ECE 615 - Lecture 6

quantization error

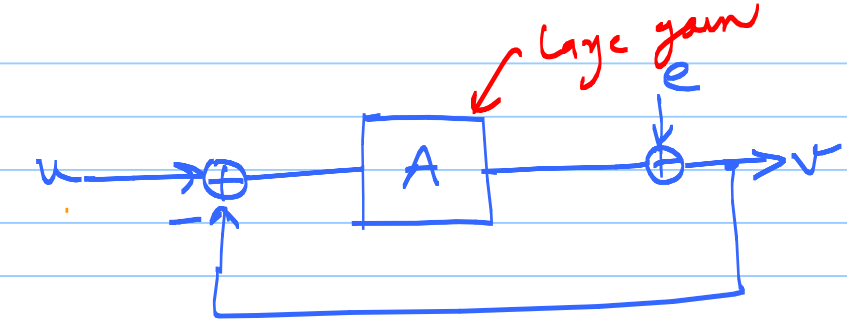
$\cancel{y} \to v$

Note Title 9/19/2013

$$y \longrightarrow \boxed{Q} \longrightarrow v \quad \Rightarrow \quad \begin{array}{c} \\ \\ y \end{array} \begin{array}{c} \\ \\ \longrightarrow \end{array} \begin{array}{c} e \\ \downarrow \\ \oplus \end{array} \begin{array}{c} \\ \\ \longrightarrow \end{array} \begin{array}{c} \\ \\ v \end{array}$$

oversampling $\longrightarrow$ $\frac{1}{2}$ bit extra per doubling in OSR

Idea → Use −ve feedback



$$\text{error} = |u - v| \to 0$$

$$A \to \infty$$

$$(u - v)A + e = v$$

$$v = \frac{uA}{1+A} + \frac{e}{1+A}$$

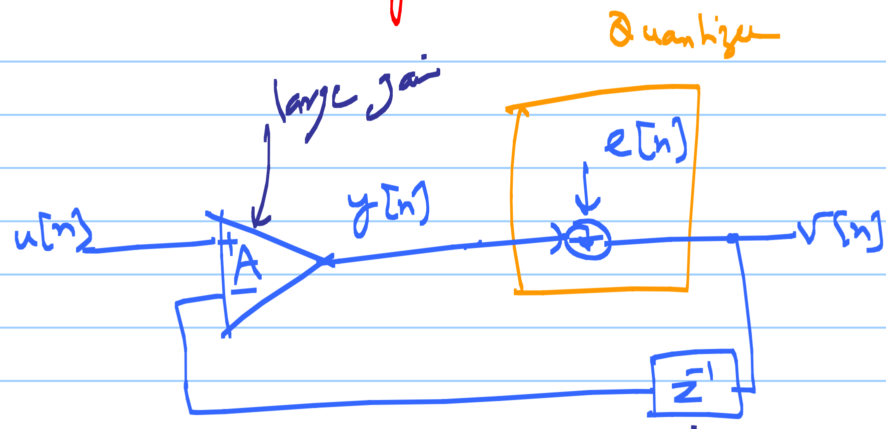
τ

$$\frac{1}{1 + Ae^{-s\tau}}$$

for $A \to \infty$

$$v \simeq u + \left(\frac{e}{1+A}\right) \to 0$$

Discrete time system



no delay-free loops

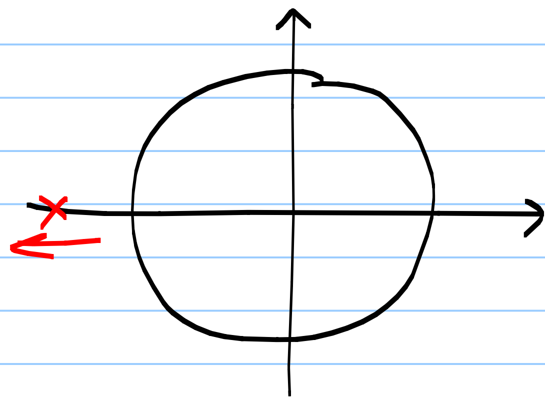
$$\left(U(z) - z^{-1}V(z)\right)A + E(z) = V(z)$$

$$V(z) = \left(\frac{A}{1 + Az^{-1}}\right)U(z) + \frac{E(z)}{1 + Az^{-1}}$$

$$TF(z) = \frac{1}{1 + A z^{-1}}$$

pole $\Rightarrow$ $z = -A$

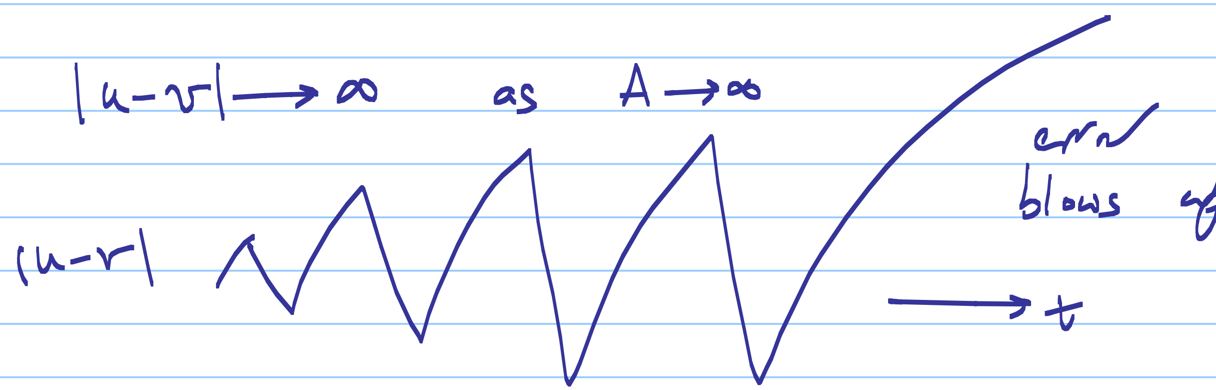
large gain $\Rightarrow$ pole at $-\infty$



pole outside the unit circle

$\hookrightarrow$ unstable system

error $|u - v| \to \infty$ as $A \to \infty$

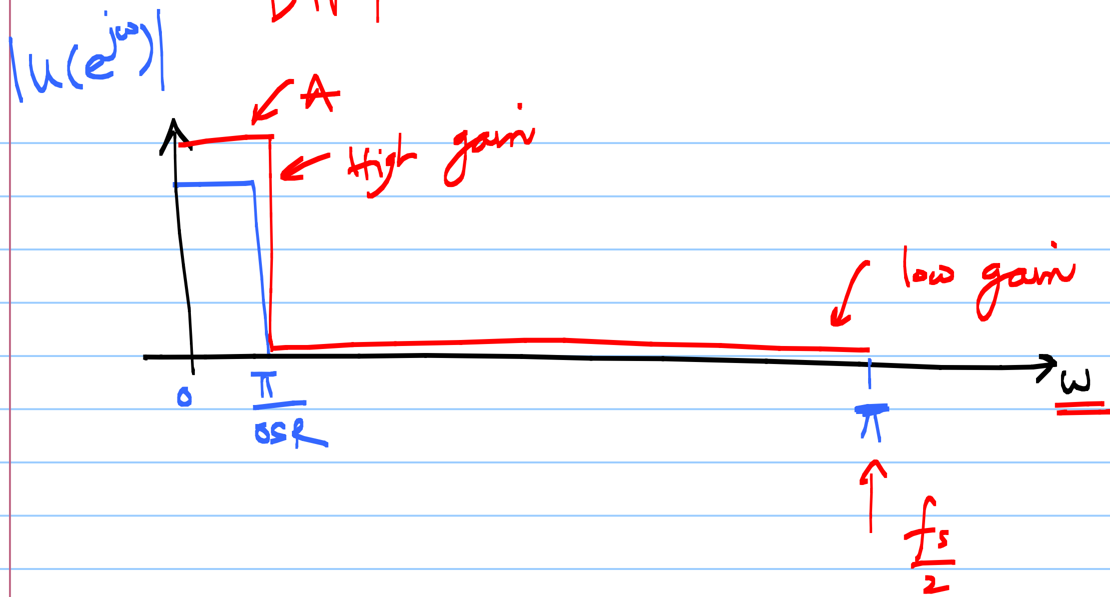


* Too much delay in the loop causes instability $(A \to \infty)$

* A constant gain (A) in the feedback loop doesn't work.

DTFT

$v = u$

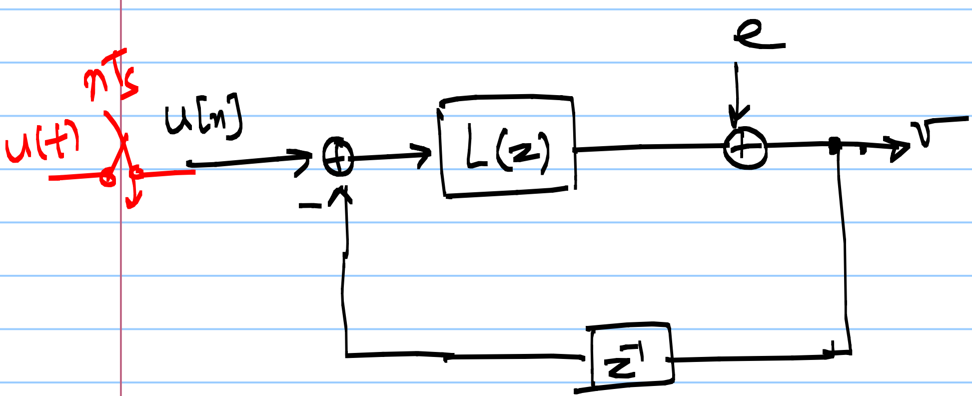


* Use frequency dependent gain

* Apply high gain at low frequencies to reduce quantization noise

* At high frequencies, keep the gain low to stabilize the loop.

$\Rightarrow$ Replace A by $L(z)$ $\Rightarrow$ loop filter



$x[n] \xrightarrow{\mathcal{Z}} X(z)$

$x[n-1] \xrightarrow{\mathcal{Z}} z^{-1} X(z)$

delay by 1 sample

$|L(z)| \to \infty$ at $z = 1$ (DC)

Example $\Rightarrow$ 1st order $\Rightarrow$ $\boxed{L(z) = \frac{1}{1 - z^{-1}}}$

$$\left[U(z) - z^{-1} V(z)\right] L(z) + E(z) = V(z)$$

$$\Rightarrow \quad U(z) L(z) + E(z) = V(z)\left(1 + z^{-1}\right)$$

$$\Rightarrow \quad V(z) = \underbrace{\boxed{\frac{L(z)}{1+z^{-1}L(z)}}}_{STF(z)} \cdot U(z) + \underbrace{\boxed{\frac{1}{1+z^{-1}L(z)}}}_{NTF(z)} \cdot E(z)$$

Uncorrelated with $U(z)$

STF(z): Signal transfer function

NTF(z): Noise transfer function

$$STF(z) = \frac{L}{1+z^{-1}L} = \frac{\frac{1}{1-z^{-1}}}{1+\frac{z^{-1}}{1-z^{-1}}} = \frac{\cancel{\frac{1}{1-z^{-1}}}}{\cancel{\frac{1-z^{-1}+z^{-1}}{1-z^{-1}}}} = 1$$

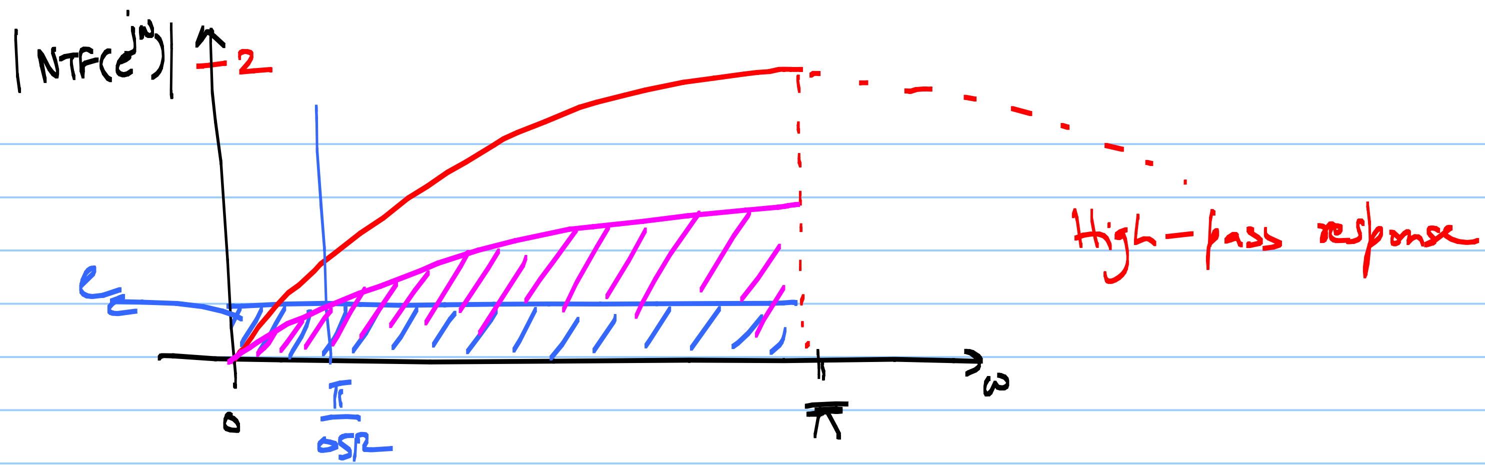
$$NTF(z) = \frac{1}{1+L(z)z^{-1}} = \frac{1}{1+\frac{z^{-1}}{1-z^{-1}}} = \boxed{(1-z^{-1})}$$

1st-order NTF

$$NTF(z) = 1 - z^{-1} \qquad z \to e^{j\omega} \qquad \boxed{e^{jx} - e^{-jx} = 2j\sin x}$$

DTFT $\Rightarrow$ $NTF(e^{j\omega}) = 1 - e^{-j\omega}$

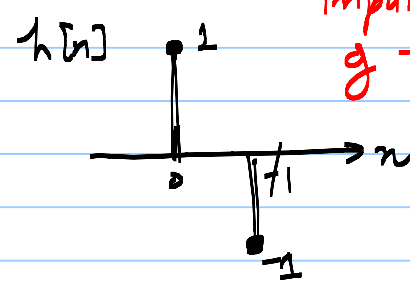
$$|NTF| = \underbrace{\cancel{\left|e^{-j\omega/2}\right|}}_{1}\left|e^{j\omega/2} - e^{-j\omega/2}\right| = \left|2j\cdot\sin\left(\frac{\omega}{2}\right)\right| = 2\left|\sin\frac{\omega}{2}\right|$$



$h[n] \overset{\mathcal{Z}}{\longleftrightarrow} NTF(z)$

quantization noise is "shaped"

↳ noise shaping



impulse response of the NTF

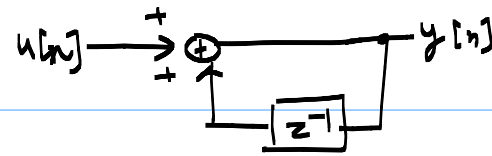
(high pass response)

$NTF(z) = 1 - z^{-1} \overset{\mathcal{Z}^{-1}}{\longleftrightarrow} h[n] = \delta[n] - \delta[n-1]$

$q[n] = e[n] - e[n-1] \Rightarrow$ DT differentiation

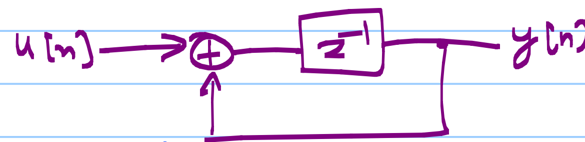
Loop filter:

$$L(z) = \frac{1}{1-z^{-1}}$$



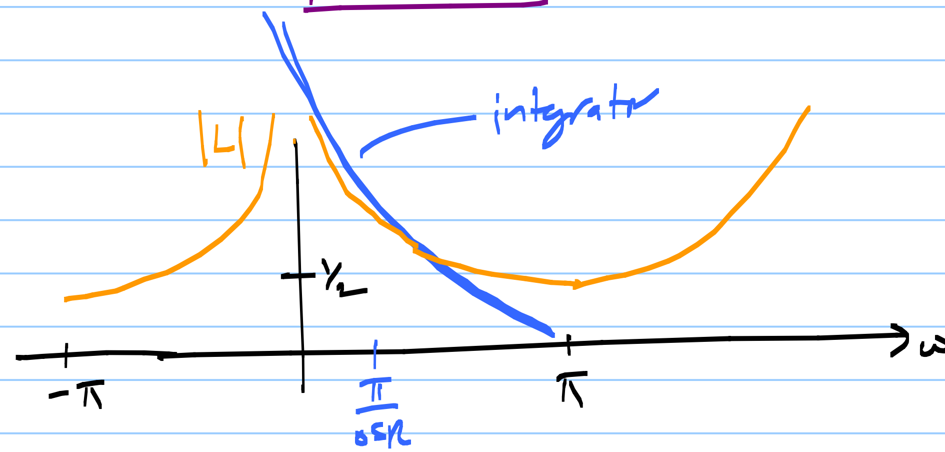
Non-delaying integrator (accumulator)

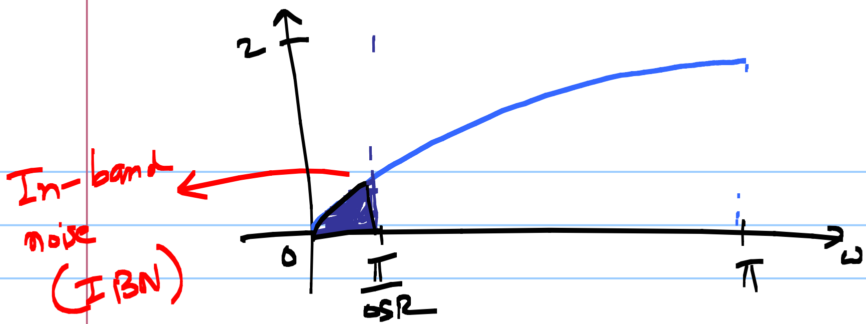
$$L_1(z) = \frac{z^{-1}}{1-z^{-1}}$$

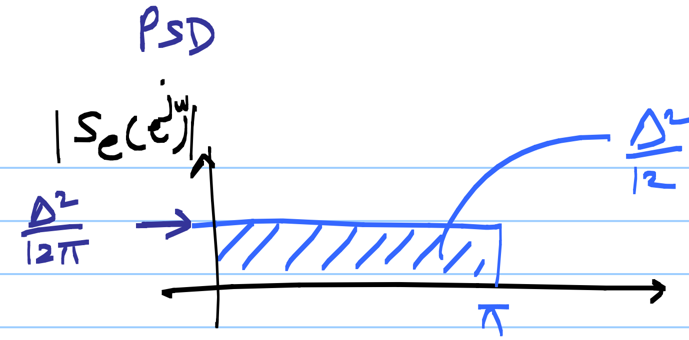


Delaying integrator

$$|L(e^{j\omega})| = \left|\frac{1}{2\sin(\frac{\omega}{2})}\right|$$







o/p quantization noise $= E(z) \cdot NTF(z)$

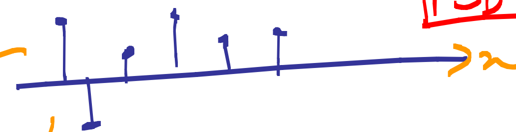
PSD of the o/p quantization noise $= S_{no}(e^{j\omega}) = S_e(e^{j\omega}) \cdot |NTF(e^{j\omega})|^2$

$$= \frac{\Delta^2}{12\pi} \cdot |1-e^{-j\omega}|^2$$

ASIDE

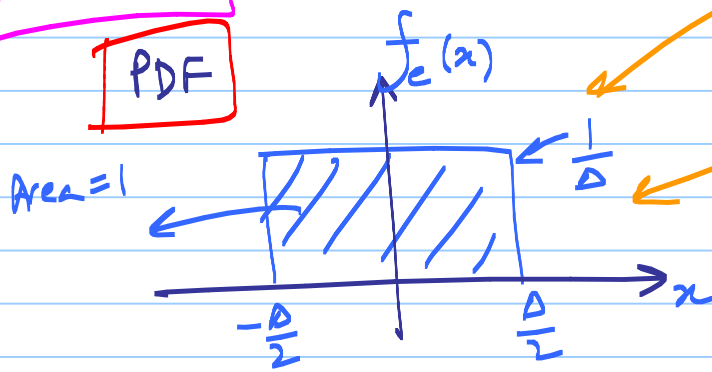
"Random Processes"

e[n]



PSD

PDF

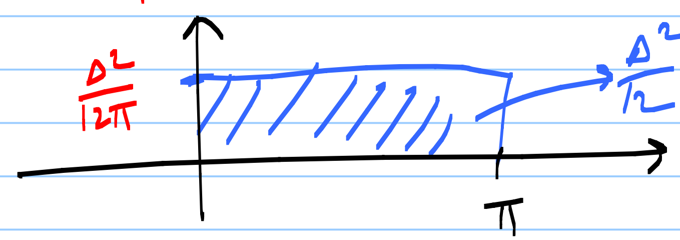


uniformly distributed

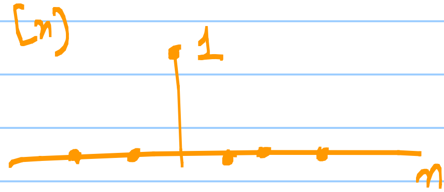
$$E[e[n]\cdot e[n+m]] = \sigma^2\delta[m]$$

$$= \begin{cases} \sigma^2 & \text{for } m=0 \\ 0 & \text{otherwise} \end{cases}$$

$|S_e(e^{j\omega})|$



$\delta[n]$



$$PSD = \frac{\Delta^2}{12\pi} \left|1 - e^{-j\omega}\right|^2$$

$2\sin\left(\frac{\omega}{2}\right) \simeq \omega$ at low frequencies

for $\frac{\pi}{OSR} \ll \pi$

$$= \frac{\Delta^2 \omega^2}{12\pi}$$

Total In Band noise

$$IBN = \int_0^{\pi/OSR} \frac{\Delta^2}{12\pi} \omega^2 \cdot d\omega$$

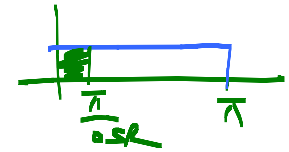
$$= \frac{\Delta^2}{12\pi} \left(\frac{\pi}{OSR}\right)^3 \cdot \frac{1}{3} = \frac{\Delta^2 \pi^2}{36 \cdot OSR^3}$$

$$\boxed{IBN = \frac{\Delta^2 \pi^2}{36} \cdot OSR^{-3}}$$

1st order noise-shaping

$$\left(\frac{\Delta^2}{12\pi}\right) \cdot \frac{\pi}{OSR}$$

$$\frac{\Delta^2}{12 \cdot OSR}$$

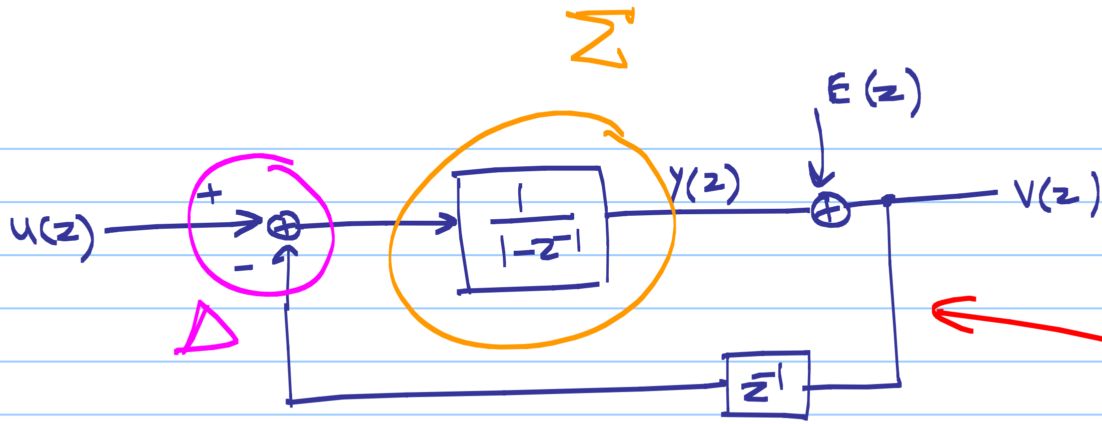


Double OSR ↑ → IBN ↓ by 9 dB

SQNR ↑ by 9dB ⇒ 1.5 bits

$$ENOB = \frac{SNR - 1.76}{6.02} \simeq \frac{SNR}{6}$$

$\boxed{\text{1.5 bits} \uparrow \text{ in resolution for doubling in OSR}}$



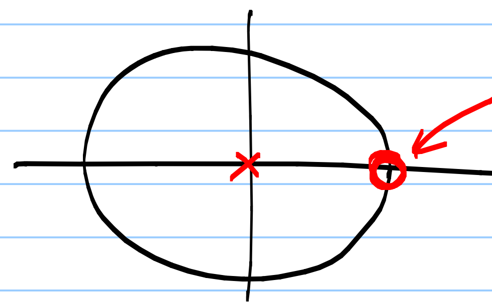
First-order $\Delta\Sigma$ modulator

$\Delta\Sigma$ Modulator

$\Sigma\Delta$

$$NTF(z) = 1 - z^{-1}$$

$$= \frac{z-1}{z}$$



NTF zero at $z=1$ (DC)

Stable system