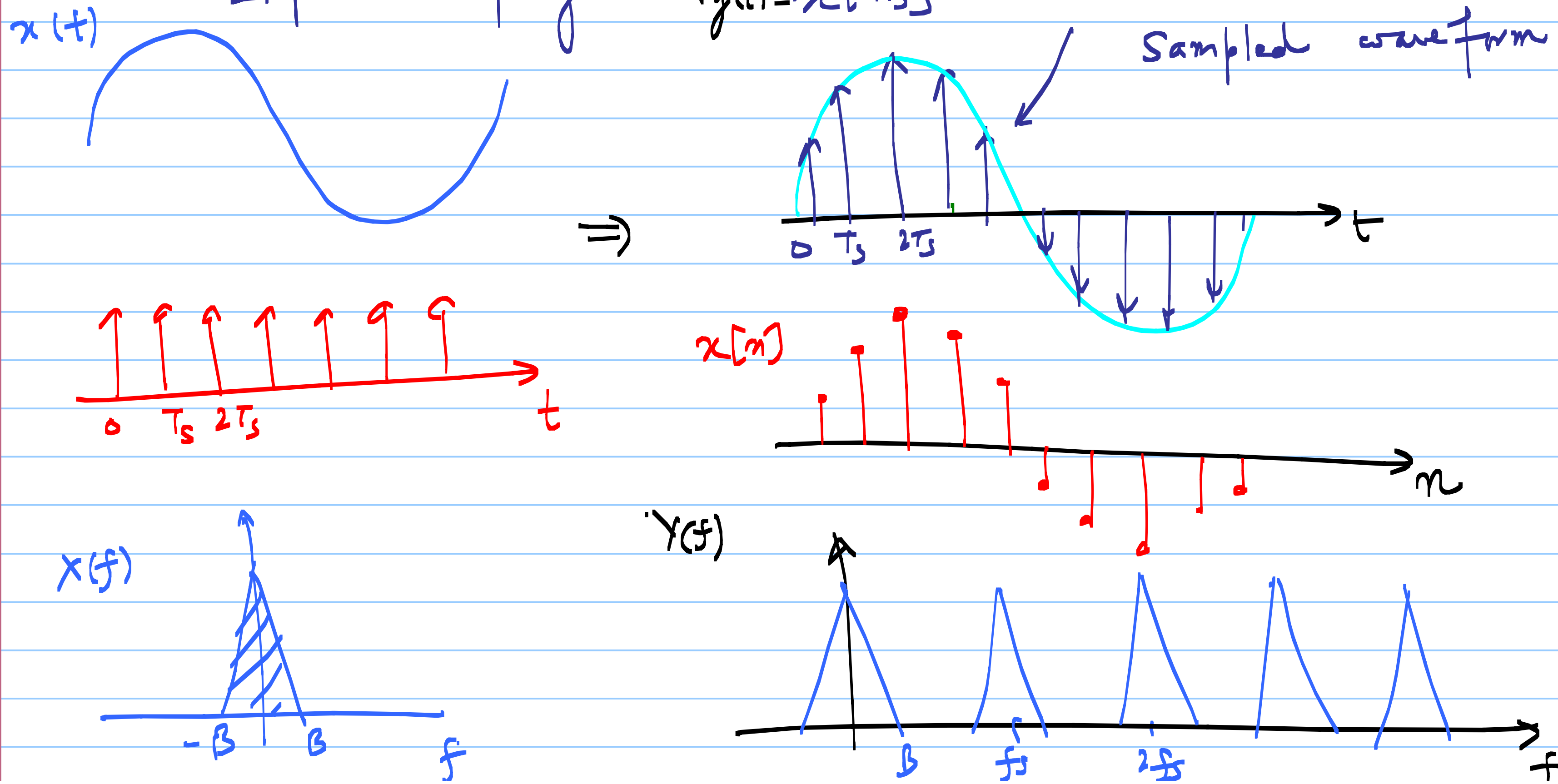
ECE 615- Lecture 2

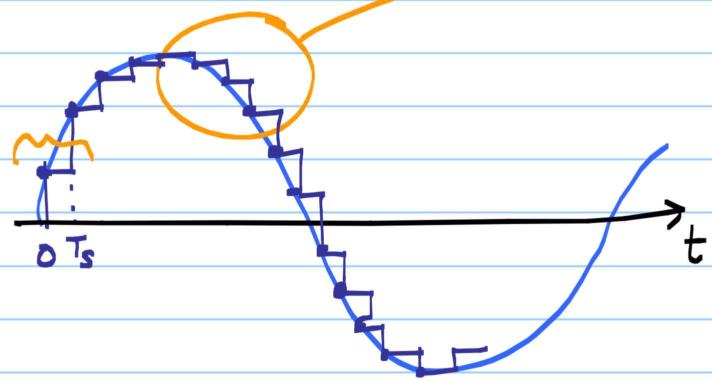
Impulse Sampling

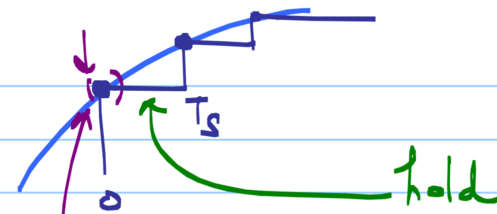


$f_s \geq 2B$

, AAF, oversampling

Sample and Hold



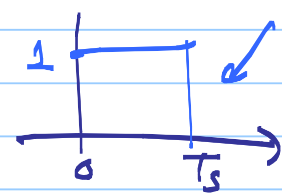


Aperture window → output tracks the input

↳ sufficiently narrow for an ideal S/H.

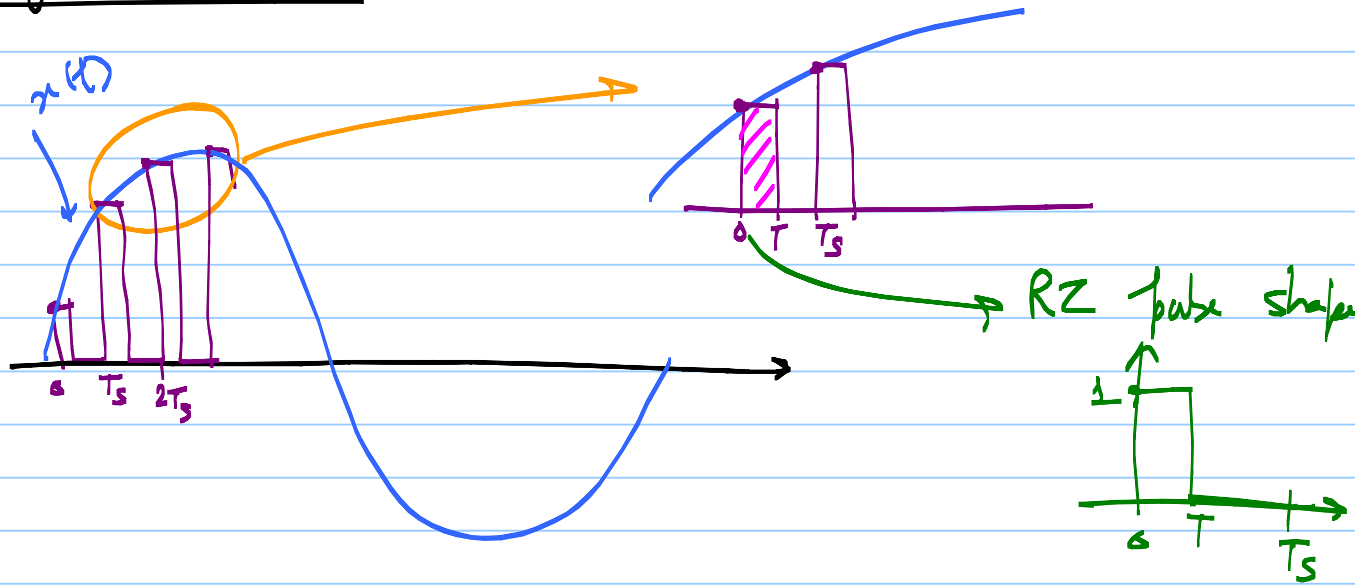
Also called zero-order Sample and hold (S/H)

↳ NRZ pulse shape



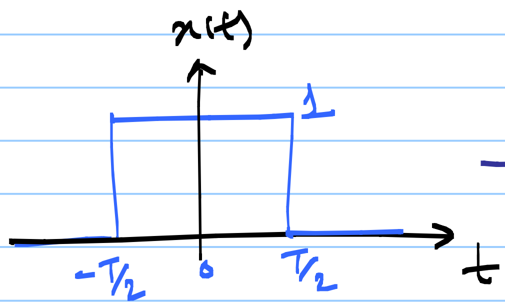
NRZ pulse shape

Generalized S/H :

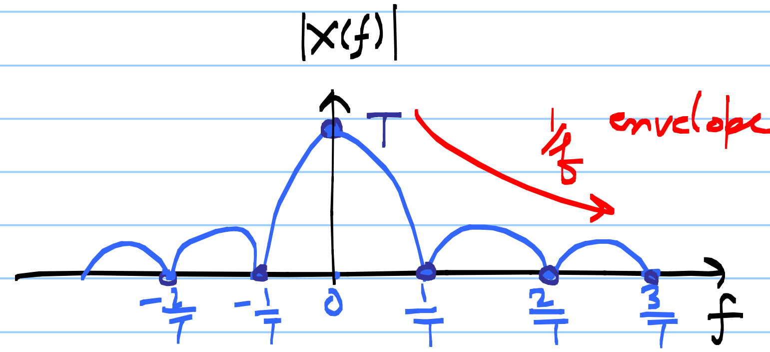


$0 < T \leq T_s$

* Recap on signals:



$$\xrightarrow{\mathcal{F}}$$



$\text{rect}\left(\frac{t}{T}\right)$

* $T\,\text{sinc}(fT)$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

* $\lim_{x \to 0} \text{sinc}(x) = 1$

impulse sampling :

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s) = x(t)\cdot p(t)$$

impulse train

Sample and Hold (zero-order)

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot \mathrm{rect}\left(\frac{t - T/2 - nT_s}{T}\right)$$

$$\Downarrow$$

$$= \sum_{n=-\infty}^{\infty} \left[x(t)\cdot\delta(t-nT_s)\right] \circledast \mathrm{rect}\left(\frac{t-T/2}{T}\right)$$

Think Here !!

$$= \left[x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right] \circledast \underbrace{\mathrm{rect}\left(\frac{t-T/2}{T}\right)}_{h(t)}$$

$$\Rightarrow \quad y(t) = \left[x(t)\cdot p(t)\right] \circledast h(t)$$

$$\Rightarrow \quad Y(f) = \left[X(f) \circledast P(f) \right] \cdot H(f) \longrightarrow ①$$

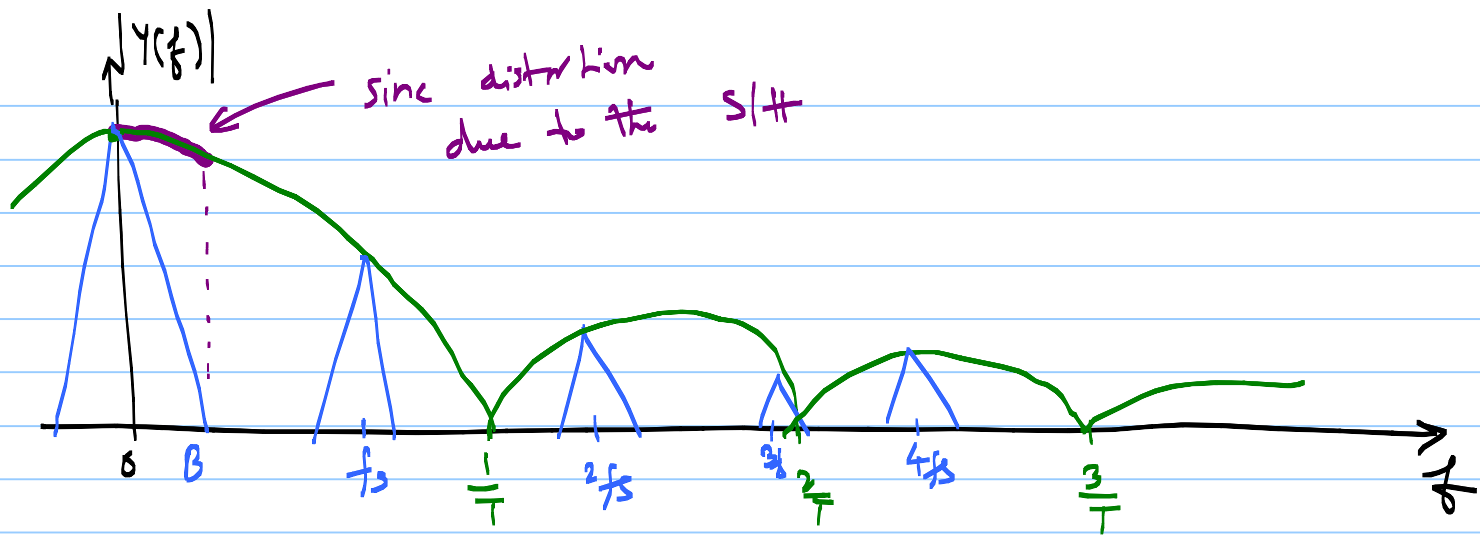
$$H(f) = \mathcal{F}\left(\mathrm{rect}\left(\frac{t - T/2}{T} \right) \right)$$

$$= T \cdot \mathrm{sinc}(fT) \cdot e^{-j\pi f T}$$

$$|H(f)| = T \cdot |\mathrm{sinc}(fT)| \longrightarrow ②$$

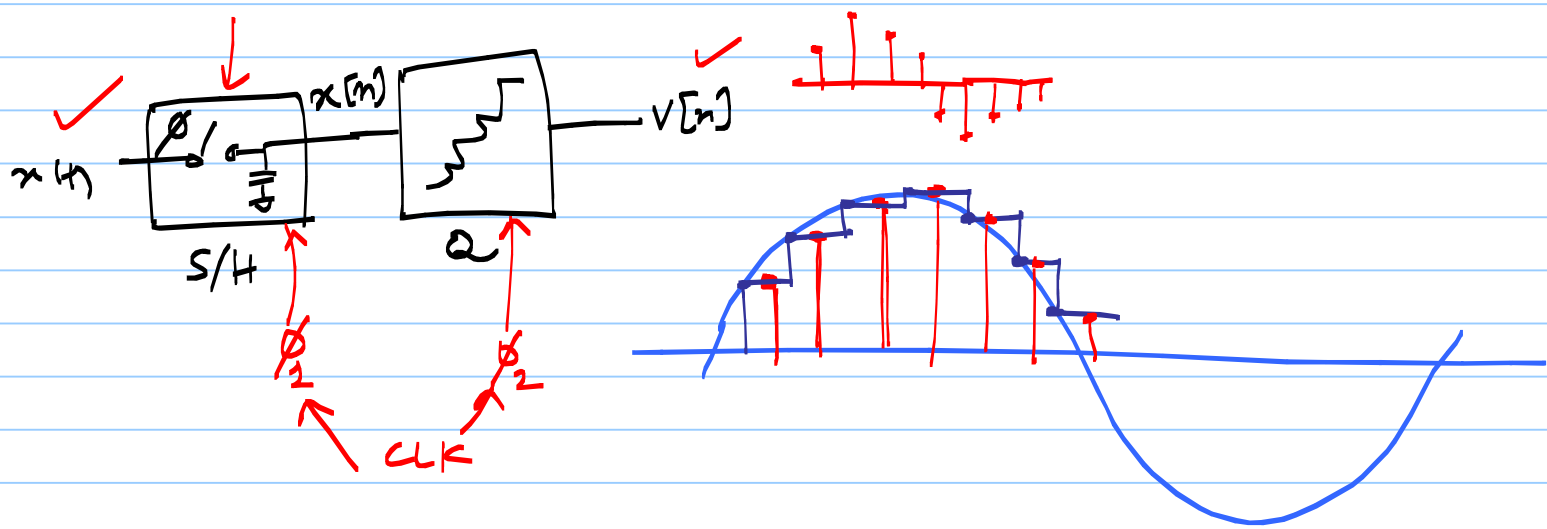
② → ①

$$\Rightarrow \quad Y(f) = \underbrace{\left(\frac{T}{T_s} \sum_{k=-\infty}^{\infty} X(f - k f_s) \right)}_{\text{replicas with BB spectrum}} \cdot \underbrace{\mathrm{sinc}(fT)}_{\text{sinc distortion!}} \cdot e^{-j\pi f T}$$



|Y(f)|
sinc distortion
due to the S/H
0
B
fs
1/T
2fs
3fs
2/T
4fs
3/T
f

Q. Is the S/H's sinc distortion a problem in an ADC with a S/H in the front-end?



* In an ADC, the quantizer senses the o/p of the S/H only during the hold mode
  → the quantized value only corresponds to the sampled

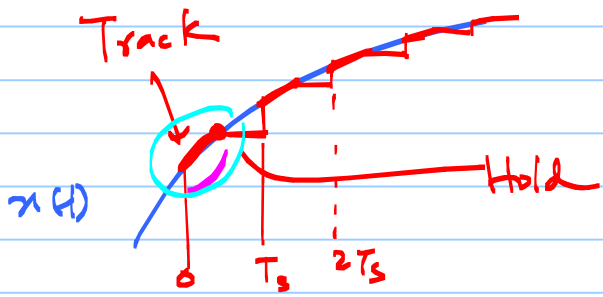
points in the input

$\Rightarrow$ Not an issue with an ADC !

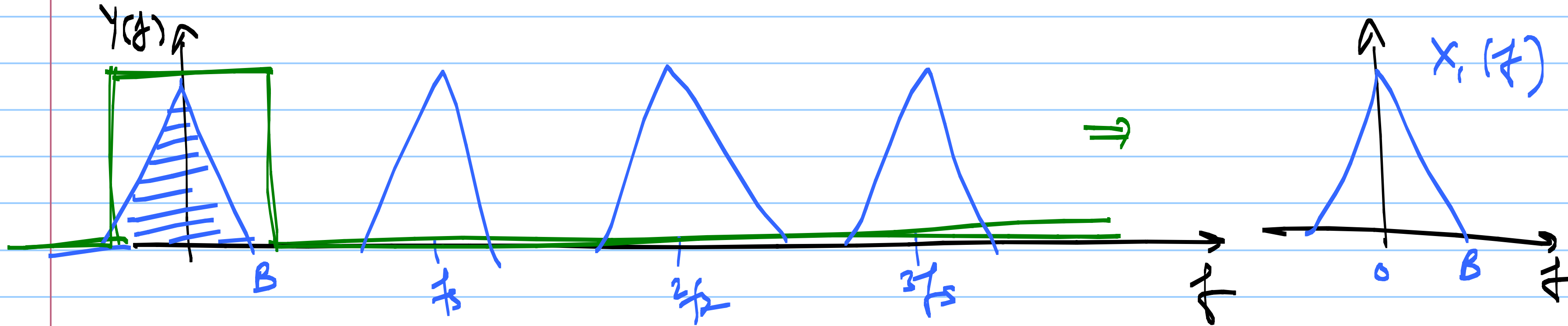
no sinc distortion in an ADC

Track and Hold (T/H):

At high speeds (100 MHz → 10 GHz),
the aperture time increases wrt the sampling period



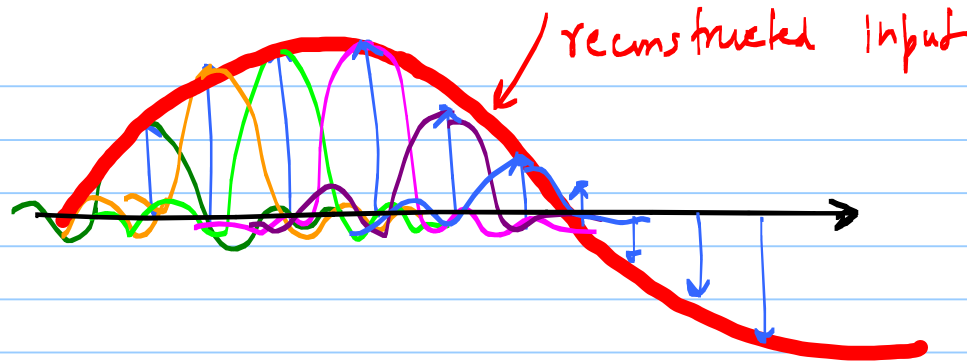
Reconstruction



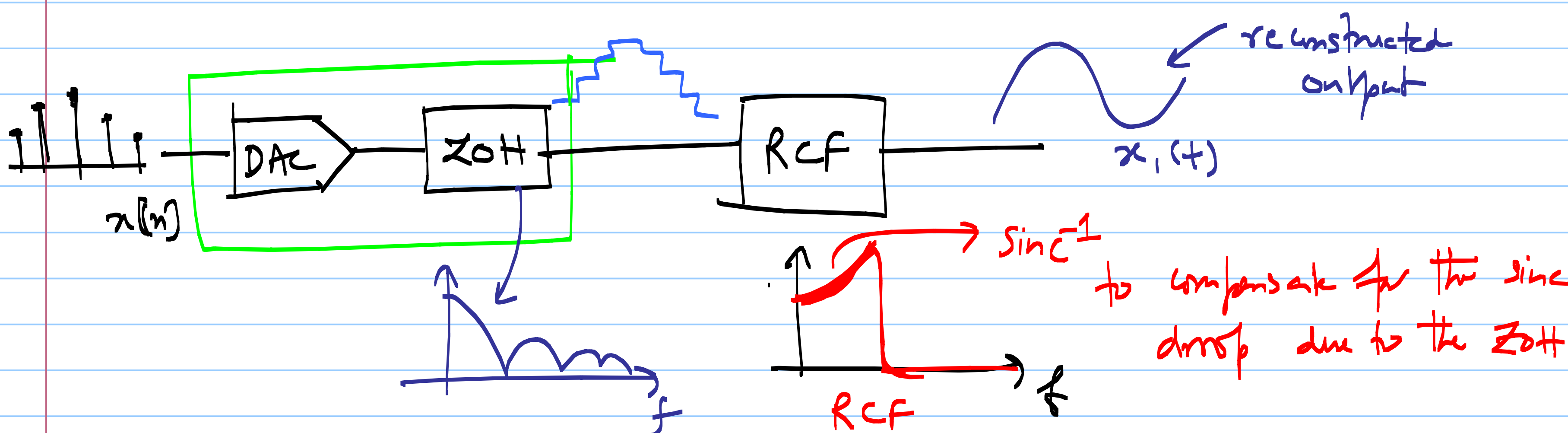
$$Y(f)\cdot \mathrm{rect}\left(\frac{f}{B}\right) \xrightarrow{\mathcal{F}^{-1}} y(t) \circledast \mathrm{sinc}(tB)$$

using Duality

$$B\,\mathrm{sinc}(tB) \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \mathrm{rect}\left(\frac{f}{B}\right)$$



⇒ In time domain
sinc impulse



Spectral Estimation Basics ← DSP course

* Oppenheim & Schafer

Variants of fourier transform

$x(t) \xrightarrow{nT_s} x(nT_s) \sim x[n]$ ← DT

↑ CT

frequency / time	Continuous	Discrete
Continuous	Fourier Transform (FT) $X(f)=\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$	×
Discrete	DTFT ✓ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$	DFT (FFT) $X(k) = \sum_{n=0}^{N-1} x[n] \cdot W_N^{nk}$

$W_N = e^{-j\frac{2\pi}{N}}$

DTFT (Discrete Time Fourier Transform)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n},$$

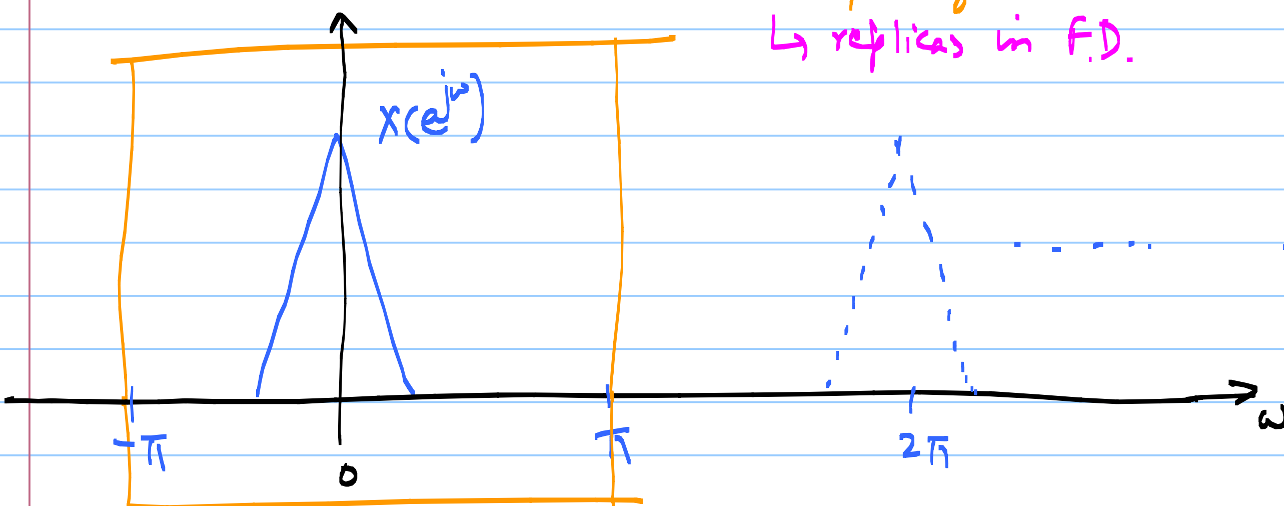
sampled signal

↳ replicas in F.D.

* $x[n]$ should be absolutely or square summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

$\Rightarrow$ $x[n]$ can not be periodic



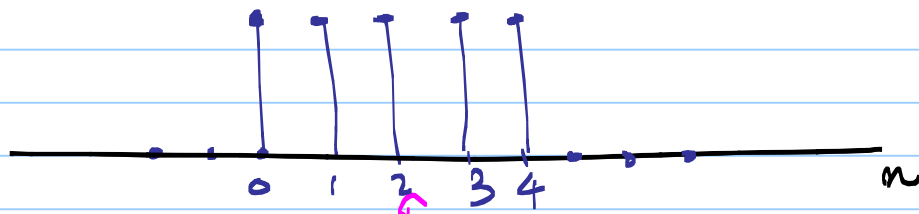
$X(e^{j\omega})$ is periodic with period 2π

$\rightarrow$ 2π corresponds to $\omega_s = 2\pi f_s$ $\leftarrow$ sample rate

$\Rightarrow$ Similar to the sampled spectrum

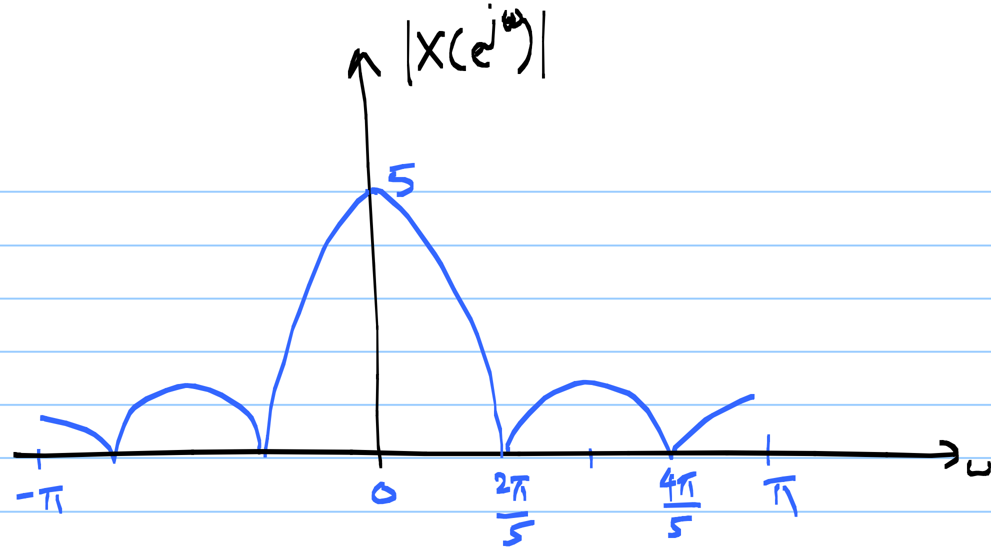
Ex.

$x[n] \leftarrow$ rect



$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = \frac{e^{-j\frac{5}{2}\omega}\left(e^{j\frac{5}{2}\omega} - e^{-j\frac{5}{2}\omega}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= e^{-j2\omega} \cdot \frac{\sin\left(\frac{5\omega}{2}\right)}{\sin\left(\omega/2\right)}$$

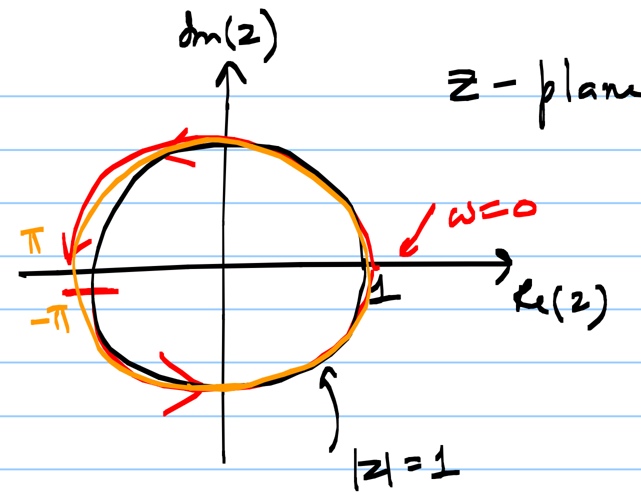


$e^{j\omega} \Rightarrow$ periodic with $\omega = 2\pi$

Relation with z-transform:

Z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

DTFT $\Rightarrow X(e^{j\omega}) = X(z)\Big|_{\boxed{z=e^{j\omega}}}$



* DTFT, $X(e^{j\omega})$ is $X(z)$ evaluated along the unit circle on the z-plane

* DTFT is continuous in frequency

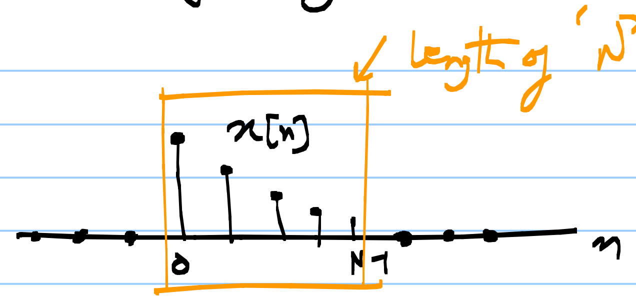
↳ Not good for frequency domain computations using digital computer (DSP processor)

↳ need some transform which is also discretized in frequency domain

↳ Recall that a Fourier Series is discrete in frequency.

Consider a finite length sequence, $x[n]$ of length N

$\Rightarrow$ $x[n]=0$ outside $0 \le n \le N-1$

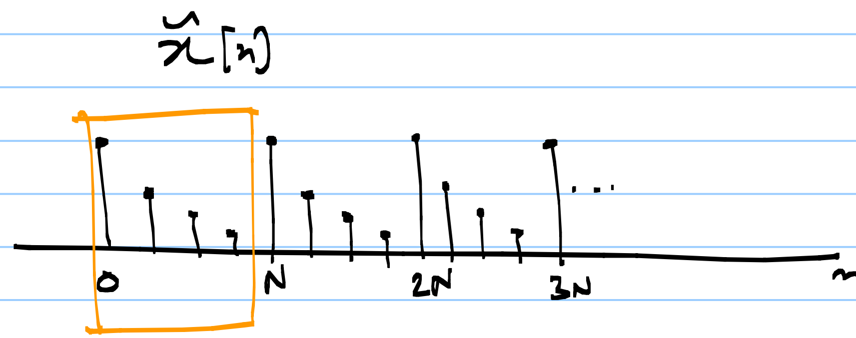


* Now create a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

$$= x[n \text{ modulo } N]$$

$$= x[(n)_N]$$



* Since $\tilde{x}[n]$ is periodic with period N,

↳ can be represented as a summation of complex exponentials with a frequency equal to the integer multiples of the fundamental frequency $\underbrace{\left(\frac{2\pi}{N}\right)}$

* periodic complex exponentials (Definition)

$$e_k[n] = e^{j\left(\frac{2\pi}{N}\right)kn} = e_k[n+rN]$$ ← periodic with N

then

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}$$ ← Discrete Fourier Series (DFS) representation of $\tilde{x}[n]$

* think of :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi n f_0 t}$$

for fourier series

Test:

$$e_{k+lN}[n] = e^{j\frac{2\pi}{N}(k+ln)}$$

$$= e^{j\frac{2\pi k}{N}} \cdot e^{j2\pi l}$$

$$= e^{j\frac{2\pi k}{N}} \cdot (e^{j2\pi})^{l} \qquad \because e^{j2\pi} = 1$$

$$= e^{j\frac{2\pi k}{N}} = e_k \leftarrow \text{periodic with } N \text{ w.r.t to the index 'k'}$$

$\Rightarrow$ Thus we need only $e_0[n]$ to $e_{N-1}[n]$ to represent $\tilde{x}[n]$

$$\hat{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{X}[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

DFS coefficient

only has N frequency components

$k = 0$ to $N-1$

$$\hat{X}[k] = \sum_{n=0}^{N-1} \hat{x}[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

Think of

$$a_k = \frac{1}{T_s}\int_{0}^{T_s} x(t)\, e^{-j2\pi n f_0 t}\, dt$$

Let $W_N = e^{-j\frac{2\pi}{N}}$ ← radix

Analysis eqn : $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, W_N^{nk}$

Synthesis eqn : $\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, W_N^{-nk}$

} Discrete Fourier Series for $\tilde{x}[n]$

Now, define

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

→ N-point sequence

'k' are the freq domain indices

$X[k] \longrightarrow$ Discrete Fourier Transform (DFT)

↳ fast algorithm for computation → FFT

Matlab has fft function

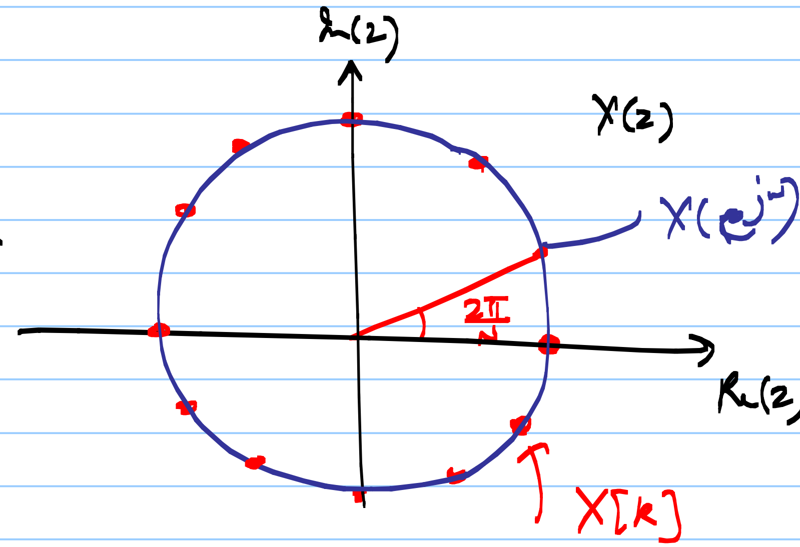
Also, we can show that

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

↓

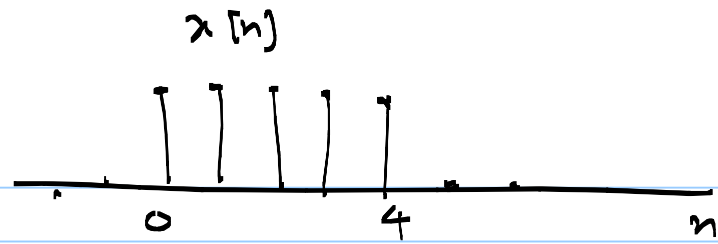
$X[k]$ for $k \in [0, N-1]$

✱ DFT is DTFT Sampled in frequency domain as $\omega = \frac{2\pi k}{N}$.



$N = 5$

Example:



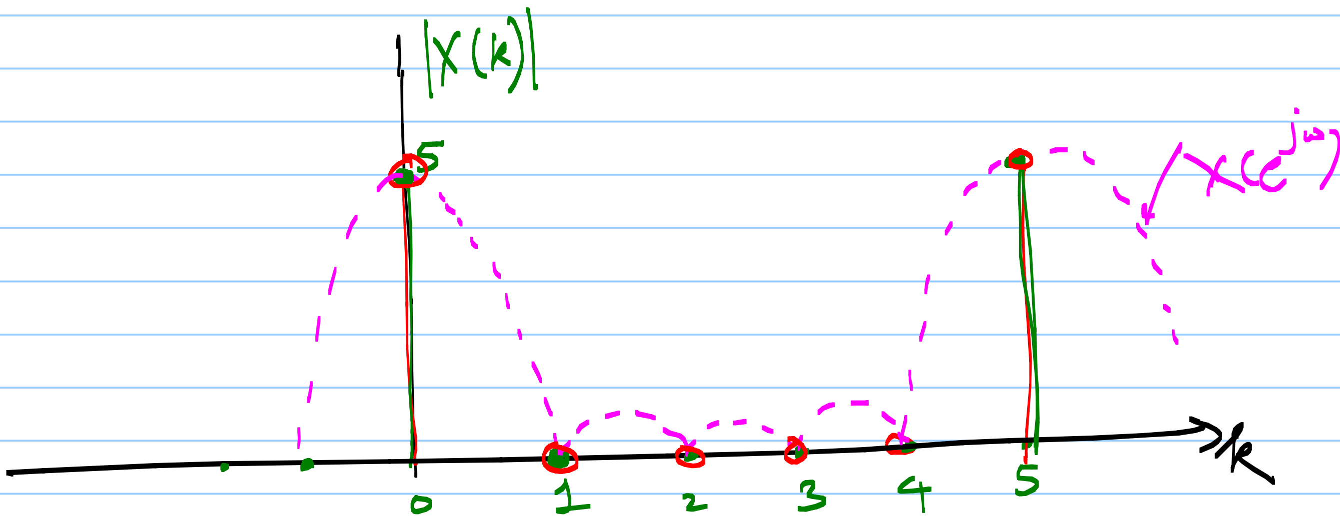
Recall $\quad X(e^{j\omega}) = e^{-j2\omega} \dfrac{\sin\left(\frac{5\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$

Now,

DFT $\quad X[k] = \sum_{n=0}^{4} W_5^{nk} = \sum_{n=0}^{4} \left(e^{-j\frac{2\pi k}{5}}\right)^n$

$$= \frac{1 - e^{-j2\pi k}}{1 - e^{-j\frac{2\pi}{5}k}} = \frac{e^{-j\frac{2\pi}{2}k}\left(e^{j\pi k} - e^{-j\pi k}\right)}{e^{-j\frac{2\pi}{10}k}\left(e^{j\frac{2\pi k}{10}} - e^{-j\frac{2\pi k}{10}}\right)}$$

$$= e^{-j\left(\frac{4\pi k}{10}\right)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}, \quad k = 0 \text{ to } 4$$



5-point DFT

⇒ DFT $X(k)$ is a discrete representation of the frequency transform of $x(n)$