\|_\infty$.

for multi-bit Modulators, we have : (Assume m-level quantizer)
$m = M+1$
$\hookrightarrow$ no. of steps for consistency with the book.

Assume the conditions:

1. The modulator is not overloaded at time $t=0$.
2. The input signal is bounded, $\|u\|_\infty \equiv \max(|u[n]|) < \infty$

observe the accumulated quantization noise at the input of the quantizer i.e. 'y'.

$$\Rightarrow \quad Y(z) = V(z) - E(z) = U(z) + (NTF(z) - 1)\cdot E(z)$$

$$\Rightarrow \quad y[n] = u[n] + e[n] \circledast (h[n] - \delta[n])$$

$$= u[n] + \sum_{i=1}^{\infty} h[i]\, e[n-i].$$

using the property that $|a+b| \le |a| + |b|$

$$\Rightarrow \quad |y[n]| \le |u[n]| + \left|\sum_{i=1}^{\infty} h[i]\, e[n-i]\right|$$

$$\le |u[n]| + \sum_{i=1}^{\infty} |h[i]|\,|e[n-i]|.$$

$$\le |u[n]| + \frac{\Delta}{2}\cdot\sum_{i=1}^{\infty} |h[i]| \qquad \because |e[k]| \le \frac{\Delta}{2}$$

$$= |u[n]| + \frac{\Delta}{2}(\|h\|_1 - 1) \longrightarrow (1)$$

To avoid overloading of the quantizer

$$|y[n]| \le \frac{m\Delta}{2} = \text{FS range} \longrightarrow (2)$$

from (1) and (2), we have

$$|u[n]| + \frac{\Delta}{2}(\|h\|_1 - 1) \le m\frac{\Delta}{2}$$

$$\Rightarrow \quad \boxed{\max_n |u[n]| \le \frac{\Delta}{2}(m - \|h\|_1 + 1).}$$

For $\Delta = 2$ and $m = M+1$, we get

$$\boxed{\|u\|_\infty \le M + 2 - \|h\|_1}$$

* Text book Page 105.

The modulator is guaranteed ~~to~~ not to experience overload for

$$\boxed{\max_n |u(n)| \overset{\Delta}{=} \|u\|_\infty \leq M + 2 - \|h\|_1}$$

↳ sufficient but 'not' necessary condition!

↳ actual u_{max} (or MSA) is ~~fo~~ determined from simulations.

Ex. ① for $M = 16$, $NTF(z) = (1-z^{-1})^3 \Rightarrow \|h\|_1 = 8$, we get

$\|u\|_\infty \leq 10 \Rightarrow \frac{10}{16} = 62.5\%$ of the full-scale range.

Summary: Even though a single-bit quantizer is the simplest to use in a DSM, multi-bit quantizers enhance modulator stability by an impressive margin for third or higher-order noise-shaping.

↳ ~~too~~ smaller LSB size ⇒ lower probability of quantizer overload.

↳ other benefits of multi-bit ~~quan~~ DSM on book pg. 179-180.

↳ Multi-bit DAC will require element mismatch shaping techniques (DEM), to be discussed later.

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* A note on simple quantizer gain modeling/estimation:

$$y \xrightarrow{k} \oplus \xrightarrow{} v, \quad e \downarrow \oplus$$

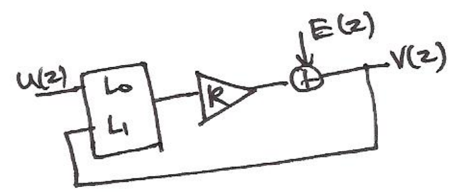
$v = ky + e$, $k = ?$

Use optimal 'k' which minimizes mean-square error

$$E(e \cdot e) = E[(v - ky)(v - ky)] = E[v \cdot v - 2k v \cdot y + k^2 y \cdot y]$$

$$\Rightarrow \frac{\partial E(e^2)}{\partial k} = 0 - 2E(v \cdot y) + 2k E[y \cdot y] = 0$$

$$\Rightarrow \boxed{k = \frac{E[v \cdot y]}{E[y \cdot y]} = \frac{E[v y]}{E[y^2]}}$$



for single-bit quantizer, ~~you can see~~ $v = \mathrm{sgn}(y)$

$$\Rightarrow v \cdot y = \mathrm{sgn}(y) \cdot y = |y|$$

$$\Rightarrow \boxed{k = \frac{E[|y|]}{E[y^2]}}$$

* A single linear loop.

* Less accurate than the describing function model.

* fails for overloading case.

• Use simulation data to find 'k'.