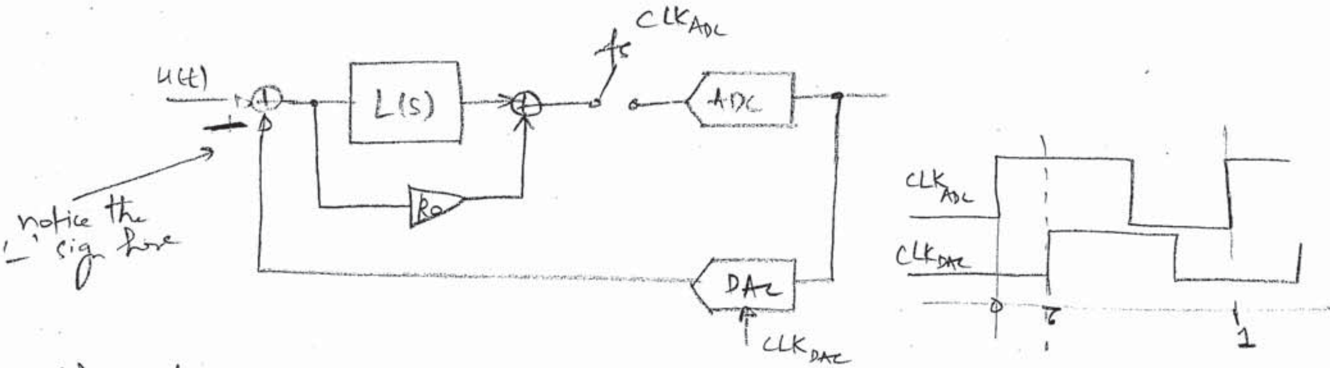
③ Direct feedback path around the quantizer



$\Rightarrow \quad L'(s) = k_0 + \frac{k_1}{s} + \frac{k_2}{s^2}$

we had $\quad L(z) = -\frac{2z+1}{(z-1)^2}$

Extra feedback path provides the extra control parameter in the loop response.

Now, if when the ELD is completely compensated:

$\Rightarrow \quad k_0 z^{-1} + k_1 \times (\text{RHS of } Ⓐ) + k_2 \times (\text{RHS of } Ⓑ) = -\left(\frac{2z+1}{(z-1)^2}\right) = \frac{2z-1}{(z-1)^2}$

Note the sign change

Going through the algebra we get:

$$\left.\begin{aligned} &0.5\tau^2 k_2 - \tau k_1 + k_0 = 0 \\ &(0.5 - \tau + 0.5\tau^2) k_2 + (1-\tau) k_1 + k_0 = 2 \\ &-(0.5 + \tau - \tau^2) k_2 + (1 - 2\tau) k_1 + 2k_0 = 1 \end{aligned}\right\} \longrightarrow ①$$

Solving this set of equation we get.

$$\{k_0', k_1', k_2'\} = \{1.5\tau + 0.5\tau^2, 1.5 + \tau, 1\}$$

verify for $\tau = 0$, $\{k_0', k_1', k_2'\} = \{0, 1.5, 1\}$ ← Same as expected.

$\Rightarrow$ k_1 is tuned and k_0 is added.

↳ This process requires one to go back and forth s- and z-domains,
↳ tedious algebra
↳ painful for higher-order modulators.

Pavan's solution for loop filter comprising of ideal integrators (no resonators).

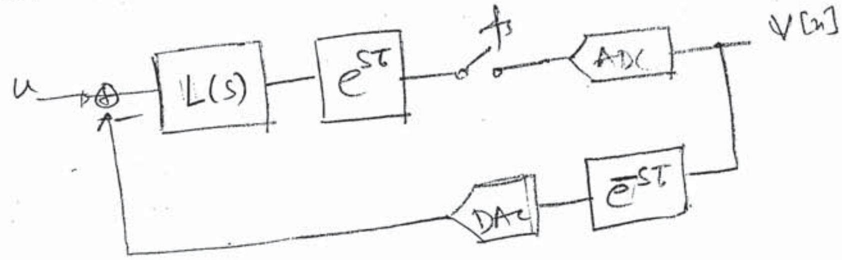
The technique:

if the CT $\Delta\Sigma$ loop-filter is

$$L(s) = \frac{k_1}{s} + \frac{k_2}{s^2} + \cdots + \frac{k_N}{s^N} \quad \text{with no excess-loop delay.}$$

What value of coefficients $\{k_0', k_1', \ldots, k_N'\}$ must be chosen so that the NTF remains same even when there is an ELD of τ.

- Conceptually we can compensate for ELD by cascading $L(s)$ with a block with TF $e^{s\tau}$



clearly $e^{s\tau} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \delta(t+\tau)$

is non-causal!

↳ not realizable in practice.

- We get an interesting case when, the input is piecewise constant

and we are only interested in the sampled output

↳ $e^{s\tau}$ can be reinterpreted to obtain

$$\hat{H}(s) = H(s)\,e^{s\tau} = k_1\frac{e^{s\tau}}{s} + k_2\frac{e^{s\tau}}{s^2} + \cdots + k_N\frac{e^{s\tau}}{s^N}$$

A NRZ DAC (Proof later)

for NRZ DAC, expand $e^{s\tau}$ as the polynomial is 's', such that

↳ for i^{th} integrator $\frac{e^{s\tau}}{s^i}$ is truncated beyond the i^{th} power of s

⇒

$$\frac{e^{s\tau}}{s} \longrightarrow \frac{1}{s}(1+s\tau) = \frac{1}{s}+\tau \longrightarrow (1)$$

$$\frac{e^{s\tau}}{s^2} \longrightarrow \frac{1}{s^2}\left(1+s\tau+\frac{s^2\tau^2}{\underline{|2}}\right) = \frac{1}{s^2}+\frac{\tau}{s}+\frac{\tau^2}{2} \longrightarrow (2)$$

$$\frac{e^{s\tau}}{s^i} \longrightarrow \frac{1}{s^i}\left(1+\cdots+\frac{s^i\tau^i}{\underline{|i}}\right) = \frac{1}{s^i}+\frac{\tau}{s^{i-1}}+\cdots+\frac{\tau^i}{\underline{|i}} \longrightarrow (N)$$

Then, the loop filter TF whose samples are identical to $L(z)$, are given by the weighted summation of the RHS of these (N) equations by $k_1, \ldots, k_N$ resp.

$$\Rightarrow \hat{L}(s) = k_1\left(\frac{e^{s\tau}}{s}\text{ exp}\right) + k_2\left(\frac{e^{s\tau}}{s^2}\text{ expan}\right)+\cdots+k_i\left(\frac{e^{s\tau}}{s^i}\text{ exp.}\right)+\cdots+\frac{k_N}{s^N}$$

$$\Rightarrow \hat{L}(s) = \left(k_1\tau + k_2\frac{\tau^2}{\underline{|2}}+\cdots+k_N\frac{\tau^N}{\underline{|N}}\right) + \frac{\left(k_1+k_2\tau+\cdots+k_N\frac{\tau^{N-1}}{\underline{|N-1}}\right)}{s}+\cdots$$

$$+\cdots+\frac{k_i+k_{i+1}\tau+\cdots+k_N\frac{\tau^{N-i}}{\underline{|N-i}}}{s^i}+\cdots+\frac{k_N}{s^N}$$

$$\Rightarrow \begin{cases} k_0' = k_1\tau+\frac{k_2}{\underline{|2}}\tau^2+\cdots+k_N\frac{\tau^N}{\underline{|N}} = \sum_{i=1}^{N} k_i\frac{\tau^i}{\underline{|i}} \\ k_1' = k_1+k_2\tau+\cdots+k_N\frac{\tau^{N-1}}{\underline{|N-1}} = \sum_{i=1}^{N} k_i\frac{\tau^{i-1}}{\underline{|i-1}} \\ \vdots \\ k_N' = k_N \end{cases}$$

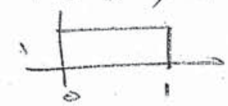
verify the result for the second order case by using $\{k_1, k_2\} = \{1.5, 1\}$

Proof :

$u_k(t) \rightarrow$ unit step function integrated 'k' times

$u_0(t) = u(t) \leftarrow$ unit step function $y(t)$ 

The DAC pulse is $u(t) - u(t-1)$, initially assume $\tau = 0$



$x_i(t) \leftarrow$ output of the i^{th} integrator.

$N \leftarrow$ order of the loop filter

$y(t) \leftarrow$ output " " " "

$\Rightarrow$ we have

$$y(t) = \sum_{i=1}^{N} k_i x_i(t) = \sum_{i=1}^{N} k_i \left(u_i(t) - u_i(t-1) \right)$$

the sampled output

$$y[n] = \sum_{i=1}^{N} k_i x[n] = \sum_{i=1}^{N} k_i \left(u_i[n] - u_i[n-1] \right).$$

when the DAC pulse is delayed by 'τ',

the integrator and loop filter outputs become $x_i(t-\tau)$ and $y(t-\tau)$.

Using Taylor series for $0 < \tau < 1$, we have the ideal sampled output of the i^{th} integrator can be expressed as:

$$x_i[n] = x_i(t)\Big|_{t=n} = x_i(t-\tau)\Big|_{t=n} + \tau \frac{dx_i(t-\tau)}{dt}\Big|_{t=n} + \cdots + \frac{\tau^l}{\underline{|l}} \frac{d^l x_i(t-\tau)}{dt^l}\Big|_{t=n} + \cdots \quad \text{①}$$

Now since,

$$\frac{dx_i(t-\tau)}{dt} = x_{i-1}(t-\tau) \;\&\; \frac{dx_0(t-\tau)}{dt}\Big|_{t=n} = 0$$

① reduces to

$$x_i[n] = \left[x_i(t-\tau) + \tau x_{i-1}(t-\tau) + \cdots + \frac{\tau^i}{\underline{|i}} x_0(t-\tau) \right]\Bigg|_{t=n}$$

$$f(t) = f(t_0) + \frac{f'(t_0)}{1!}(t-t_0) + \frac{f''(t_0)}{2!}(t-t_0)^2 + \cdots + \frac{f^{(l)}(t_0)}{l!}(t-t_0)^l + \cdots$$

$$f(t) = x_i(t)$$

$$t_0 = t - \tau \Big|_{t=n} \quad \cancel{\text{with}} \quad \text{i.e. } \underline{\underline{\text{here}}}\ t = n$$

$$\Rightarrow \quad x_i(t) = x_i(t-\tau) + \frac{x_i'(t-\tau)}{1!}\tau + \cdots + x_i^{(l)}(t-\tau)\cdot\tau^l + \cdots$$

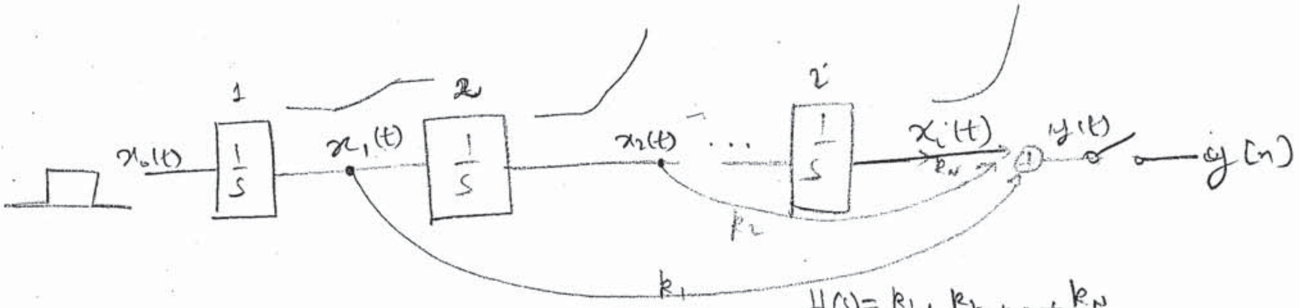
$$\underbrace{x_i(t)\Big|_{t=n}}_{\text{ideal integrator output}} = x_i(t-\tau)\Big|_{t=n} \underbrace{+ \tau\, x_i'(t-\tau)\Big|_{t=n} + \cdots + \tau^l x_i^{(l)}(t-\tau)}_{\text{current integrator outputs}}\Big|_{t=n} + \cdots$$

$$x_i'(t-\tau) = x_{i-1}$$

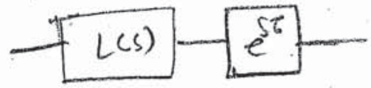
$$\underline{x_0'(t-\tau) = 0}$$

$$\Rightarrow \quad x_i[n] = \left[x_i(t-\tau) + \tau\cdot x_{i-1}(t-\tau) + \cdots + \frac{\tau^i}{i!}x_0(t-\tau)\right]\Bigg|_{t=n}$$

$$\frac{1}{s^i} + \tau\frac{s}{s^i} + \cdots + \frac{\tau^i}{i!}\frac{s^i}{s^i} = \frac{1}{s^i}\cdot e^{s\tau}\Big|_{\underline{(i+1)\text{ terms}}}$$



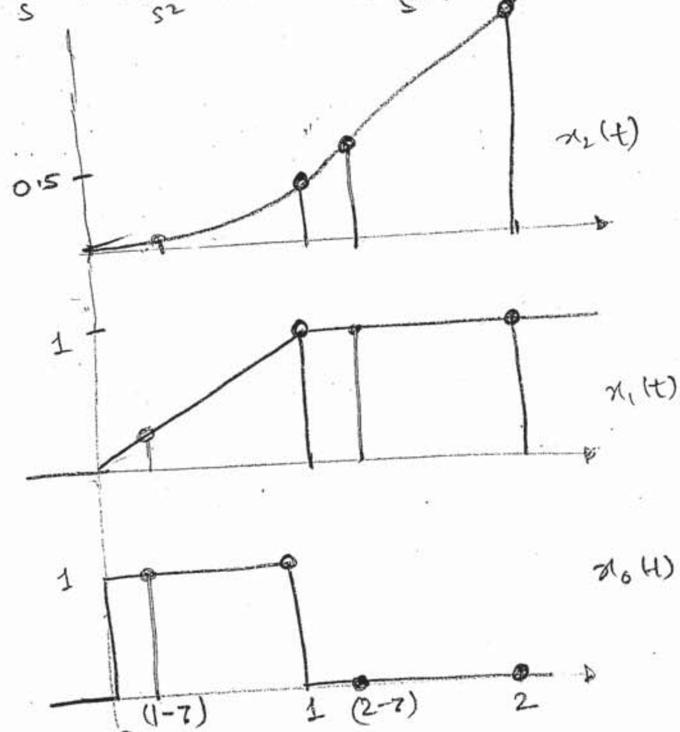
$H(s) = \frac{k_1}{s} + \frac{k_2}{s^2} + \dots + \frac{k_N}{s^N}$



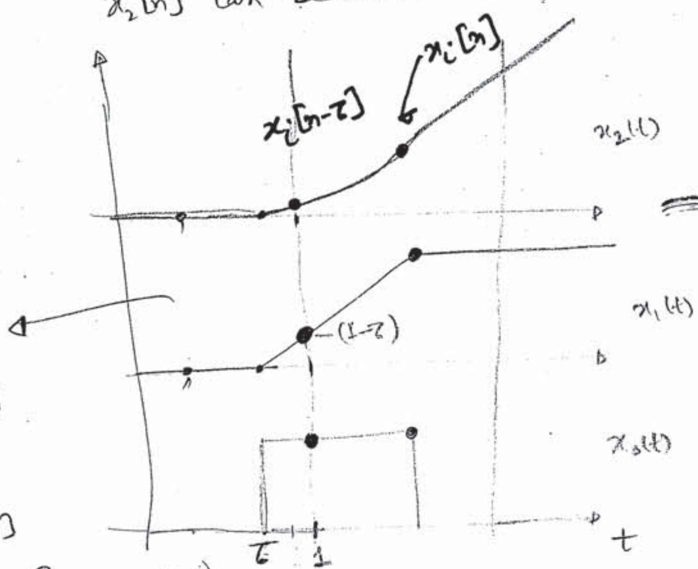
$\hat{H}(s) = L(s)e^{s\tau} = k_1\frac{e^{s\tau}}{s} + \frac{k_2 e^{s\tau}}{s^2} + \dots + k_N\frac{e^{s\tau}}{s^N}$

$x_0(t) = u(t) - u(t-1)$

$x_1(t)$



=> for a second-order modulator, and NRZ DAC, $x_2[n]$ can be determined from $x_2[n-\tau]$, $x_1[n-\tau]$ and $x_0[n-\tau]$.



reds can be linearly combined to estimate green dots.

$x_1[n] = x_1[n-\tau] + \tau \cdot x_0[n-\tau]$

$x_2[n] = x_2[n-\tau] + \tau x_1[n-\tau] + \frac{\tau^2}{2} x_0[n-\tau]$

$\Rightarrow$ Even with delayed DAC pulse, the ideal output samples of the i^{th} integrator can be generated by combining the output samples of the i^{th} integrator and the preceding $(i-1)$ integrators as well as the input to the loop filter $x_0(t-\tau)$.

In frequency-domain, we can say that the ideal output samples of the of the i^{th} integrator, can be obtained by sampling the output of a filter whose transfer function is given by

$$\frac{1}{s^i} + \tau\frac{s}{s^i} + \cdots + \frac{1}{\lfloor i}\tau^i\frac{s^i}{s^i} = \frac{1}{s^i}e^{s\tau}\Big|_{i+1 \text{ terms}}$$

$\Rightarrow$ TF of the compensated loop filter is given by

$$\hat{L}(s) = \frac{k_N}{s^N}\left(1 + s\tau + \frac{s^2\tau^2}{\lfloor 2} + \cdots + \frac{s^N\tau^N}{\lfloor N}\right) + \cdots + \frac{k_i}{s^i}\left(1 + s\tau + \frac{s^2\tau^2}{\lfloor 2} + \cdots + \frac{s^i\tau^i}{\lfloor i}\right) + \cdots + \frac{k_1}{s}(1 + s\tau) \longrightarrow (2)$$

$\Rightarrow$ from (2) the direct path comes out as a direct consequence

$$k_0' = k_1\tau + k_2\frac{\tau^2}{\lfloor 2} + \cdots + k_N\frac{\tau^N}{\lfloor N}.$$

(B) RZ feedback DAC (proof omitted).

Two cases:

(i) $\tau < 0.5$, the order of the system doesn't increase

↳ ELD compensation done by coeff tuning.

Same as NRZ derivation, but the expansion of $e^{s\tau}$ in $\frac{e^{s\tau}}{s^i}$ is truncated after $(i-1)^{th}$ power of s.

$$\Rightarrow \frac{e^{s\tau}}{s^i} \rightarrow \frac{1}{s^i}\left(1 + \cdots + \frac{(s\tau)^{i-1}}{\lfloor i-1}\right) = \frac{1}{s^i} + \frac{\tau}{s^{i-1}} + \cdots + \frac{\tau^{i-1}}{s\lfloor i-1}$$

→ the compensated loop TF is

$$\hat{L}(s) = \frac{k_1 + k_2\tau + \cdots + k_N \frac{\tau^{N-1}}{\lfloor N-1}}{s} + \frac{k_i + k_{i+1}\tau + \cdots + k_N \frac{\tau^{N-i}}{\lfloor(N-i)}}{s^i} + \cdots + \frac{k_N}{s^N}.$$

(ii) $\tau > 0.5$, a direct path is necessary in addition to coefficient tuning

↳ see ref for details.

→

⇒ simple hand-calcs all in s-domain!

↳ method requires that the higher order derivatives of the i^{th} integrator output become 0 when driven by a piecewise constant DAC pulse.

↳ Not true when NTF with complex zeros is used.

↳ pulse responses now contain sine and cosine.

↳ no easy solution exist :(

↳ The low pass formulae hence do an acceptable job of stabilizing the loop-delay for large OSR, low-pass DSMs.

↳ No solution for BP-DSMs.

Issues with Table based methods :

↳ (+) Certainly need mathematical analysis for better understanding/modeling of the system.

(−) The algebra (for the general case) is tedious and unwieldy

↳ TF of real op-amp have several poles/zeros due to finite A_{DC}, f_{un} effects. (assuming Active-RC implementation).

↳ obtaining these pole zero locations not an easy task!

↳ The system may not have a solution when the integrators are non-ideal and the poles of $L(z)$ are different from the poles of the integrator paths

eg for integrators with finite gain, poles of real loop filter will not be at $z=1$.

⇒ can not be solved.

Numerical fitting approach

↳ implemented in the realizeNTF_ct function.

$l[n] \leftarrow$ column vectors of N samples

Eg. for $NTF(z) = (1-z^{-1})^2$, we have

$l[n] = [0\ 2\ 3\ \cdots]^T$.

- The colⁿ vectors formed by N samples of the pulse responses of the direct-path and the integrator outputs are denoted as

$l_0[n] = [0\ 1\ 0\ \cdots\ 0]^T$

$l_1[n] = [0\ (1-\tau)\ 1\ \cdots\ 1]^T$

$l_2[n] = [0\ \ 0.5(1-\tau)^2\ \ (1.5-\tau)\ \cdots\ (N-0.5-\tau)]^T$

Choose 'N' such that it is much larger than the number of unknowns to be determined. Then we have the weighting coefficient $K = [k_0\ k_1\ k_2]^T$, determined by solving

$$[l_0[n]\ \ l_1[n]\ \ l_2[n]]\,K = l[n].$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1-\tau & 0.5(1-\tau)^2 \\ 0 & 1 & 1.5-\tau \\ \vdots & \vdots & \vdots \\ 0 & 1 & N-0.5-\tau \end{bmatrix}}_{N\times N} \underbrace{\begin{bmatrix} k_0 \\ k_1 \\ k_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{N\times 1} = \underbrace{\begin{bmatrix} \cancel{0} \\ 2 \\ 3 \\ 4 \\ \vdots \\ N+1 \end{bmatrix}}_{N\times 1}$$

more equations than unknown

with ideal integrators, the above set of equations admit a unique solution. ⇒ K is independent of N.

↳ does away with tedious Algebra

↳ easily obtained from simulation

(-) Has issues when real opamps are used

↳ read paper by Shanthi.