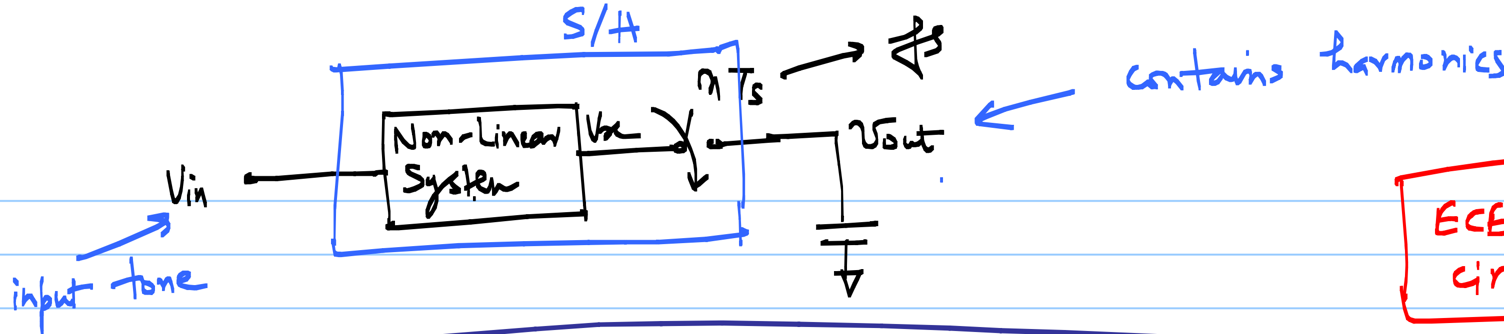
ECE 615 – Lecture 3

DFS, DFT (FFT)

Example: Characterizing the distortion of a S/H using a single tone input.



ECE 614 for circuit details

* Use a discrete time periodic sequence $v[n]$ with period N

$$v[n] = v[n+N] \leftarrow \text{periodic with 'N'}$$

can be represented as a DFS

$$v[n] = \sum_{k=0}^{N-1} V[k]\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$V[k]$ are complex DFS coefficients

* $V[k]$ are easily computed using FFT → Cadence, Matlab

└→ $\underline{\underline{N}}$ → FFT size

Recall that frequency axis is discretized in a DFT

frequencies $\left\{0, \frac{2\pi}{N}, \frac{2\pi}{N}\times 2, \ldots, \frac{2\pi}{N}(N-1)\right\}$

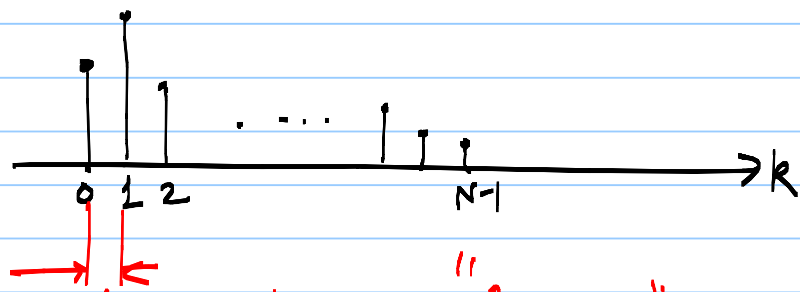
T_s → sample period ⇒ $\frac{1}{T_s} = f_s$ is the sampling frequency

N-point FFT ⇒ $\frac{f_s}{N}$ → resolution of the FFT

└→ spacing between the FFT tones

└→ total time in continuous time axis ⇒ $NT_s = \frac{N}{f_s}$

FFT $|V[k]|$



⇒ discretized frequency axis

resolution or "bin size" of the FFT $= \frac{f_s}{N}$

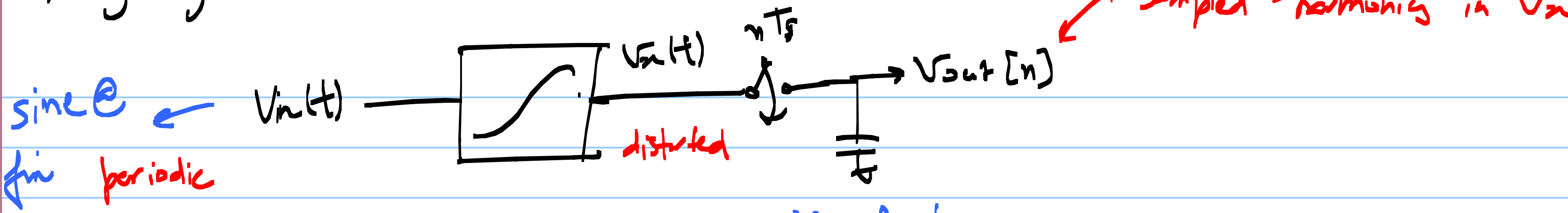
* "M" ⇒ record length ⇒ size of the data collected from simulation or measurements

↳ if FFT is taken over the whole record length

⇒ $N = M$

otherwise $N < M$

* getting back to our S/H distortion



$$V_{in}(t) = A \sin(2\pi f_{in} t) = \Im m\{A e^{j2\pi f_{in} t}\}$$

$$V_{out}[n] = V_{out}(nT_s) = V_{out}\left(\frac{n}{f_s}\right) = \sum_k a_k e^{j2\pi k f_{in}\left(\frac{n}{f_s}\right)}$$

* $V_{out}[n]$ is periodic only when

$$e^{j2\pi k f_{in}\frac{(n+N)}{f_s}} = e^{j2\pi k \frac{f_{in}}{f_s} n}$$

$$\Rightarrow \quad e^{j2\pi \frac{f_{in}}{f_s} N} = 1$$

$$\Rightarrow \quad 2\pi \frac{f_{in}}{f_s} N = 2m\pi, \qquad m \in I$$

$$\Rightarrow \quad \boxed{\frac{f_{in}}{f_s} = \frac{m}{N}} \longrightarrow (1)$$

← $v_{out}[n]$ is a periodic sequence & DFS expansion is valid

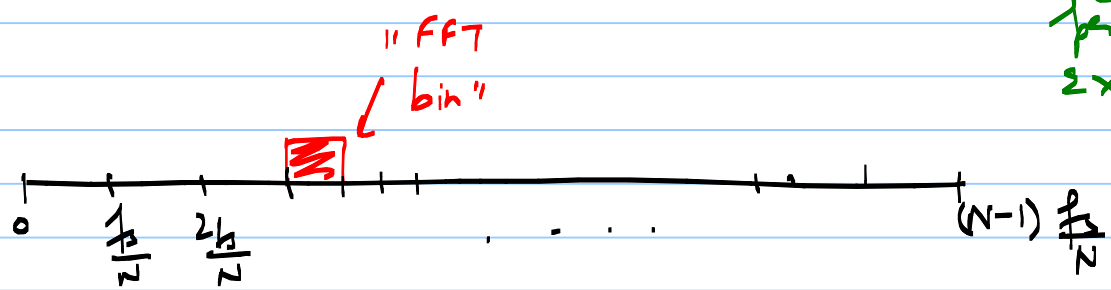
In time-domain

$$\frac{m}{f_{in}} = \frac{N}{f_s}$$

$\Rightarrow$ 'm' cycles of f_{in} = 'N' cycles of f_s

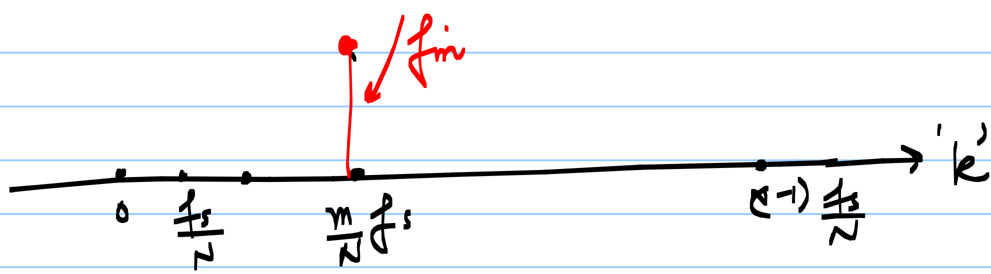
$\rightarrow$ only then $V_{out}[n]$ is periodic & can be expanded as a DFS

$\rightarrow$ can evaluate an FFT over $V_{out}[n]$



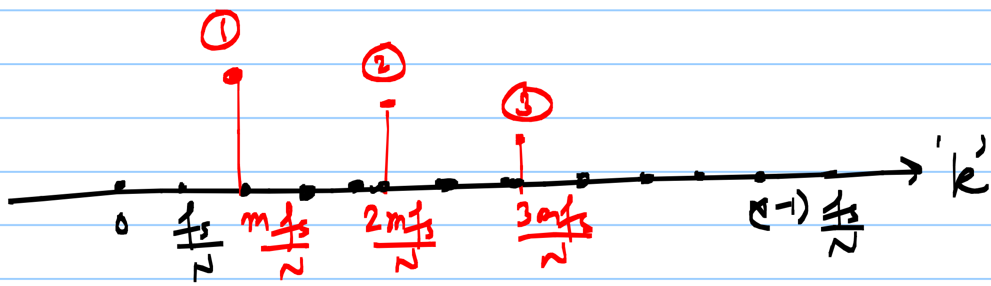
$\Rightarrow \frac{f_s}{N} \Rightarrow$ FFT resolution

For $f_{in} = \frac{m}{N} f_s$



* Now, with distortion

component at multiple of $\frac{m f_s}{N}$



* Choose 'f_{in}' and 'f_s' carefully, else the sampled sequence will not be periodic ⇒ FFT will show misleading results.

Example :

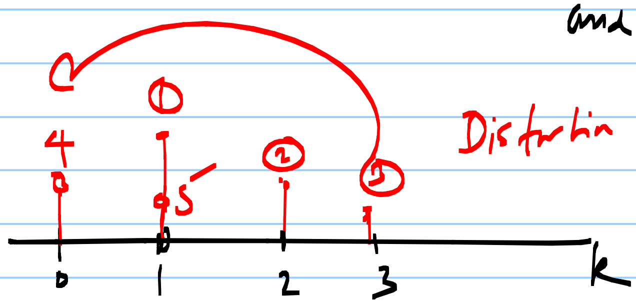
Select $f_{in} = \frac{m}{N} f_s = \frac{f_s}{4} = 2\left(\frac{f_s}{8}\right) = 4\left(\frac{f_s}{16}\right)$

$\Rightarrow$ harmonic components at $0, \frac{f_s}{4}, \frac{2f_s}{4}, \frac{3f_s}{4}, f_s$

record length $N = 4$

$\Rightarrow$ make sure that f_{in} & f_s are mutually prime

and $f_{in} < f_s$



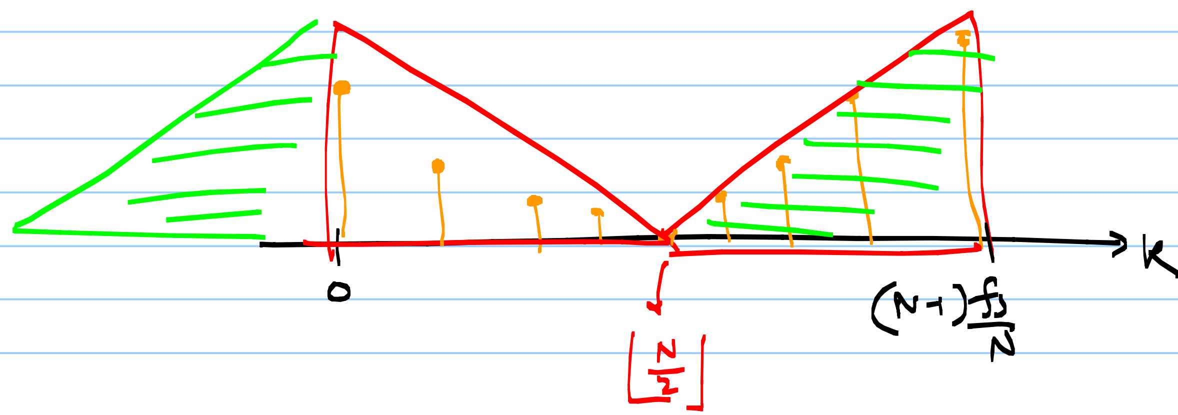
Distortion folding back

* For FFT
  * Conjugate symmetry

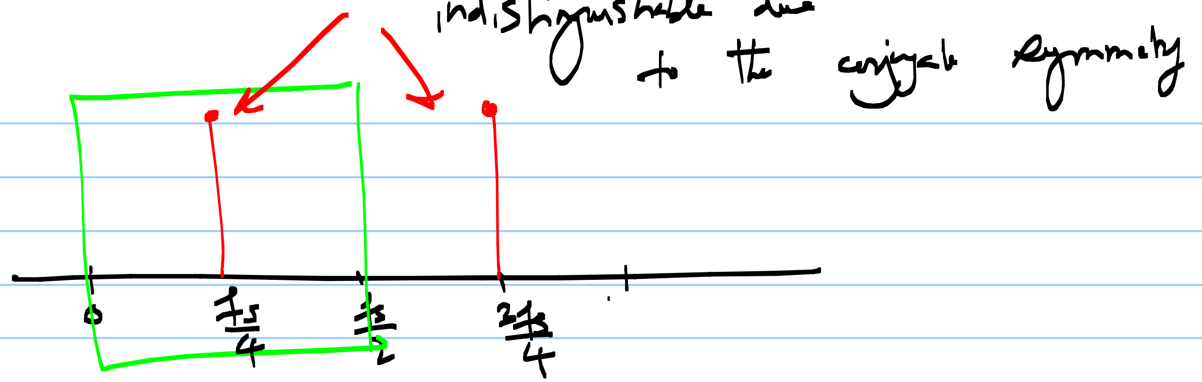
$$X^*[k] = X[N-k]$$

$$\Rightarrow |X[k]| = |X[N-k]|$$

$$\& \ \angle X[k] = -\angle X[N-k]$$



for $f_{in} = \frac{f_s}{4}$



$\Rightarrow$ It is not just enough that if

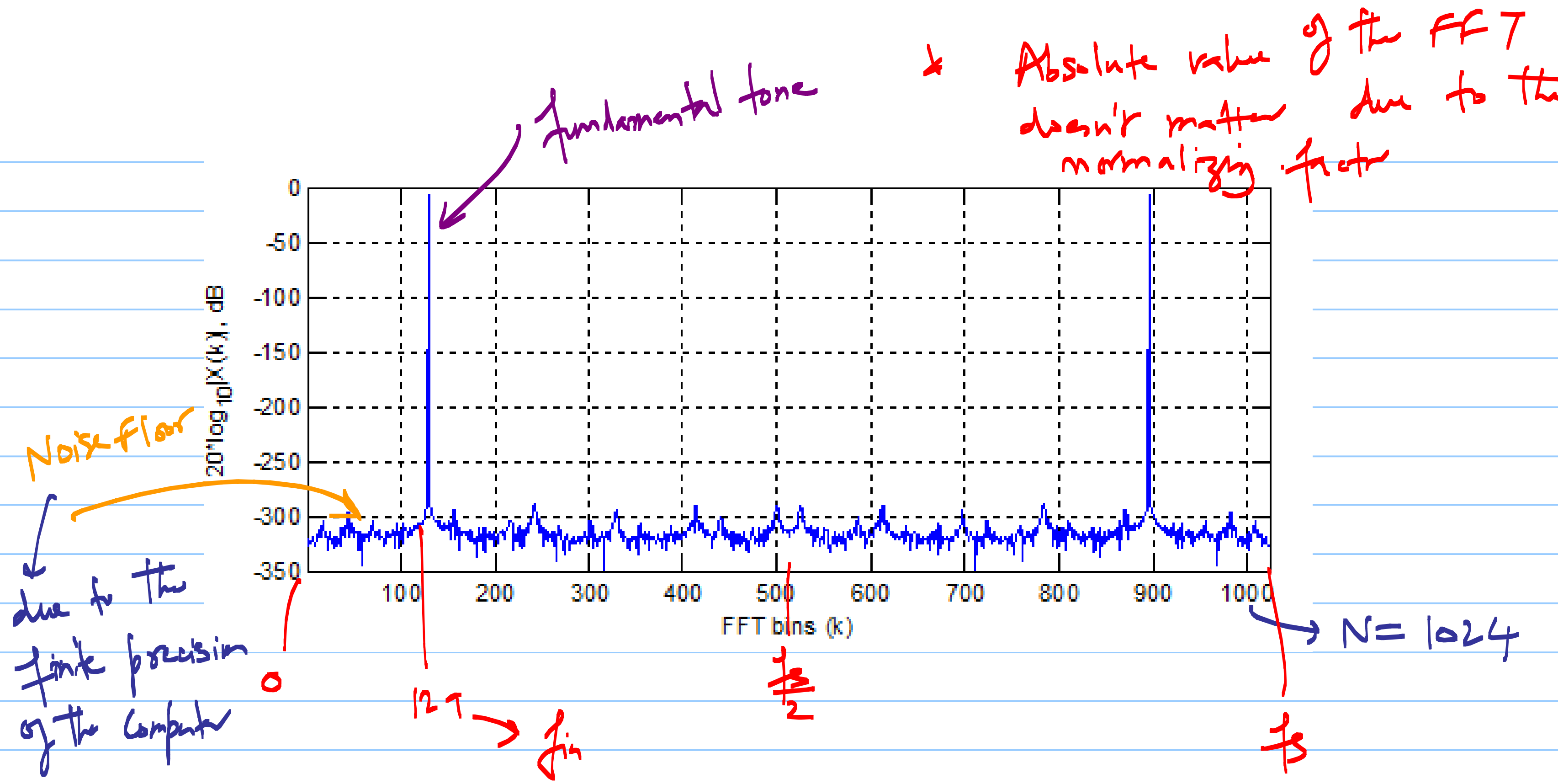
$\frac{f_{in}}{f_s} = \frac{m}{N}$, necessary but not sufficient

$\Rightarrow$ Use $f_{in} = \frac{m}{N} f_s$, such that 'm' is prime w.r.t. to 'N' (N > m), so that the harmonics don't alias back to the fundamental tone.

* Also choose $N = 2^p$ for FFT computation, $p \in I$

Eg. $f_{in} = \frac{m}{2^{10}} f_s = \frac{m}{1024} f_s$

$$f_{in} = \frac{129}{1024} f_s$$



* Absolute value of the FFT doesn't matter due to the normalizing factor
fundamental tone
Noise floor
due to the finite precision of the computer
0
-50
-100
-150
-200
-250
-300
-350
20*log10|X(k)|, dB
100
200
300
400
500
600
700
800
900
1000
FFT bins (k)
0
127 → fin
fs/2
fs
N = 1024

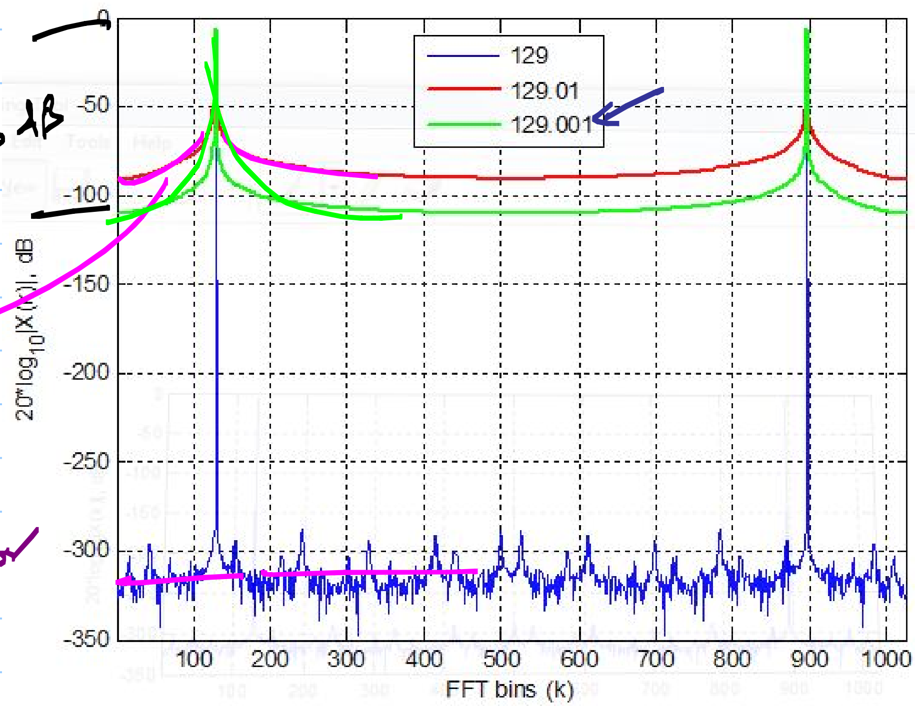
What happens when

$$f_{in} = \frac{129.01}{1024} f_s$$

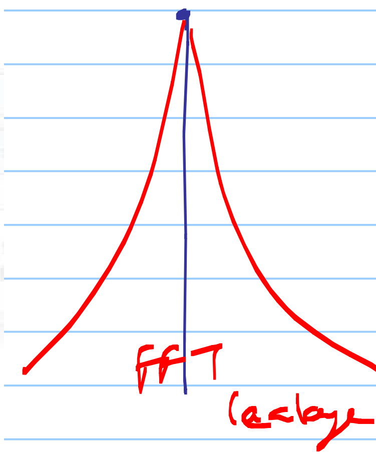
80 dB

FFT leakage

Eiffel Tower Effect



$$\frac{f_{in}}{f_s} = \frac{129}{1024}, \frac{129.01}{1024}, \frac{129.001}{1024}$$

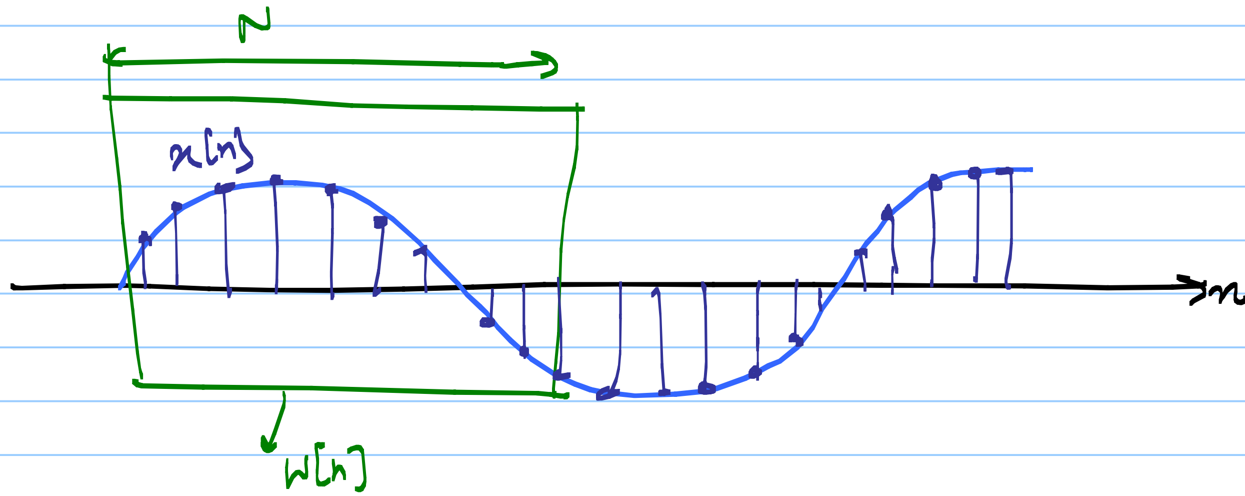


Here $\frac{m}{f_{in}} \neq \frac{N}{f_s}$

*

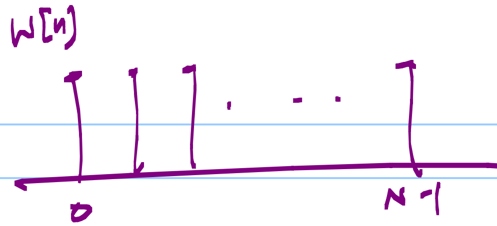
$x[n] = \sin\left(2\pi \frac{f_{in}}{f_s} n\right)$ is a periodic signal of infinite length

$\Rightarrow$ we restrict it to a length 'N'.

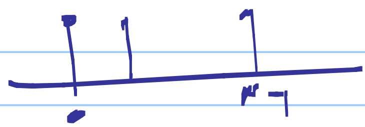


$x[n] \cdot w[n] \Rightarrow$ the signal $x[n]$ is windowed by window $w[n]$

($w[n]$: rectangular window)

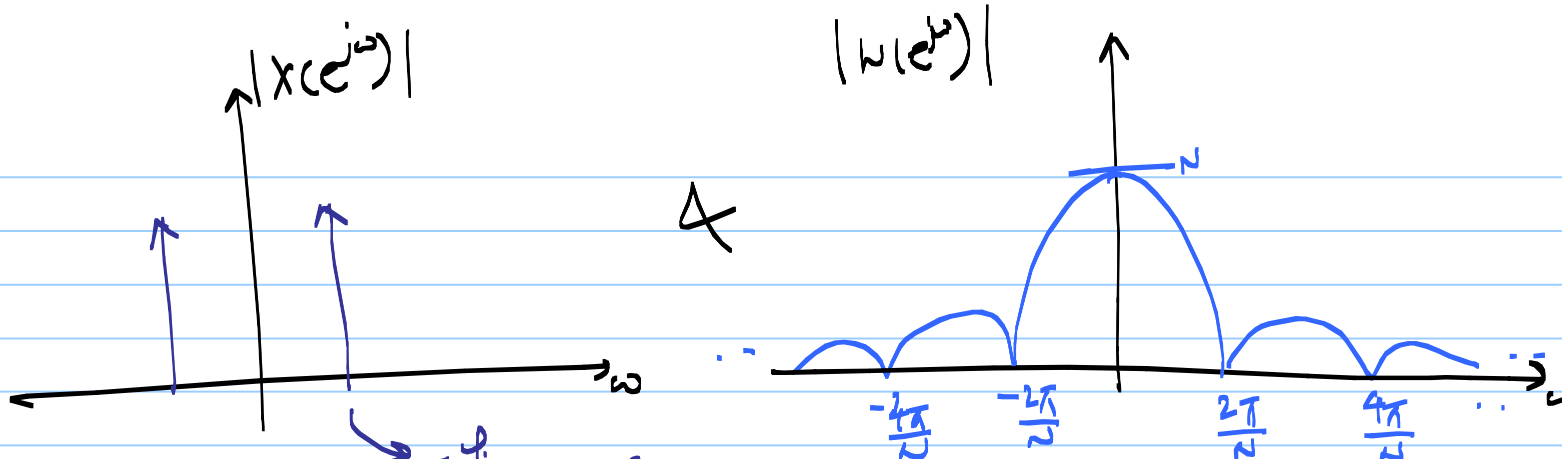


$$p[n] = x[n] \cdot w[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \circledast W(e^{j\omega})$$



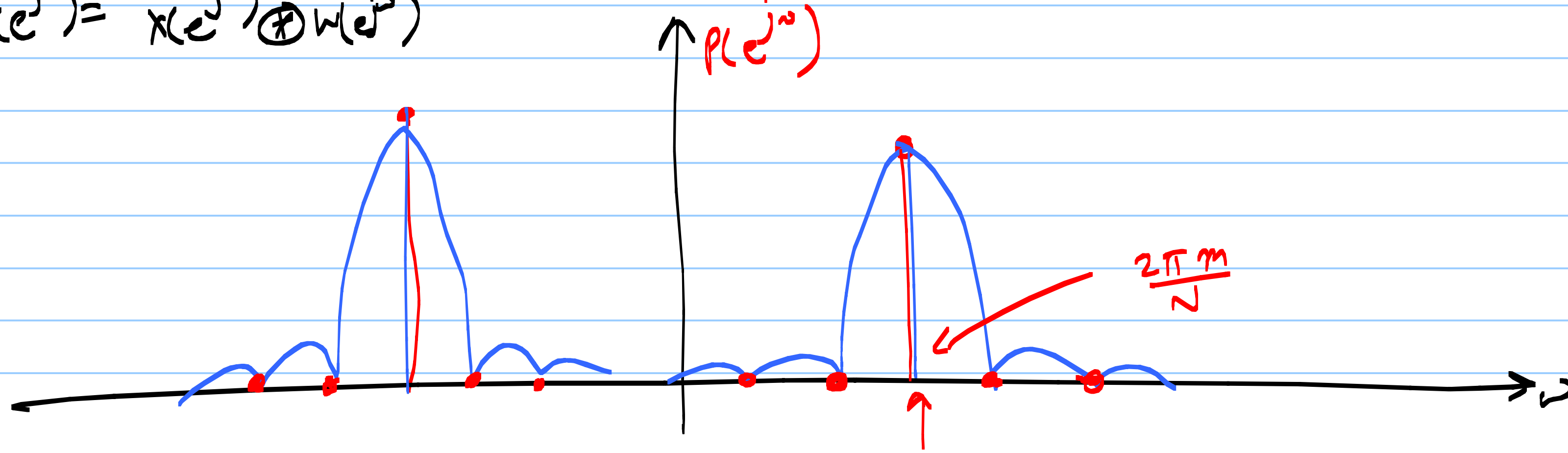
$$\Rightarrow \quad e^{-j\omega\frac{(N-1)}{2}} \cdot \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin(\omega/2)}$$

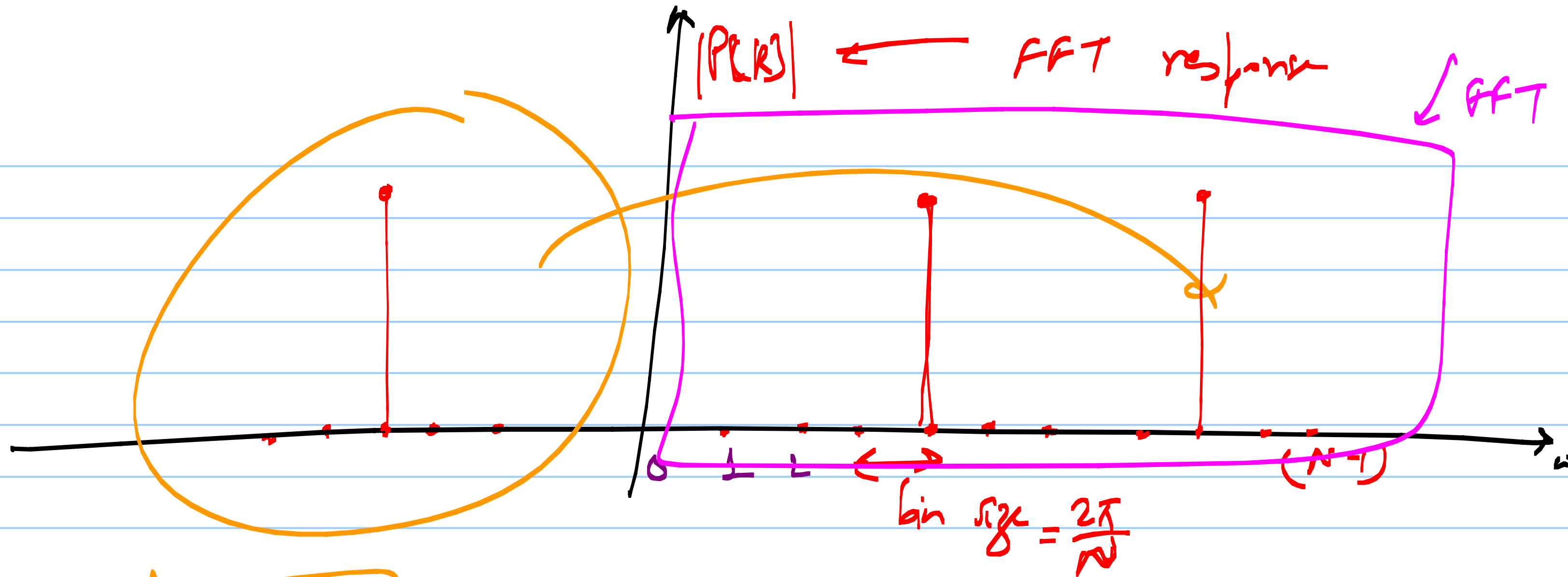
avg delay of $\frac{(N-1)}{2}$ samples



$$\pi \frac{f_{in}}{(f_s/2)} = \frac{2\pi m}{N}$$

$$P(e^{j\omega}) = X(e^{j\omega}) \circledast W(e^{j\omega})$$

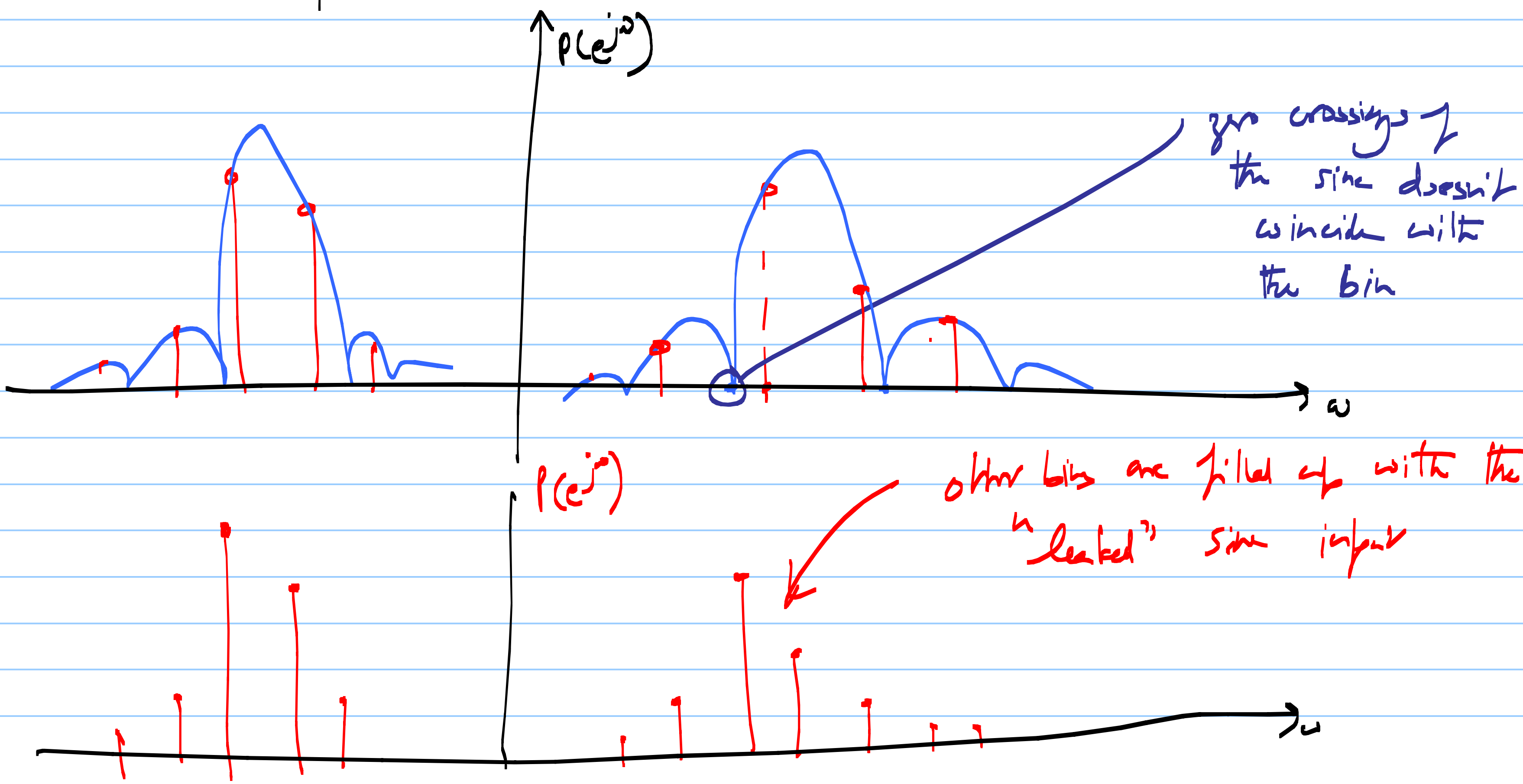




* If $f_{in} = \frac{m}{N} f_s$ ie the input is rationally related to the sampling rate, we get a single peak and other bins are zero

⇒ FFT computes DFS properly

* for $\frac{f_{in}}{f_s} \neq \frac{m}{N}$



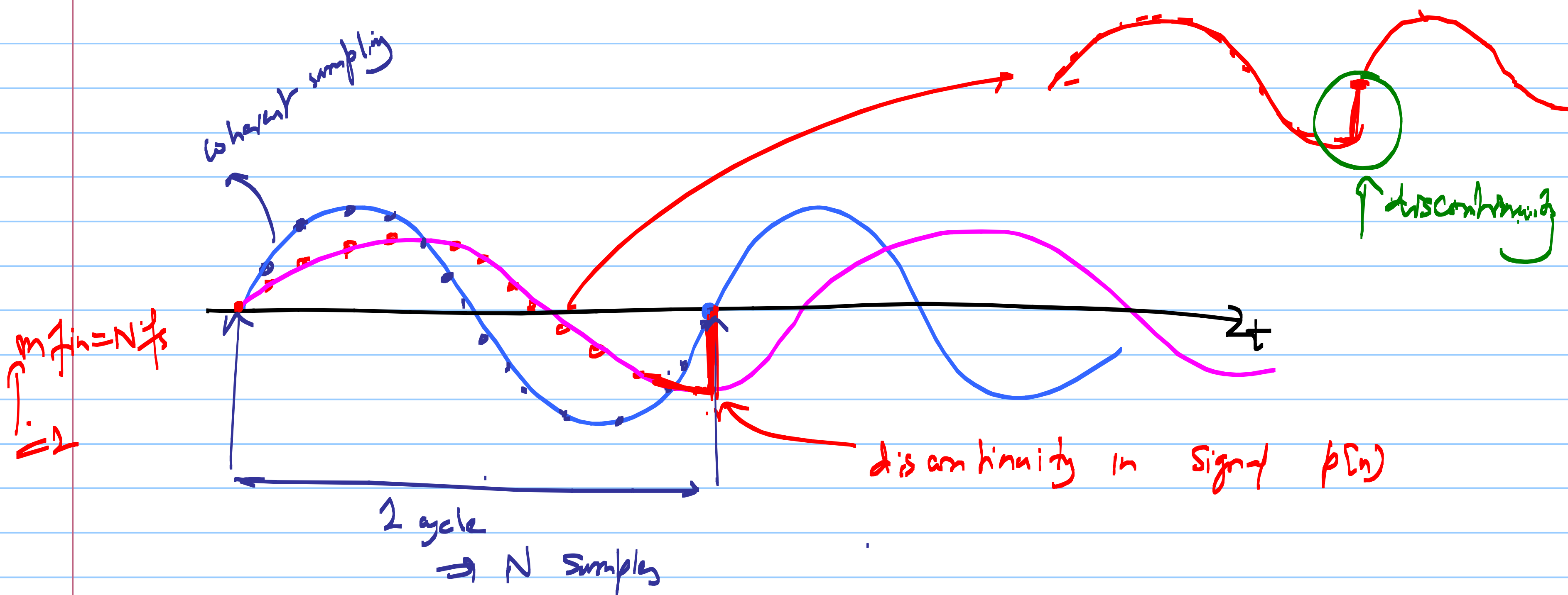
⇒ As $\frac{f_{in}}{f_s} \to \frac{m}{N}$, the Eiffel tower become taller

↳ FFT leakage is reduced.

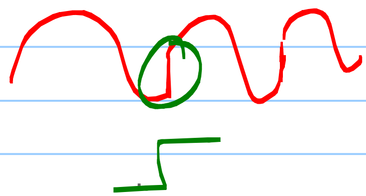
* Synchronous Sampling or Coherent sampling

$$\boxed{\frac{f_{in}}{f_s} = \frac{m}{N}}$$

Time-domain understanding



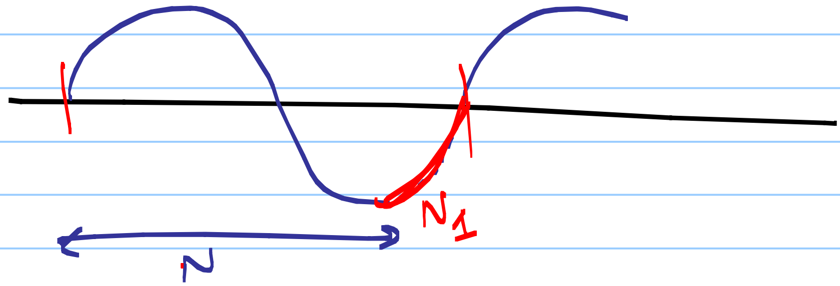
* Temporal discontinuity in p[n], while taking DFS causes FFT leakage



discontinuity → step function → has all frequency components

Q. How to subdue the effect of the discontinuity in non-coherent sampling.

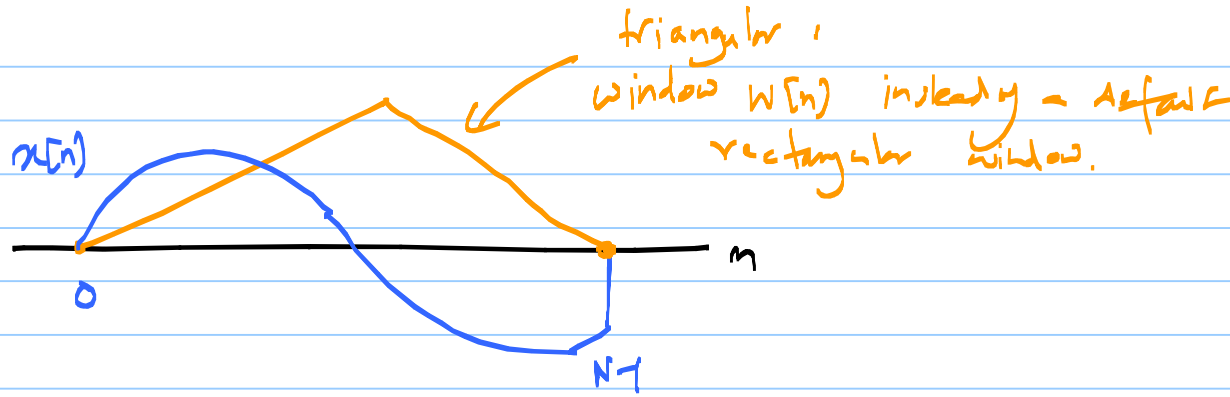
① Signal reconstruction

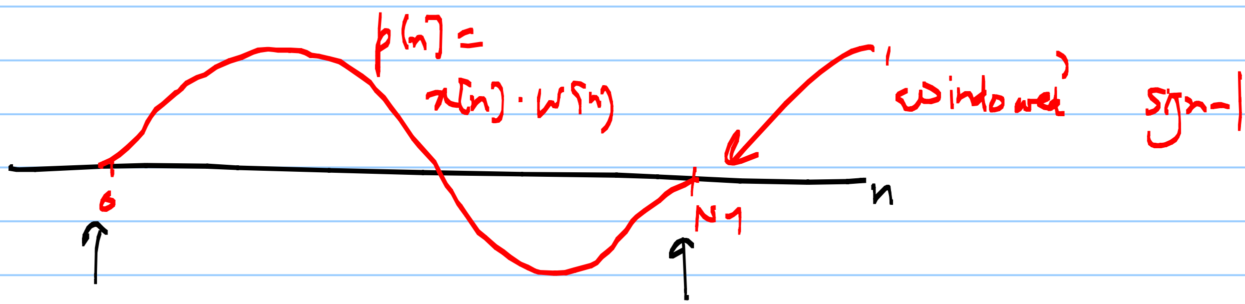


Adjust N such that it has full cycles of $x(n)$

↳ impractical to implement

② Attenuate the discontinuity



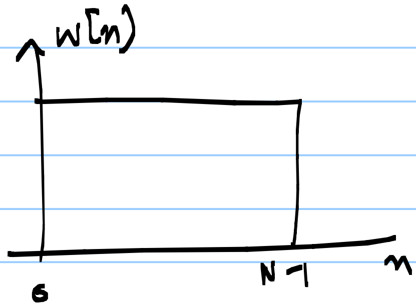


↳ give lesser emphasis on the ends and more importance to the signal in the middle.

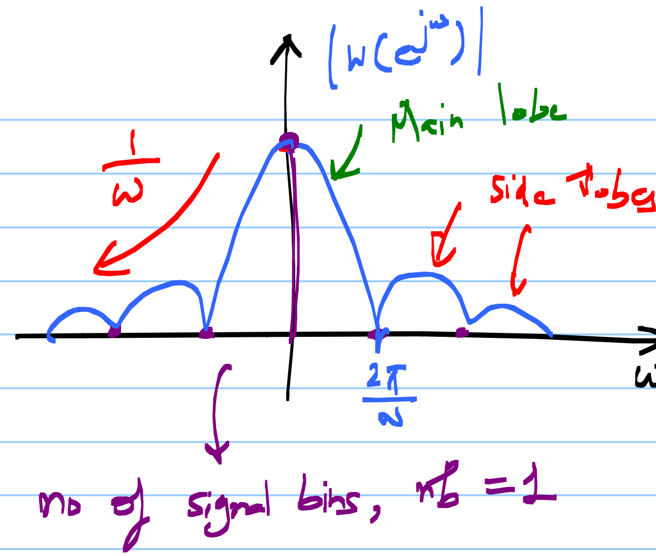
↳ A large number of windows available in literature

↳ Matlab

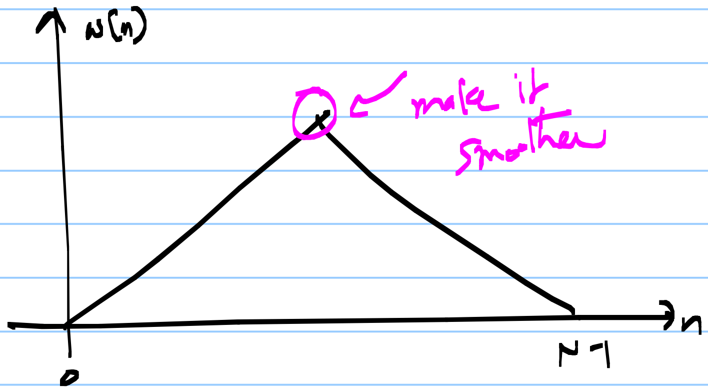
Rectangular window



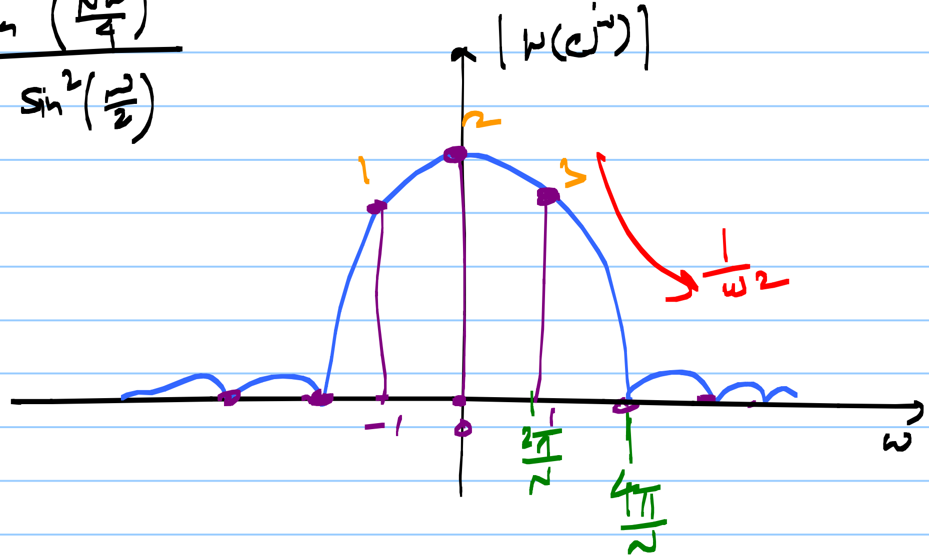
$$|W(e^{j\omega})| \propto \frac{\sin\left(\frac{N\omega}{2}\right)}{\sin\left(\omega/2\right)}$$



Triangular Window (Bartlett window)



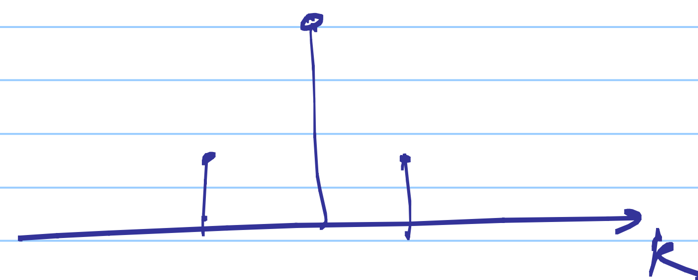
$$\propto \frac{\sin^2\left(\frac{N\omega}{4}\right)}{\sin^2\left(\frac{\omega}{2}\right)}$$



* Double the main lobe width
* larger side-lobe suppression

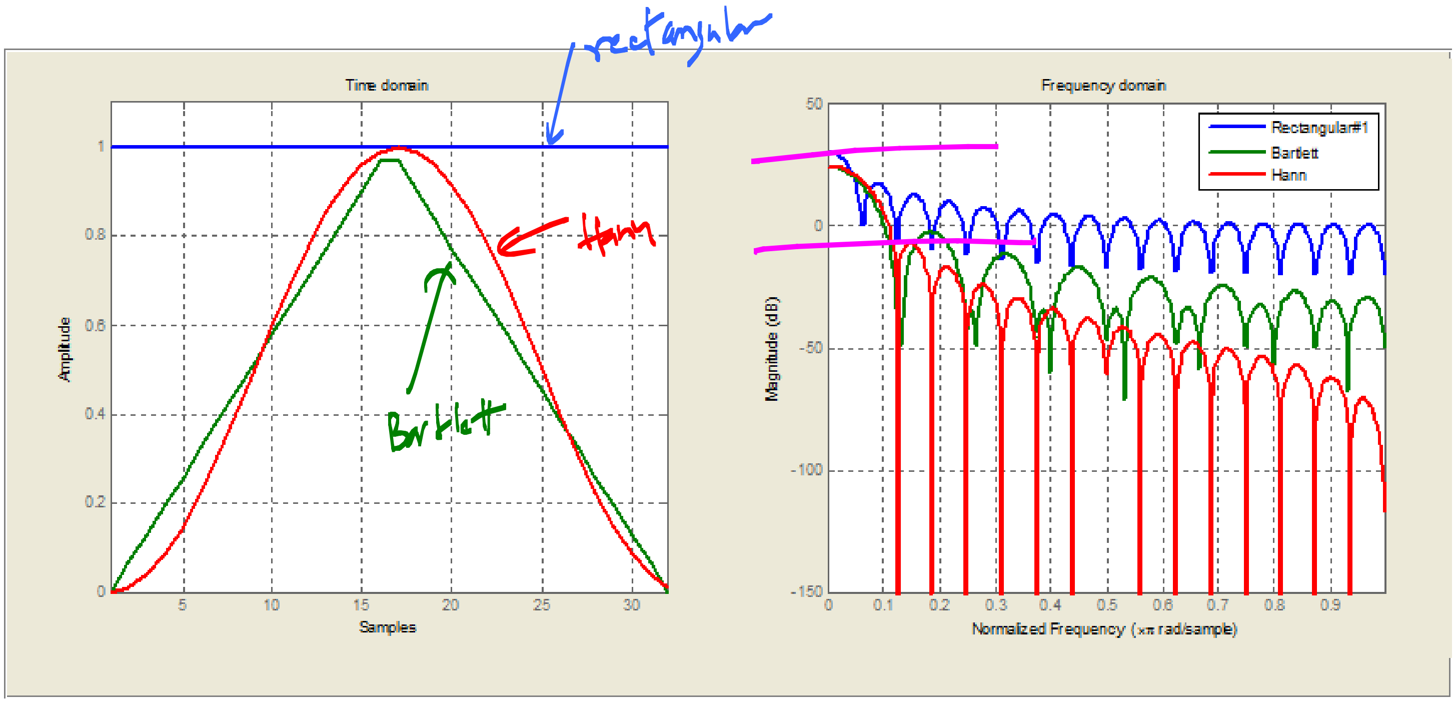
$n_b = 3$

Raised Cosine Window (Hann, Hanning)



$$w[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N}\right)\right] \quad 0 \le n \le (N-1)$$

→ Used in simulations where $\frac{f_{in}}{f_s} = \frac{m}{N}$ can be precisely controlled



rectangular
Time domain
Amplitude
Samples
Hann
Bartlett
Frequency domain
Magnitude (dB)
Normalized Frequency (×π rad/sample)
Rectangular#1
Bartlett
Hann

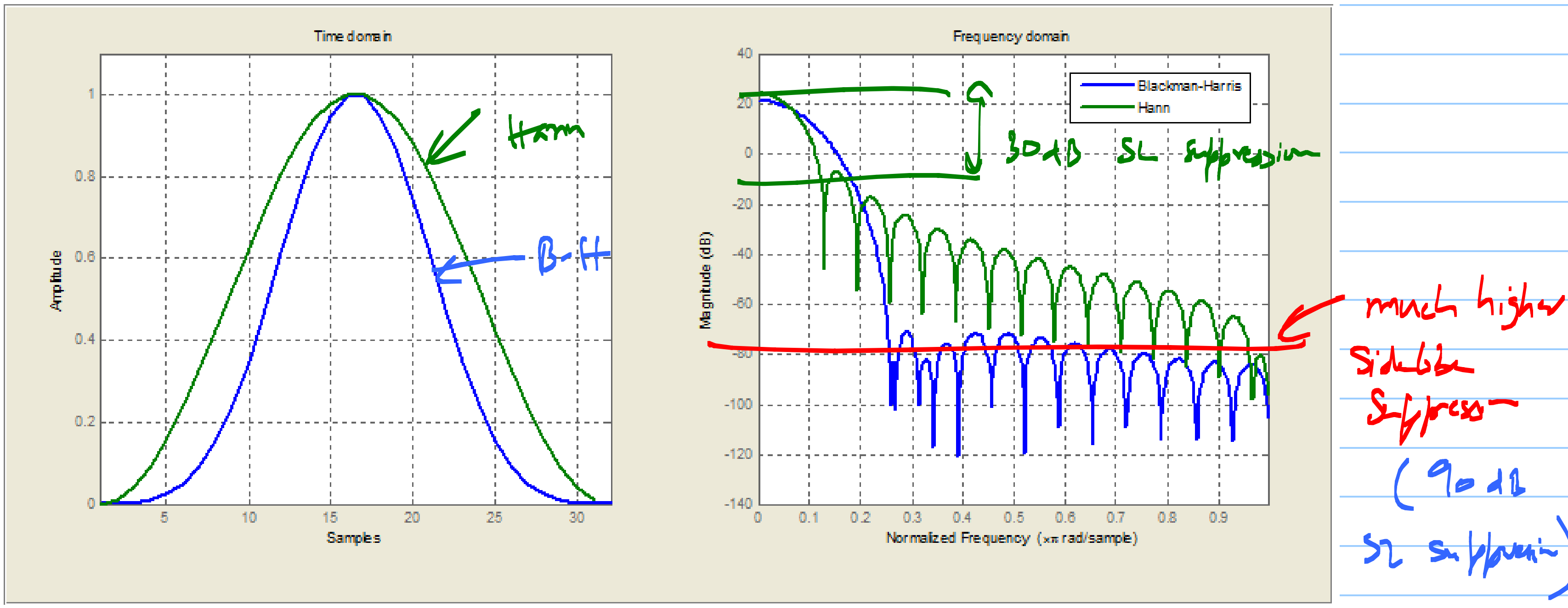
+ In experiments, its hard to enforce $\frac{f_{in}}{f_s} = \frac{m}{N}$

Blackmann-Harris window → non-coherent sampling

$$w[n] = a_0 + a_1 \cos\left(\frac{2\pi}{N} n\right) + a_2 \cos\left(\frac{2\pi}{N} 2n\right) + a_3 \cos\left(\frac{2\pi}{N} \cdot 3n\right)$$

$$-\frac{N}{2} \le n \le \frac{N}{2}$$

mb = 5



Time domain
Amplitude
Samples
Hann
B-H
Frequency domain
Magnitude (dB)
Normalized Frequency (×π rad/sample)
Blackman-Harris
Hann
30 dB SL suppression
much higher sidelobe suppression
(90 dB SL suppression)