ECE 615 - Lecture 4

Note Title 1/21/2016

$x[n]$ of length L

$\tilde{x}[n]$ ← periodic

$\big\downarrow$ DFS

$\tilde{X}(k) \rightarrow$ periodic for $k = rN$

Discrete Fourier Transform (DFT)

$$X(k) = \begin{cases} \tilde{X}(k) & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] \xrightarrow{DFT} X(k)$$ N-point sequence

N-point N coefficients

'k' frequency domain indices

$0 \le k \le N-1$

N-point DFT

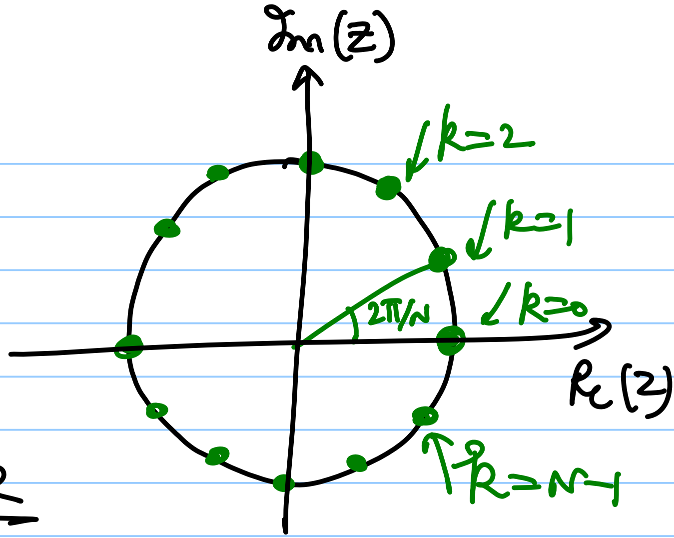
└→ fast algorithms for computation ⇒ FFT

Matlab → fft(.)

Also, we can show that

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$

DFT is DTFT sampled in frequency domain as $\omega = \frac{2\pi k}{N}$



Summary :

① make periodic extensions of $x[n]$ to obtain $\tilde{x}[n]$ with period N

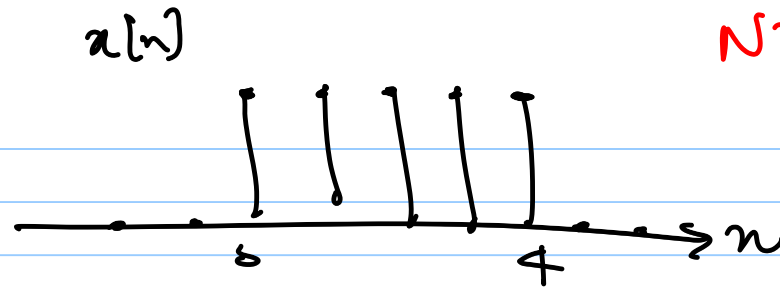
② find DFS coefficients $\tilde{X}[k]$ of $\tilde{x}[n]$

③ DFT $X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$

→ N-point FFT ⇒ usually $N = 2^L$, $L \in \mathbb{Z}^+$

DFT : $X[k] = \sum_{k=0}^{N-1} x[n] W_N^{nk}$ $W_N = e^{-j\frac{2\pi}{N}}$

Example:



$N=5$

We had -

$$X(e^{j\omega}) = e^{-j\omega}\,\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

Analysis FFT: $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

$$W_N = e^{-j\frac{2\pi}{N}}$$

Synthesis IFFT: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-nk}$

5-point FFT

$$X[k] = \sum_{n=0}^{4} W_5^{nk} = \sum_{n=0}^{4} \left(e^{-j\frac{2\pi}{5}}\right)^{nk}$$

$$= \sum_{n=0}^{4} \left(e^{-j\frac{2\pi}{5}k}\right)^{n}$$

$$= \frac{1 - e^{-j2\pi k}}{1 - e^{-\frac{2\pi k}{5}}}$$

$$= \frac{e^{-j\frac{2\pi k}{2}}\left(e^{j\pi k} - e^{-j\pi k}\right)}{e^{-j\frac{2\pi k}{10}}\left(e^{+j\frac{2\pi k}{10}} - e^{-j\frac{2\pi k}{10}}\right)}$$

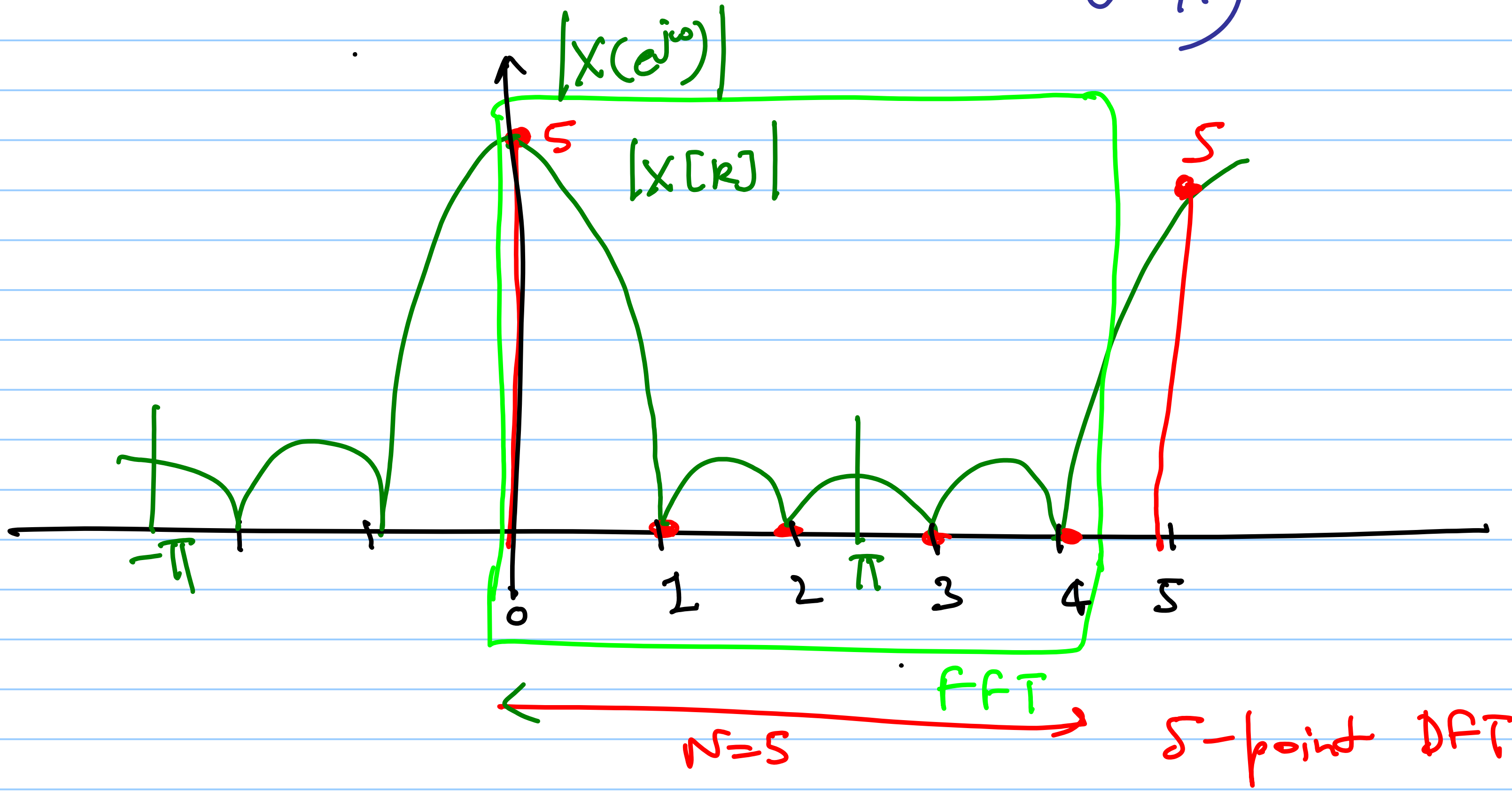
G.S.

$|r| \leq 1$

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

$$\frac{1 - e^{j\theta_1}}{1 - e^{j\theta_2}}$$

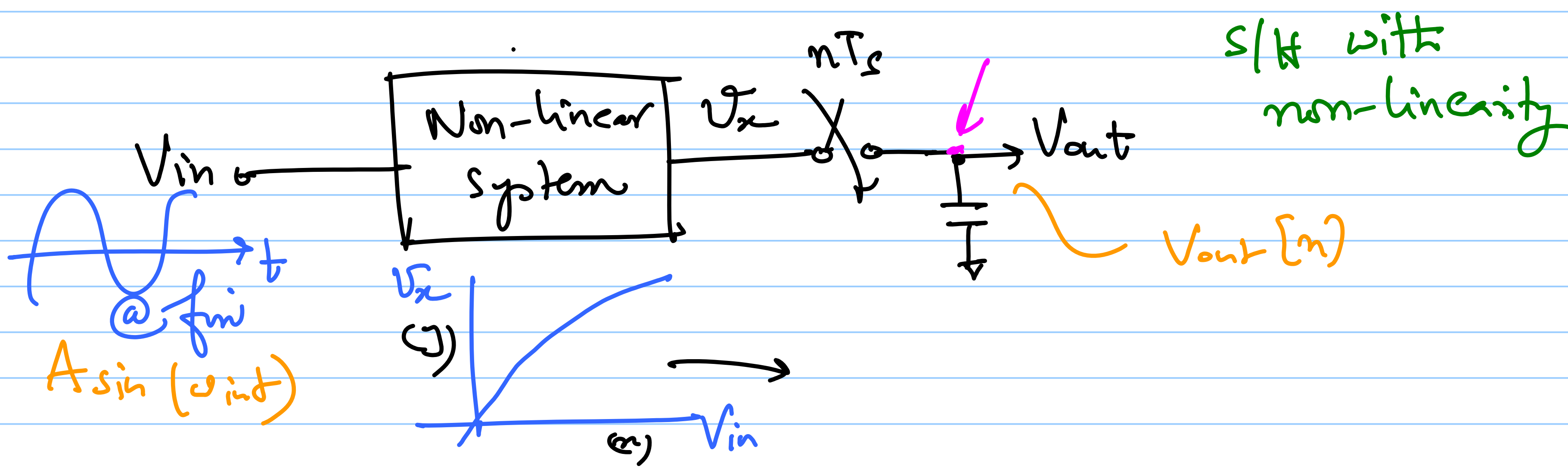
$$= e^{-j\left(\frac{2\pi k}{5}\right)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$



DFT/FFT is a discrete representation of $X(e^{j\omega})$ discretized at $\omega = \frac{2\pi k}{N}$

Spectral Estimation using FFT / Matlab

Example: Characterizing the distortion of a S/H using single tone (sine wave) input



$$V_x = \alpha_1 V_{in} - \alpha_3 V_{in}^3 \qquad V_{in} = A \sin(\omega_{in} t)$$

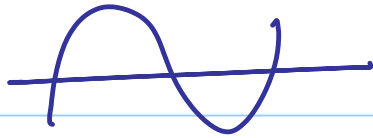
$$= \alpha_1 A \sin(\omega_{in} t) - \alpha_3 A^3 \sin^3(\omega_{in} t)$$

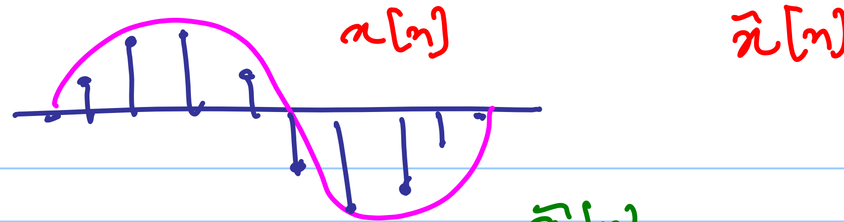
$$= \alpha_1 A \sin(\omega_{in} t) - \frac{3\alpha_3 A^3}{4} \sin(\omega_{in} t) + \frac{\alpha_3 A^3}{4} \sin(3\omega_{in} t)$$

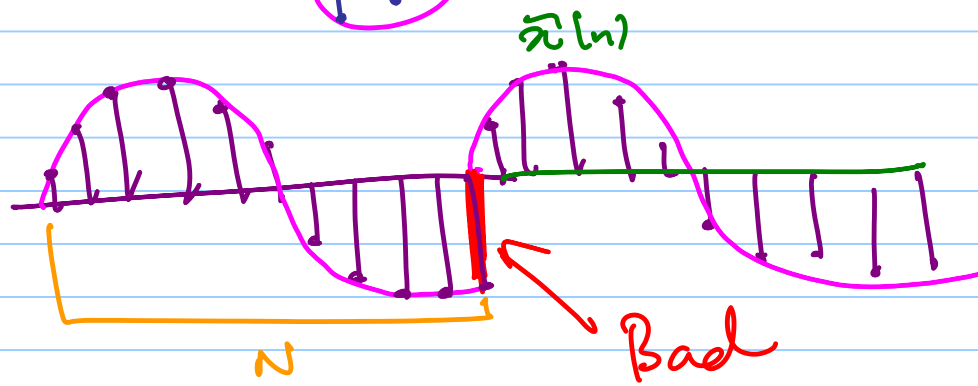
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$= \left(\alpha_1 A - \frac{3\alpha_3}{4} A^3\right) \sin(\omega_{in} t) + \frac{\alpha_3}{4} A^2 \sin(3\omega_{in} t)$$







Use a discrete time periodic sequence with period N

$$\tilde{v}_{out}[n] = v_{out}[n + rN]$$

This $\tilde{v}_{out}[n]$ can be represented as DFS

$$\hat{v}_{out}[n] = \sum_{k=0}^{N-1} V[k] e^{j\left(\frac{2\pi}{N}\right)kn}$$

complex DFS coefficients

V[k] is easily computed using FFT.

"bins"

"N"

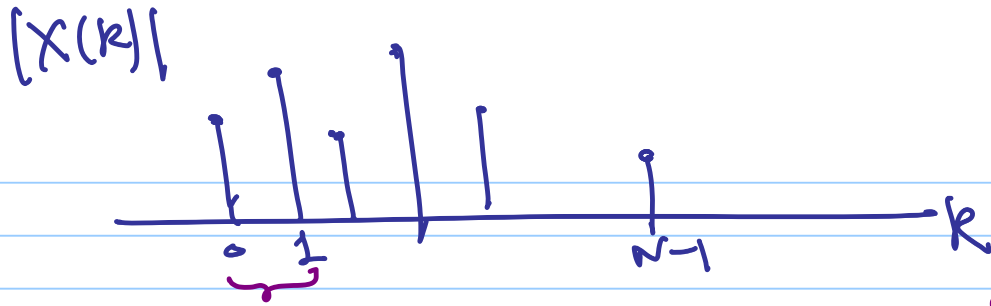
frequencies

$$\omega \Rightarrow \left\{0, \frac{2\pi}{N}, \frac{2\pi}{N}\cdot 2, \ldots, \left(\frac{2\pi}{N}\right)(N-1)\right\}$$

$\frac{N}{f_s} \rightarrow$ time in continuous-time axis

$\frac{f_s}{N} \Rightarrow$ resolution of the FFT

spacing between the tones



$\frac{f_s}{N}$ ← resolution or bin size

$M \Rightarrow$ size of data collected from the simulation or measurement

↳ record length

If FFT is taken over the whole record length

FFT size $(N) = M$

$$v_{in} = A \sin(2\pi f_{in} t) = \Im\{A e^{j2\pi f_{in} t}\}$$

$$v_x = \sum_k a_k e^{j2\pi k f_{in} t}$$

$a_1 \Rightarrow$ fundament

$a_{2,3,\ldots} \Rightarrow$ harmonics

sample

$$\tilde{v}_{out}[n] = \sum_k a_k e^{j2\pi k f_{in} \frac{n}{T_s}}$$

$t = \frac{n}{T_s}$

$\tilde{v}_{out}[n]$ is periodic only when

$$e^{j\frac{2\pi k f_{in}(n+N)}{f_s}} = e^{j2\pi k \frac{f_{in}}{f_s} n}$$

$$e^{j\frac{2\pi k f_{in} n}{f_s}} \times \underbrace{e^{j\frac{2\pi k f_{in} N}{f_s}}}_{\triangleq 1} \triangleq e^{j 2\pi k \frac{f_{in}}{f_s} n}$$

$$e^{j2\pi}$$

$$\Rightarrow \cancel{2\pi} \frac{f_{in}}{f_s} N = 2m\cancel{\pi}, \quad m \in I$$

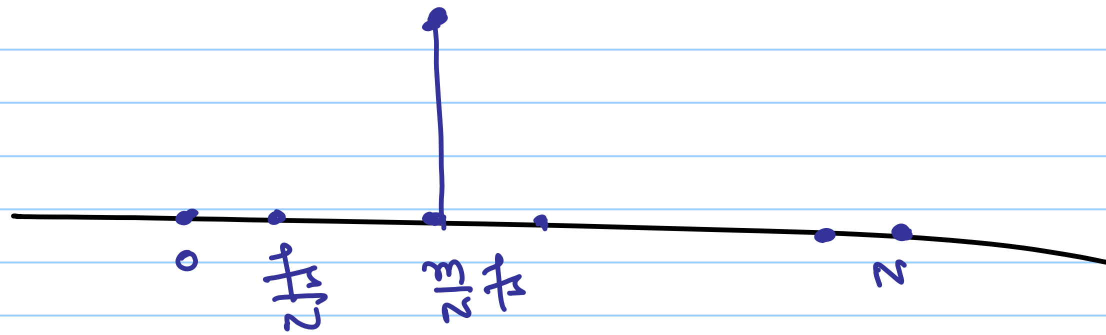
$$\Rightarrow \boxed{\frac{f_{in}}{f_s} = \frac{m}{N}} \Rightarrow V_{out}[n] \text{ is a periodic sequence only if}$$

(1)

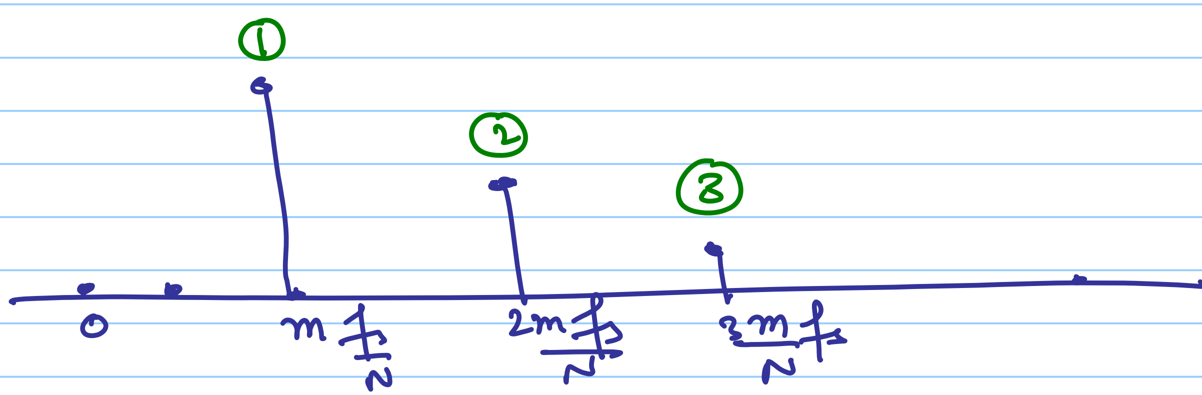
In time-domain

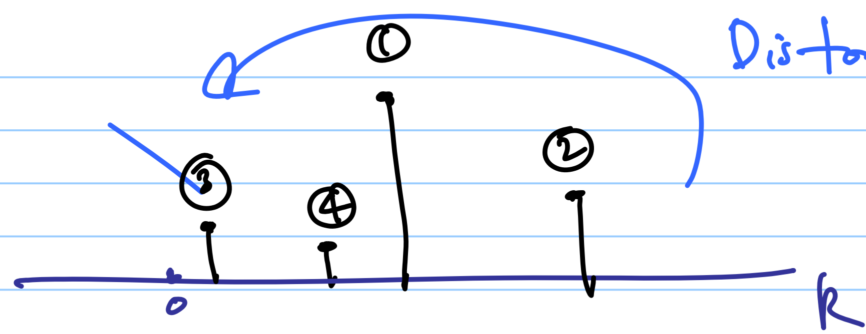
$$\frac{m}{f_{in}} = \frac{N}{f_s} \Rightarrow \text{'m' cycles of } f_{in} \triangleq \text{'N' cycles of } f_s$$

If (1) is satisfied, $V_{out}[n]$ can be expressed as a DFS for $f_{in} = \frac{m}{N} f_s$ $\qquad$ $V_{in} = e^{j\omega_{in} t}$



with distortion





Distortion folds backs