ECE 615 - Lecture 1

Signals Refresher

Fourier Series: For a periodic signal $g(t)$, with period T_0

$$g(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k}\, e^{j2\pi k f_0 t}$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-j2\pi k f_0 t}\, dt$$

$\left(e^{j2\pi f_0 t}\right) \rightarrow$ fundamental tone

$Re\left(\quad\right) = \cos(2\pi f_0 t)$

Fourier Transform :

$x(t) \longrightarrow$ Linear time Invariant

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

frequency domain

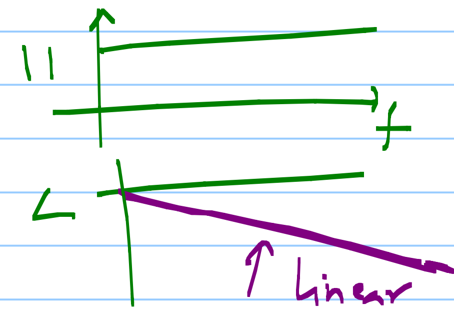
time domain

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

Linearity ⇒ $a x(t) + b x(t) \xrightarrow{\mathcal{F}} aX(f) + bY(f)$

time delay ⇒ $x(t - \underline{t_0}) \xrightarrow{\mathcal{F}} X(f)\, \boxed{e^{-j2\pi f t_0}}$

$1 \underline{/ -2\pi f t_0}$



frequency translation ⇒

$\underbrace{e^{j2\pi f_0 t}}_{\text{complex sinusoid}}\, x(t) \xleftrightarrow{\mathcal{F}} X(f - f_0)$

complex sinusoid

frequency shift

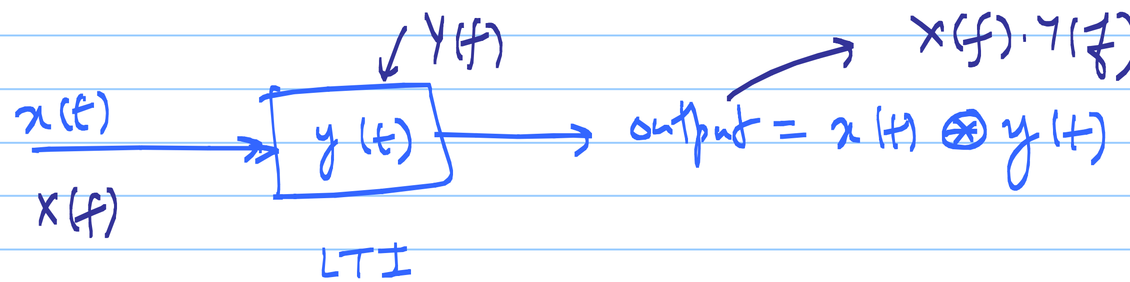
$x(-t) \xleftrightarrow{\mathcal{F}} X(-f)$

Scaling ⇒ $x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$

slow down → frequency component will shrink

Speed up → max frequency component

Convolution ⇒ $x(t) \circledast y(t) \xrightarrow{\mathcal{F}} X(f) \cdot Y(f)$



Multiplication ⇒ $x(t) \cdot y(t) \xleftrightarrow{\mathcal{F}} X(f) \circledast Y(f)$

Duality $\Rightarrow$

$$x(t) \xrightarrow{\mathcal{F}} X(f)$$

$$X(t) \xrightarrow{\mathcal{F}} x(-f)$$

Oppenheim & Wilsky
Signals & Systems

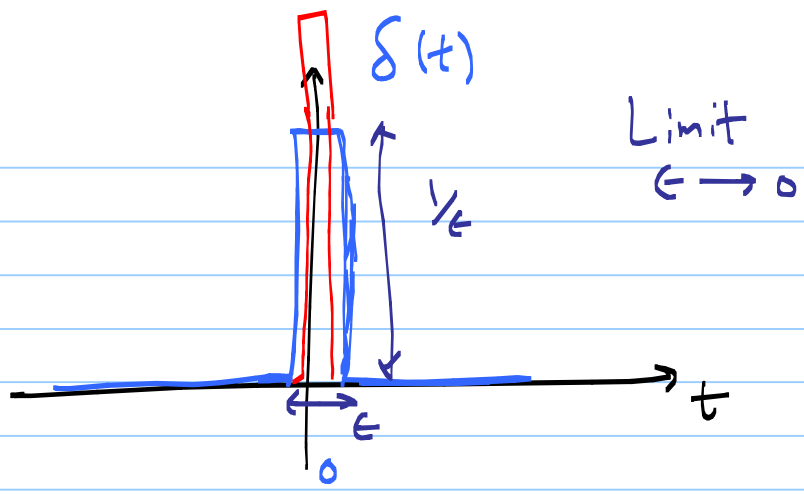
Parseval's Theorem : (Energy Conservation)

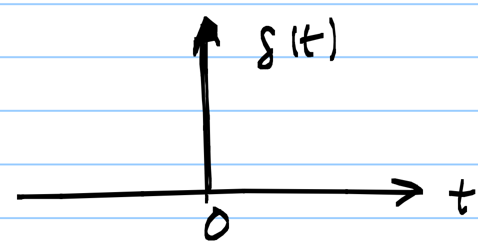
$$\int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} X(f)\, X^*(f)\, df$$

Complex conjugate

Delta Function (?)

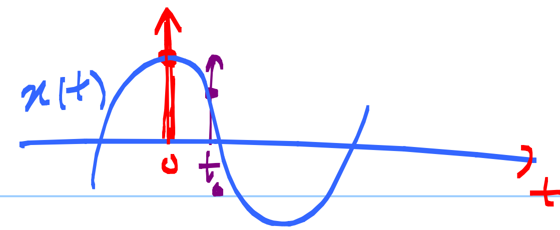
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$





* $x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$

picks the value of $x(t)$ at $t = 0$



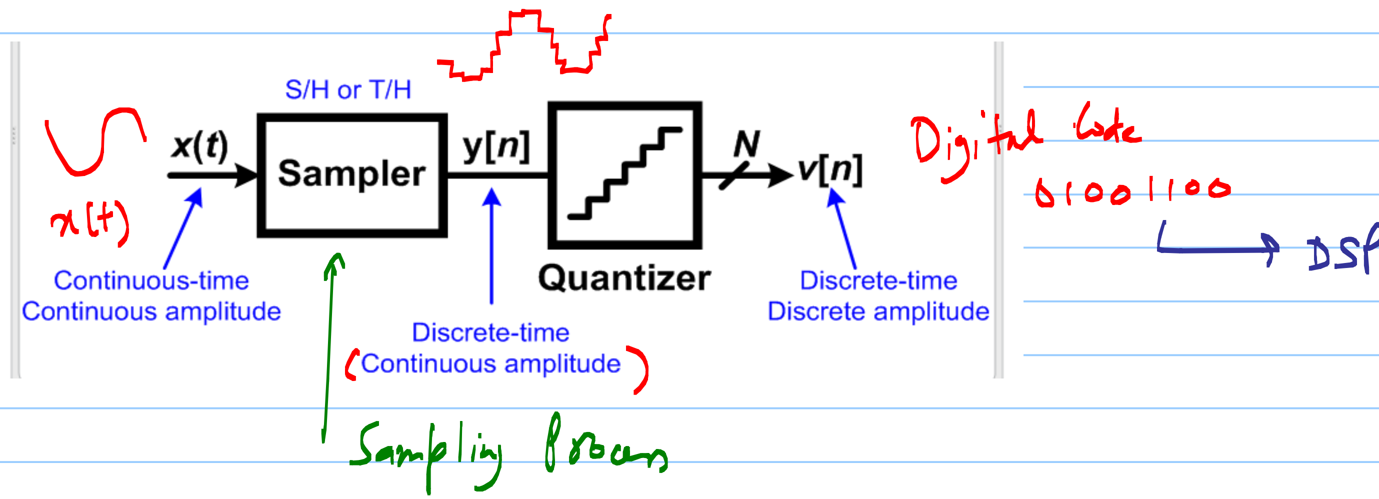
* $x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$

* $x(t) \circledast \delta(t) = x(t)$

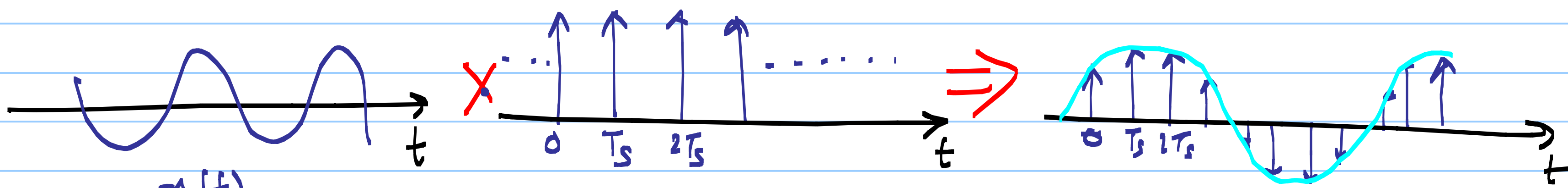
  $\hookrightarrow \; x(t) \circledast \delta(t - t_0) = x(t - t_0)$

* $\delta(t) \xrightarrow{\mathcal{F}} 1$

Analog to Digital Converter



Ideal Sampling (impulse sampling)



$x(t)$ impulse train, $p(t)$ $y(t) = x_s(t)$

Sampled signal

$$y(t) = x(t)\cdot \overbrace{\sum_{n=-\infty}^{\infty}\delta(t-nT_s)}^{p(t)} = x(t)\cdot p(t)$$

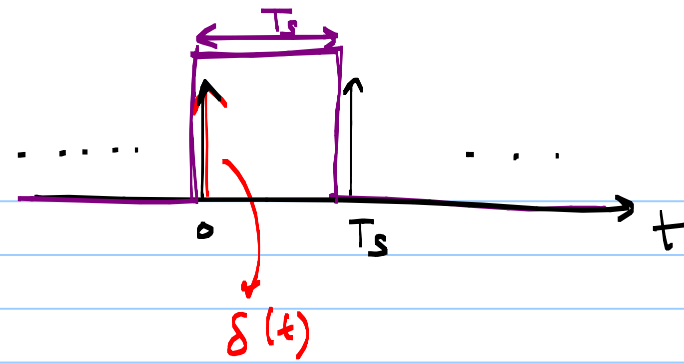
$$\Rightarrow \quad Y(f) = X(f) \circledast P(f)$$

How to find $P(f) = ?$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

↳ periodic signal

↳ fourier series components



$$p(t) = \sum_{k=-\infty}^{\infty} \boxed{a_k}\, \underbrace{e^{j2\pi k f_s t}}_{\left(e^{j2\pi f_s t}\right)^k}$$ ← fourier series representation of $p(t)$

$$\Rightarrow \quad a_k = \frac{1}{T_s}\int_0^{T_s^-} p(t)\, e^{-j2\pi k f_s t} \cdot dt = \frac{1}{T_s}\int_0^{T_s^-} \delta(t)\, \underbrace{e^{-j2\pi k f_s (0)}}_{1}\, dt$$

$t=0$

$$= \frac{1}{T_s}$$

$a_k = \frac{1}{T_s}$ for all the harmonics $\forall k$

$\delta(t) \xrightarrow{\mathcal{F}} 1$

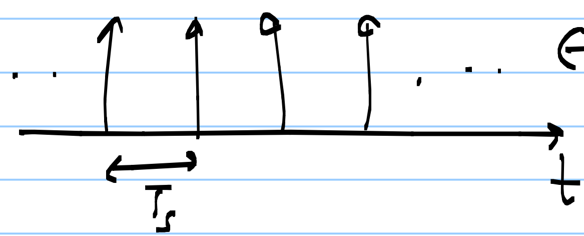
$1 \xrightarrow{\mathcal{F}} \delta(f)$

$$\hookrightarrow p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{j2\pi k f_s t} \cdot 1$$

Lets take Fourier transform on both sides

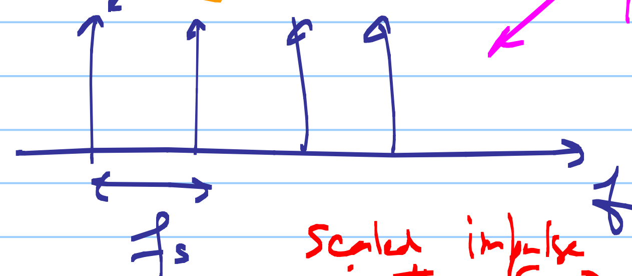
$$P(f) = \boxed{\frac{1}{T_s}} \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

also looks like impulse train in the frequency domain



$\xleftrightarrow{\mathcal{F}}$

$\boxed{\frac{1}{T_s}} = f_s$



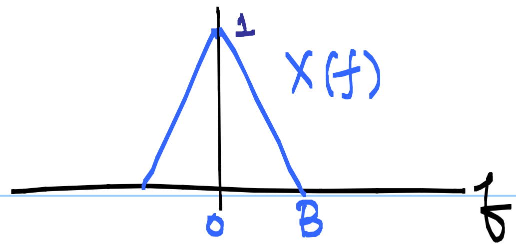
important!

Scaled impulse train in the F.D.

$$Y(f) = X(f) \circledast P(f)$$

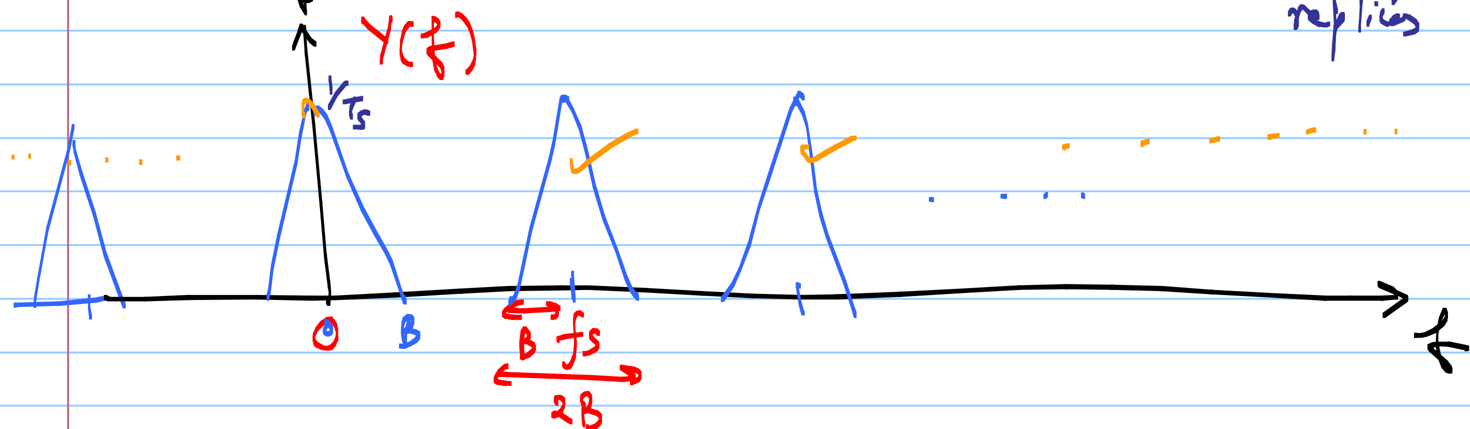
$$= X(f) \circledast \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

$$\boxed{Y(f) = \underbrace{\frac{1}{T_s}} \sum_{k=-\infty}^{\infty} X(f - kf_s)} \Leftarrow \text{Fundamental Result}$$



ideal Sampling

frequency domain replicas due to sampling

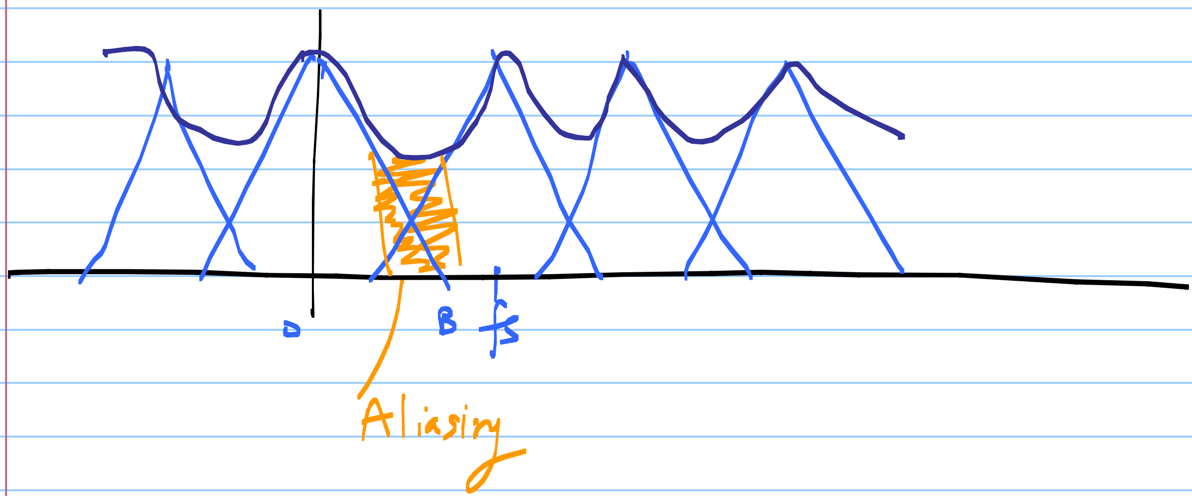


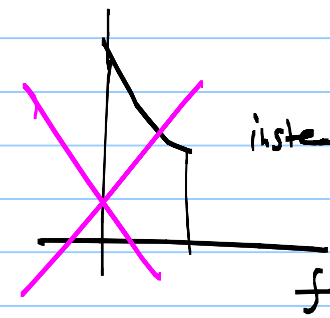
⇒ To avoid Aliasing

$$\boxed{f_s \geq 2B}$$

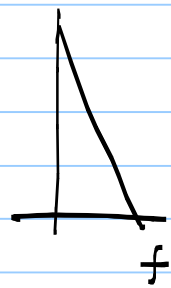
Nyquist Sampling Theorem

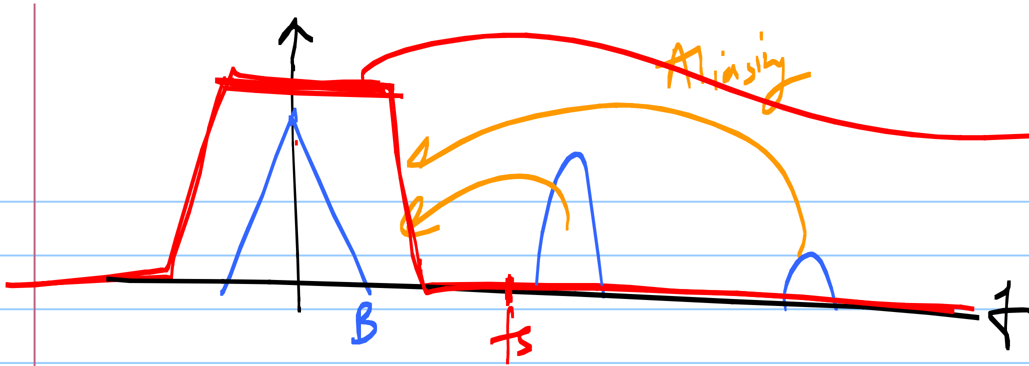
* what happens if $f_s < 2B$





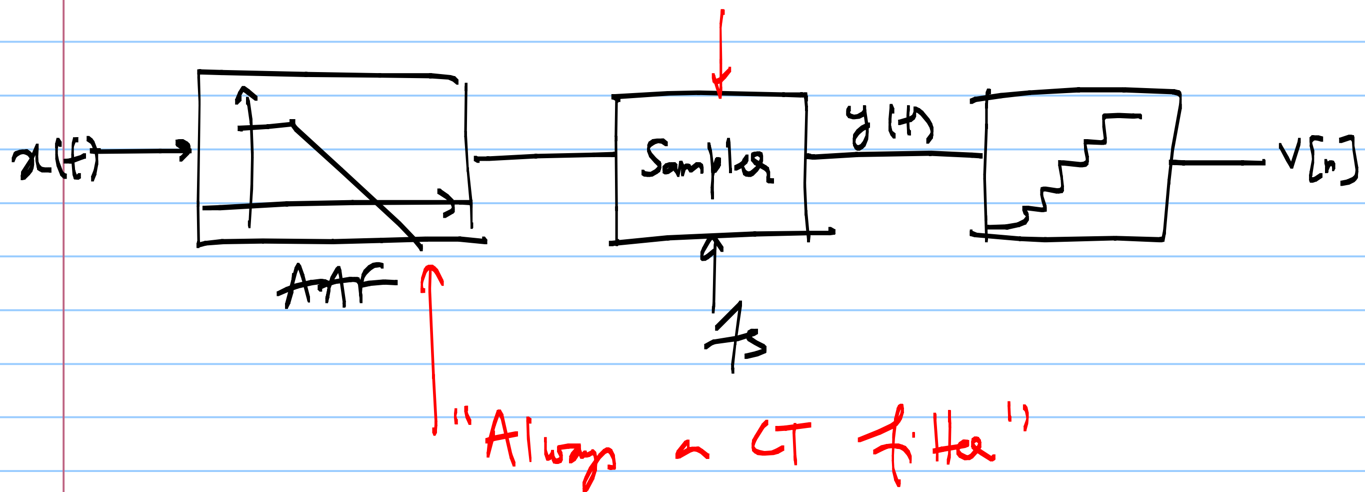
instead of



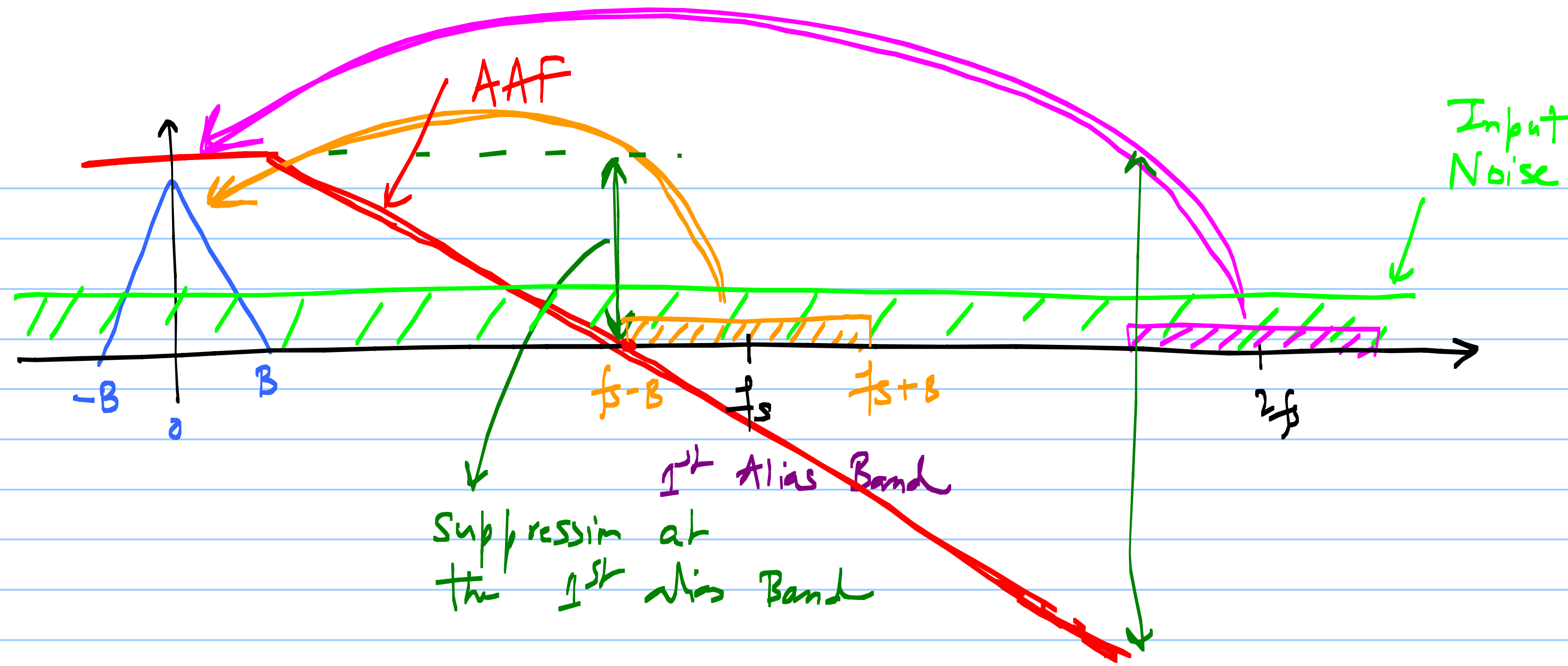


Anti-Alias filter
(AAF)

* Bandlimit the input signal

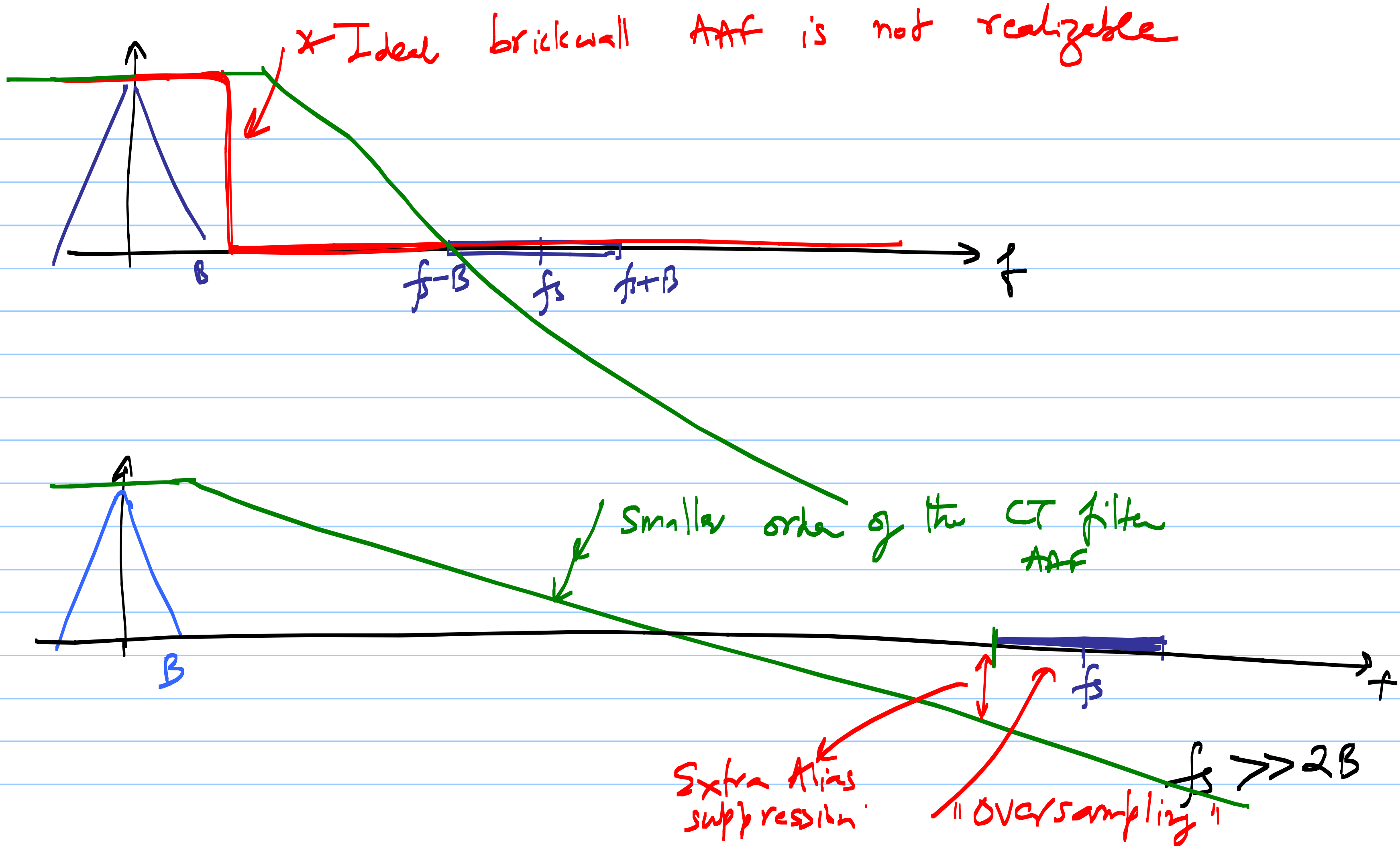


x(t)
AAF
Sampler
y(t)
fs
V[n]
"Always a CT filter"



* AAF Suppresses the noise in the alias bands

↳ AAF is a must before a sampler



× Ideal brickwall AAF is not realizable
f
B
fs−B
fs
fs+B
Smaller order of the CT filter
AAF
B
fs
f
fs >> 2B
Extra Alias suppression
"Oversampling"

⇒ Oversampling results in

↳ better alias rejection with the same AAF

↳ lower order AAF for the same amount of alias rejection

⇒ Oversampling relaxes the design requirements on AAF

$$\text{Oversampling ratio} = \frac{f_s}{f_{s,\,Nyquist}} = \frac{f_s}{2B}$$

$$\boxed{OSR = \frac{f_s}{2B}}$$

$$f_s = OSR \cdot f_{s,\,Nyquist}$$