

# Generalized Digital Butterworth Filter Design\*

Ivan W. Selesnick and C. Sidney Burrus  
*Selesnick is with Polytechnic University, Brooklyn, NY.*  
*Burrus is with Rice University, Houston, TX.*

December 7, 1997

## Abstract

This paper introduces a new class of IIR digital filters that unifies the classical digital Butterworth filter and the well known maximally flat FIR filter. New closed form expressions are provided, and a straight forward design technique is described. The new IIR digital filters have more zeros than poles (away from the origin) and their (monotonic) square magnitude frequency responses are maximally flat at  $\omega = 0$  and at  $\omega = \pi$ . Another result of the paper is that, for a specified cut-off frequency and a specified number of zeros, there is only one valid way in which to split the zeros between  $z = -1$  and the passband. This technique also permits continuous variation of the cut-off frequency. IIR filters having more zeros than poles are of interest, because often, to obtain a good trade-off between performance and implementation complexity, just a few poles are best.

*EDICS: SP 2.1.2 DSP/Filter Design/IIR Systems*

Contact Address:  
*Ivan W. Selesnick*  
*Electrical Engineering*  
*Polytechnic University*  
*6 Metrotech Ctr*  
*Brooklyn, NY 11201-3840*  
*Phone: (718) 260 - 3416*  
*FAX: (718) 260 - 3906*  
*email: selesi@radar.poly.edu*

Submitted to: IEEE Transactions on Signal Processing - Sept 1995

Revised - May 1996, November 1996

---

\*This work has been supported by BNR and by NSF grant MIP-9316588.

## List of Figures

1	Magnitudes of the three digital IIR filters shown in Figures 2, 3 and 4. . . . .	13
2	$L = 4, M = 0, N = 4$ . . . . .	14
3	$L = 6, M = 0, N = 4$ . The poles at the origin are not shown in the figure. . . .	14
4	$L = 16, M = 7, N = 4$ . The poles at the origin are not shown in the figure. . . .	14
5	Generalized Butterworth filters. $L + M + N = 20, \omega_o = 0.6\pi$ . $N$ is varied from 0 to 10 in increments of 2. $N = 10$ corresponds to the filter having the steepest transition and the most peaked group delay. The values of $L, M$ , and $N$ are shown in Table 3. . . . .	15
6	Generalized Butterworth filters for special values of $\omega_o$ . $L + M = 22, N = 4$ . $L$ is varied from 5 to 21. The widest band filter corresponds to $L = 5$ . . . . .	16

## List of Tables

1	Notation. . . . .	11
2	For the choice $L, M$ , and $N$ shown in the table, the interval of achievable half-magnitude frequencies $\omega_o$ is given by $[\omega_{min}, \omega_{max}]$ . $L + M$ is the numerator degree (number of zeros) and $N$ is the denominator degree (number of poles). . . . .	11
3	For the half-magnitude frequency $\omega_o = 0.6\pi$ and $L + M + N = 20$ , the table shows the correct values of $L$ and $M$ , and the derivative of the magnitude response at $\omega_o$ . The filter responses are shown in Figure 5. . . . .	11
4	Permissible ranges for $c$ for the first generalization. . . . .	12
5	The number and locations of the real roots of $\mathcal{T}_N\{(1-x)^L\} + cx^N - 4(1-x)^L$ for $L > N > 0$ . . . . .	12
6	Auxiliary polynomials. For negative values of $n$ , the convention [11], $\binom{n+k-1}{k} = (-1)^k \binom{-n}{k}$ for $k \geq 0$ is used. In addition, note that $\binom{n}{k} = 0$ for $n \geq 0, k < 0$ ; and that $\binom{n}{k} = 0$ for $n \geq 0, k > n$ . . . . .	12
7	Permissible ranges for $c$ for the second generalization. . . . .	13
8	The expression for $F(x)$ gives the magnitude squared function in the $x$ domain in terms of a constant $c$ . When $c$ is chosen according to the expression given in the table, $F(x_o)$ equals $1/4$ . . . . .	13

# 1 Introduction

Probably the best known and most commonly used method for the design of IIR digital filters is the bilinear transformation of the classical analog filters (the Butterworth, Chebyshev I and II, and Elliptic filters) [9]. One advantage of this technique is the existence of formulas for these filters. However, the numerator and denominator of such IIR filters have equal degree. It is sometimes desirable to be able to design filters having more zeros than poles (away from the origin) to obtain an improved compromise between performance and implementation complexity.

The new formulas introduced in this paper unify the classical digital Butterworth filter and the well known maximally flat FIR filter described by Herrmann [3]. The new maximally flat lowpass IIR filters have an unequal number of zeros and poles and possess a specified half-magnitude frequency. It is worth noting that not all the zeros are restricted to lie on the unit circle, as is the case for some previous design techniques for filters having an unequal number of poles and zeros. The method consists of the use of a formula and polynomial root finding. No phase approximation is done; the approximation is in the magnitude squared — as are the classical IIR filter types.

Another result of the paper is that for a specified number of zeros and a specified half-magnitude frequency, there is only one valid way to divide the number of zeros between  $z = -1$  and the passband. The paper also describes how to construct a table, from which it is simple to determine the correct way in which to split the zeros between these two bands.

Given a half-magnitude frequency  $\omega_o$ , the filters produced by the formulas described below are optimal (maximally flat) in the following sense: the maximum number of derivatives at  $\omega = 0$  and  $\omega = \pi$  are set to zero, under the constraint that the filter possesses the half-magnitude frequency  $\omega_o$  and a monotonic frequency response magnitude. The classical digital Butterworth filter and the well known maximally flat FIR filter [3, 5, 6, 20, 23] are both special cases of the filters produced by the formulas given in this paper.

Several authors have addressed the design and the advantages of IIR filters with an unequal number of (nontrivial) poles and zeros. While [14, 22] give formulas for IIR filters with Chebyshev stopbands having more zeros than poles, these methods require that all zeros lie on the unit circle. This restriction limits the range of achievable cut-off frequencies. In [4] Jackson notes that the use of just 2 poles “is often the most attractive compromise between computational complexity and other performance measures of interest.” In [13] Saramäki discusses the trade-offs between numerator and denominator order and describes an iterative algorithm in which zeros are not constrained to lie on the unit circle for the design of filters having Chebyshev stopbands. In [12, 13], Saramäki finds that the classical Elliptic and Chebyshev filter types are seldom the best choice.

## 2 Notation

Let  $H(z) = B(z)/A(z)$  denote the transfer function of a digital filter. Its frequency response magnitude is given by  $|H(e^{j\omega})|$ . Throughout this paper, the degree of  $B(z)$  will be denoted by  $L + M$ , where  $L$  is the number of zeros at  $z = -1$  and  $M$  is the number of remaining zeros. The zeros at  $z = -1$  produce a flat behavior in the frequency response magnitude at  $\omega = \pi$ , while the remaining zeros, together with the poles, are used to produce a flat behavior at  $\omega = 0$ . The half-magnitude frequency is that frequency at which the magnitude equals one half. Like the 3 dB point, it indicates the location of the transition band. The meanings of the parameters are shown in Table 1. It should be noted that by ‘degree of flatness’, we mean the number of derivatives that are made to match the desired response, including the  $0^{th}$  derivative.

## 3 Examples

The classical digital Butterworth filters (defined by  $L = N$  and  $M = 0$ ) are special cases of the filters discussed in this paper. Figures 1 and 2 illustrate a classical digital Butterworth filter of order 4 ( $L = 4$ ,  $M = 0$ ,  $N = 4$ ). The first generalization of the classical digital Butterworth filter described below permits  $L$  to be greater than  $N$ , with  $M = 0$ . Figure 3 illustrates an IIR filter with  $L = 6$ ,  $M = 0$ ,  $N = 4$ . It was designed to have the same half-magnitude frequency. It turns out that when  $L > N$ , the restriction that  $M$  equal zero limits the range of achievable half-magnitude frequencies, as will be elaborated below. This motivates the second generalization. In addition to permitting  $L$  to be greater than  $N$ , the second generalization permits  $M$  to be greater than zero:  $L > N$  and  $M > 0$ . Figure 4 illustrates an IIR filter with  $L = 16$ ,  $M = 7$ ,  $N = 4$ .

As mentioned above, for a specified half-magnitude frequency  $\omega_o$  and specified numerator and denominator degrees, there is only one correct way to split the zeros between  $z = -1$  and the passband. To illustrate this property, it is helpful to construct a table that indicates the appropriate values for  $L$ ,  $M$  and  $N$ . When  $N = 4$  and  $L + M$  is varied from 4 to 7, Table 2 gives the values  $L$  and  $M$  required, to achieve a desired half-magnitude frequency. As can be seen from the table, the intervals cover the interval (0,1) and do not overlap. This will be true in general, as long as at least one pole is used. In the FIR case, the intervals cover an interval  $(a, b)$  with  $a > 0$  and  $b < 1$ . (Neither the passband nor the stopband can be arbitrarily narrow). Notice that in the case of the classical Butterworth filter ( $L + M = N$ ),  $M$  equals zero, regardless of the specified half-magnitude frequency. As will be explained below, these intervals can be unambiguously computed by inspecting the roots of an appropriate set of polynomials.

To illustrate the trade-offs that can be achieved with the generalized Butterworth filters

described in this paper, it is useful to examine a set of filters all of which have the same half-magnitude frequency and the same total number of poles and zeros ( $L + M + N$ ). For example, when  $L + M + N$  is fixed at 20 and the half-magnitude  $\omega_o$  is fixed at  $0.6\pi$ , the filters shown in Figure 5 are obtained. The number of poles of the filters in this figure vary from 0 to 10 in steps of 2. It is interesting to measure the slope of the magnitude  $|H(e^{j\omega})|$  at the half-magnitude frequency. The figure shows the negative reciprocal of the slope of  $|H(e^{j\omega})|$  at  $\omega_o$  — this indicates the approximate width of the transition band. Notice from Table 3 and Figure 5, that for this example, as the number of poles and zeros become more equal, the slope of the magnitude at  $\omega_o$  becomes more negative and the transition region becomes sharper. However, it is interesting to note that the improvement in magnitude is greatest when the number of poles is increased from 0 to 2.

## 4 Design Formulas

The approach described below uses the mapping  $x = \frac{1}{2}(1 - \cos \omega)$  and provides formulas for two nonnegative polynomials  $P(x)$  and  $Q(x)$ . A stable IIR filter  $B(z)/A(z)$  is obtained having a magnitude squared frequency response  $|H(e^{j\omega})|^2$  given by

$$|H(e^{j\omega})|^2 = \frac{P\left(\frac{1}{2} - \frac{1}{2}\cos \omega\right)}{Q\left(\frac{1}{2} - \frac{1}{2}\cos \omega\right)}$$

as in [3]. Accordingly,  $F(x) = P(x)/Q(x)$  is designed to approximate a lowpass response over  $x \in [0, 1]$ .  $B(z)$  and  $A(z)$  are most conveniently found by first computing the roots of  $P(x)$  and  $Q(x)$  and by then mapping those roots to the  $z$ -plane via

$$z = 1 - 2x \pm \sqrt{1 - 2x - 1}. \quad (1)$$

For stable minimum-phase solutions, take the sign of the square root yielding points inside the unit circle. We begin with the classical digital Butterworth filter. This establishes notation and makes the generalization more clear.

### 4.1 Classical Digital Butterworth Filter

Assume  $L \leq N$  and  $M = 0$ , then the rational function  $F(x) = P(x)/Q(x)$  is given by

$$F(x) = \frac{(1-x)^L}{(1-x)^L + cx^N}. \quad (2)$$

The classical Butterworth filter is obtained when  $N = L$ . Note that  $|H(e^{j\pi/2})|^2 = F(1/2) = 1/(1 + c \cdot 2^{L-N})$ . Clearly,  $c$  should be chosen so that this value lies between 0 and 1. Therefore,  $c$  must be greater than zero.

To choose  $c$  to achieve a specified half-magnitude frequency is straight-forward. The equation  $|H(e^{j\omega_o})| = 1/2$  becomes  $F(x_o) = 1/4$  where  $x_o = \frac{1}{2}(1 - \cos \omega_o)$ . Solving this equation for  $c$ , one obtains  $c = 3(1 - x_o)^L/x_o^N$ . Because this expression is positive for all  $x_o \in (0, 1)$ , any half-magnitude  $\omega_o \in (0, \pi)$  is achievable when  $L \leq N$  and  $M = 0$ .

## 4.2 First Generalization

For the first generalization, assume  $L > N$ ,  $M = 0$ . Then, introducing the notation  $\mathcal{T}_N$  for polynomial truncation (discarding all terms beyond the  $N^{\text{th}}$  power),  $F(x)$  can be written as

$$F(x) = \frac{(1-x)^L}{\mathcal{T}_N\{(1-x)^L\} + cx^N}. \quad (3)$$

The term,  $c$ , is the free parameter that, as in the classical case, can be chosen to achieve a specified half-magnitude frequency, and must be chosen to lie within an appropriate range. The allowable ranges for  $c$  are given in Table 4. When  $c$  is chosen to lie in the ranges shown in the table, then  $0 < F(x) < 1$  for  $x \in (0, 1)$ . See [16] for a proof.

To choose  $c$  to achieve a specified half-magnitude frequency  $\omega_o$ , solve  $F(x_o) = 1/4$  for  $c$ . This yields

$$c = \frac{4(1-x_o)^L - \mathcal{T}_N\{(1-x)^L\}(x_o)}{x_o^N}. \quad (4)$$

The value this expression gives for  $c$  may or may not lie in the appropriate range shown in Table 4. If it does not, then the specified half-magnitude frequency is too high for the current choice of  $L$  and  $N$ . It should be noted that, although the passband can be made arbitrarily narrow, it can not be made arbitrarily wide for a fixed  $L$  and  $N$  (when  $L > N$ ).

The greatest half-magnitude frequency achievable for a fixed  $L$  and  $N$  can be found by setting  $c$  equal to the appropriate value shown in Table 4 and solving (4) for  $x_o$ . It is seen that  $x_o$  is a root of the polynomial

$$\mathcal{T}_N\{(1-x)^L\} + cx^N - 4(1-x)^L = 0. \quad (5)$$

Note that  $x_o$  should lie in  $(0, 1)$ . When  $L > N > 0$ , this polynomial has exactly one real root in  $(0, 1)$ , see [16] for a proof. The number and locations of the real roots of (5) are given in Table 5.

**Example:** For  $L = 6$  and  $N = 4$ , the boundary value for  $c$  from Table 4 is 0 ( $N$  is even), so the polynomial equation (5) becomes:  $\mathcal{T}_4\{(1-x)^6\} - 4(1-x)^6 = 0$ . Its roots are:  $3.9476, 0.3798 \pm 1.1659j, 0.4262 \pm 0.3245j, 0.4404$ . Therefore, for this choice of  $L$  and  $N$ ,  $x_o$  must lie in  $(0, 0.4404]$ , so  $\omega_o$  must lie in  $(0, 0.4620\pi]$ . To obtain filters having wider passbands with the same number of zeros and (nontrivial) poles, it is necessary to move at least one zero from  $x = 1$  ( $z = -1$ ) to the passband.

### 4.3 Second Generalization

For the second generalization, assume  $L > N$ ,  $M > 0$ . The zeros lying off the unit circle are used to obtain a higher degree of flatness at  $\omega = 0$ . Such a filter is shown in Figure 4. In this case,  $F(x)$  is given by

$$F(x) = \frac{(1-x)^L(R(x) + cT(x))}{\mathcal{T}_N\{(1-x)^L(R(x) + cT(x))\}} \quad (6)$$

where  $R(x)$  and  $T(x)$  are given in Table 6. Table 6 also provides expressions for  $(1-x)^L R(x)$  and  $(1-x)^L T(x)$ . These polynomials are such that the numerator of  $F(x) - 1$  is divisible by  $x^{M+N}$ . Again, the free parameter  $c$  can be chosen to precisely position the location of the transition band. However,  $c$  must lie in the ranges shown in Table 7. (When  $N$  is even, for example, the positive endpoint of this interval is that point beyond which  $F(x)$  is no longer monotonic — and the negative endpoint of this interval is that point beyond which  $F(x)$  is no longer nonnegative.)

To choose  $c$  to achieve a specified half-magnitude frequency, solve  $F(x_o) = 1/4$  for  $c$ . This yields

$$c = \frac{4(1-x_o)^L R(x_o) - \mathcal{T}_N\{(1-x)^L R(x)\}(x_o)}{\mathcal{T}_N\{(1-x)^L T(x)\}(x_o) - 4(1-x_o)^L T(x_o)}. \quad (7)$$

The value this expression gives for  $c$  may or may not lie in the appropriate range given by Table 7. If it does not, then the specified half-magnitude frequency is either too high or too low for the current choice of  $L$ ,  $M$  and  $N$  — it is necessary to alter the distribution of zeros between  $x = 1$  ( $z = -1$ ) and the passband.

For fixed  $L$ ,  $M$ , and  $N$ , the minimum and maximum permissible values of the half-magnitude frequency  $\omega_o$  can be computed by (i) setting  $c$  to the values in Table 7, (ii) solving (7) for  $x$  and (iii) using  $\omega = \arccos(1 - 2x)$ . When  $c$  is finite, it is seen that  $x$  is a root of the polynomial:

$$\mathcal{T}_N\{(1-x)^L(R(x) + cT(x))\} - 4(1-x)^L(R(x) + cT(x)) = 0. \quad (8)$$

Note that when  $N$  is odd,  $c$  can be chosen to be arbitrarily large. Letting  $c$  approach infinity, one obtains instead of (8), the polynomial

$$\mathcal{T}_N\{(1-x)^L T(x)\} - 4(1-x)^L T(x) = 0. \quad (9)$$

Therefore, for both even and odd  $N$ , the range of achievable half-magnitude frequencies can be found by computing the roots of appropriate polynomials. It was found that each relevant polynomial has exactly one real root in the interval  $(0,1)$ , so there is no ambiguity regarding root selection. A table, similar to Table 5, indicating the number and the location of the real roots of the relevant polynomials is given in [16].

## 4.4 Special Values

For fixed values  $N$  and  $L + M$ , as the specified frequency  $\omega_o$  is varied over  $(0, \pi)$ , the values of  $L$  and  $M$  must be varied according to a table such as Table 2. For the boundary values of  $\omega_o$  (for example  $\omega_o = 0.5349$  when  $L + M = 5$  and  $N = 4$ ), an extra degree of flatness is achieved, when  $N$  is even. For those filters, the rational function  $F(x)$  is given by

$$F(x) = \frac{(1-x)^L S(x)}{\mathcal{T}_N\{(1-x)^L S(x)\}} \quad (10)$$

where  $S(x)$  is given in Table 6. The exact location of the half-magnitude frequency is entirely determined by the parameters  $L$ ,  $M$  and  $N$ . Fixing  $L + M = 22$  and  $N = 4$ , the frequency response magnitudes of the filters obtained using (10) as  $L$  is varied from 5 to 21, are shown in Figure 6.

The FIR solution obtained, when  $N = 0$ , is a special case well established in the literature. When  $N = 0$ , the function (10) specializes to

$$F(x) = (1-x)^L \sum_{k=0}^M \binom{L+k-1}{k} x^k \quad (11)$$

which was given by Herrmann in [3] for the design of symmetric (Type 1) FIR filters. It is worth noting that recently, formulas for all four types of symmetric FIR filters have been given [1].

When  $L = M + N + 1$ , with  $N$  even, the function (10) is useful in the design of IIR orthogonal wavelets with a maximal number of vanishing moments [2, 17]. In this case, the transfer function  $H(z)$  obtained from (10) satisfies  $H(z)H(1/z) + H(-z)H(-1/z) = 1$ , an equation central to the design of orthogonal two-channel filter banks and orthogonal wavelet bases.

## 5 Further Remarks

To summarize, the design procedure described above requires three parameters: the denominator degree ( $N$ ), the numerator degree ( $L + M$ ), and the half-magnitude frequency ( $\omega_o$ ). By making a table such as Table 2, the way to split the number of zeros between  $z = -1$  and the passband ( $L$  and  $M$ ) can be determined. The corresponding formulas can then be used to compute  $F(x)$ . After polynomial root finding and the mapping (1), the filter coefficients can be obtained. To clarify the design process presented in this paper, we list the steps:

1. Specify the numerator and denominator degrees of  $H(z)$ , and the frequency  $\omega_o$ .
2. Construct a specification table, like Table 2, using the equations discussed above.

3. Locate  $\omega_o$  in the specification table. This gives  $L$  and  $M$  individually — thereby indicating how to split the zeros between  $z = -1$  and the passband.
4. Use formulas given above to construct the rational function  $F(x) = P(x)/Q(x)$ .
5. Compute roots of  $P(x)$  and  $Q(x)$ .
6. Map roots to  $z$ -plane via (1).
7. Compute coefficients by forming polynomials from roots.

Using a specification table like Table 2, in conjunction with the formulas, the half-magnitude frequency  $\omega_o$  can be varied continuously in the interval  $(0, \pi)$ . If desired, a frequency other than the half-magnitude frequency can be specified. To specify a frequency  $\omega_o$  for which  $|H(e^{j\omega_o})| = H_o$  is possible for any  $H_o$ ,  $0 < H_o < 1$ . The resulting design formulas differ only in that they contain slightly different constants. Also note that, while the examples illustrate minimum-phase solutions, nonminimum-phase solutions can also be obtained by reflecting ‘passband’ zeros about the unit circle. This is equivalent to using different signs of the square root in (1).

Note that when  $N$  is odd, one of the poles must lie on the real line. When there are zeros that lie off the unit circle, in the passband ( $M > 0$ ), it is expected that the pole lying on the real line does little to contribute to the performance of the frequency response. This is indeed true. In some situations, a pole and a zero will lie close together on the real line and, depending on the specified half-magnitude frequency, almost cancel. For this reason, it is expected that generalized digital Butterworth filters having an odd number of poles and passband zeros will be of little interest — they have been presented in this paper for completeness.

It should be noted that for the classical Butterworth filter, explicit solutions for the locations of the poles are known [9]. For the generalized case, however, it appears that the roots of  $P(x)$  and  $Q(x)$  must be found numerically. It should also be realized, that a filter formed by cascading (*i*) a classical Butterworth digital filter and (*ii*) a maximally flat FIR digital filter, is not optimal in the maximally flat sense in general. To obtain a true maximally flat solution, all the degrees of freedom must be considered together.

It is also worth noting that the classical Butterworth filter can be realized as a parallel sum of two allpass filters [24], a structure that has received much attention recently. The approach taken in this paper did not attempt to preserve this property, however, it is possible to obtain a quite different generalization of the Butterworth filter, by structurally imposing this property [15]. Finally, if phase linearity is important and a maximally flat response is desired, then it is more appropriate to use symmetric FIR filters [1], nearly symmetric FIR filters (with reduced delay) [16, 19], or approximately linear phase IIR filters [15].

## 6 Conclusion

By using appropriate formulas, by computing polynomial roots, and by employing a transformation (1), maximally flat IIR filters having more zeros than poles (away from the origin) can be easily designed — and without the restriction that all zeros lie on the unit circle. The technique presented allows for the continuous variation of the half-magnitude frequency. In addition, for fixed numerator and denominator degrees and a fixed half-magnitude frequency, the formulas above give a direct way of finding the correct way to split the number of zeros between  $z = -1$  and the passband.

The maximally flat FIR filter described by Herrmann [3] and the classical Butterworth filter are special cases of the filters given by the formulas described in this paper. Table 8 gives a summary of the filter design formulas. Table 6 gives auxiliary polynomials. An earlier version of this paper is [18]. A more detailed description is given in [16]. Matlab programs are available on the World Wide Web at URL <http://www.dsp.rice.edu/>.

## A Connection to a Series of Gauss

The polynomials  $R(x)$ ,  $T(x)$ , and  $S(x)$  given in Table 6, are special cases of the Gauss hypergeometric series [7],  $\mathcal{F}(a, b; c; z)$ , given by

$$\mathcal{F}(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \quad (12)$$

where the pochhammer symbol  $(a)_k$  denotes the rising factorial:  $(a)_k = (a) \cdot (a+1) \cdot (a+2) \cdots (a+k-1)$ . When  $a$  or  $b$  is a negative integer,  $\mathcal{F}(a, b; c; z)$  is a polynomial. The polynomials  $R(x)$ ,  $T(x)$ , and  $S(x)$  can be written as

$$S(x) = \frac{(M+1)_N}{N!} \cdot \mathcal{F}(-M, L-N; -M-N; x) \quad (13)$$

$$R(x) = \frac{(M)_N}{N!} \cdot \mathcal{F}(1-M, L-N; 1-M-N; x) \quad (14)$$

$$T(x) = \frac{(M)_{N-1}}{(N-1)!} \cdot x \cdot \mathcal{F}(2-M, L-N-1; 2-M-N; x). \quad (15)$$

There are many recurrence formulas for the hypergeometric series; with them recursion formulas for  $R(x)$ ,  $S(x)$ , and  $T(x)$  can be obtained. Those relationships may also facilitate the computation of the roots of the polynomials, as suggested in [8, 21].

## References

- [1] T. Cooklev and A. Nishihara. Maximally flat FIR filters. In *Proc. IEEE Int. Symp. Circuits and Systems (ISCAS)*, volume 1, pages 96–99, Chicago, May 3-6 1993.

- [2] C. Herley and M. Vetterli. Wavelets and recursive filter banks. *IEEE Trans. on Signal Processing*, 41(8):2536–2556, August 1993.
- [3] O. Herrmann. On the approximation problem in nonrecursive digital filter design. *IEEE Trans. on Circuit Theory*, 18(3):411–413, May 1971. Also in [10].
- [4] L. B. Jackson. An improved Martinez/Parks algorithm for IIR design with unequal numbers of poles and zeros. *IEEE Trans. on Signal Processing*, 42(5):1234–1238, May 1994.
- [5] B. C. Jinaga and S. C. Dutta Roy. Coefficients of maximally flat low and high pass non-recursive digital filters with specified cutoff frequency. *Signal Processing*, 9(2):121–124, September 1985.
- [6] J. F. Kaiser. Design subroutine (MXFLAT) for symmetric FIR low pass digital filters with maximally-flat pass and stop bands. In IEEE ASSP Soc. Digital Signal Processing Committee, editor, *Programs for Digital Signal Processing*, chapter 5.3, pages 5.3–1 – 5.3–6. IEEE Press, 1979.
- [7] F. Oberhettinger. Hypergeometric functions. In M. Abramowitz and I. A. Stegun, editors, *Handbook of Mathematical Functions*, chapter 15. Dover, 1970.
- [8] H. J. Orchard. The roots of maximally flat-delay polynomials. *IEEE Trans. on Circuit Theory*, 12(3):452–454, September 1965.
- [9] T. W. Parks and C. S. Burrus. *Digital Filter Design*. John Wiley and Sons, 1987.
- [10] L. R. Rabiner and C. M. Rader, editors. *Digital Signal Processing*. IEEE Press, 1972.
- [11] J. Riordan. *Combinatorial Identities*. John-Wiley and Sons, 1968.
- [12] T. Saramäki. Design of optimum recursive digital filters with zeros on the unit circle. *IEEE Trans. on Acoust., Speech, Signal Proc.*, 31(2):450–458, April 1983.
- [13] T. Saramäki. Design of digital filters with maximally flat passband and equiripple stopband magnitude. *International Journal of Circuit Theory and Applications*, 13(2):269–286, April 1985.
- [14] T. Saramäki and Y. Neuvo. IIR filters with maximally flat passband and equiripple stopband. In *Europ. Conf. on Circuit Theory and Design*, volume ECCTD 80, Pt. 2, pages 468–473, 1980.
- [15] I. W. Selesnick. Maximally flat lowpass filters realizable as allpass sums. *IEEE Trans. on Circuits and Systems II*, submitted December 1996.

- [16] I. W. Selesnick. *New Techniques for Digital Filter Design*. PhD thesis, Rice University, 1996.
- [17] I. W. Selesnick. Explicit formulas for orthogonal IIR wavelets. *IEEE Trans. on Signal Processing*, submitted January 1997.
- [18] I. W. Selesnick and C. S. Burrus. Generalized digital Butterworth filter design. In *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing (ICASSP)*, volume 3, pages 1367–1370, Atlanta, May 7-10 1996.
- [19] I. W. Selesnick and C. S. Burrus. Nonlinear-phase maximally-flat FIR filter design. In *Proceedings of the Seventh IEEE DSP Workshop*, pages 374–377, Loen, Norway, September 2-4, 1996.
- [20] P. Thajchayapong, M. Puangpool, and S. Banjongjit. Maximally flat FIR filter with prescribed cutoff frequency. *Electron. Lett.*, 16(13):514–515, Jun 19 1980.
- [21] J. P. Thiran. Recursive digital filters with maximally flat group delay. *IEEE Trans. on Circuit Theory*, 18(6):659–664, November 1971.
- [22] R. Unbehauen. On the design of recursive digital low-pass filters with maximally flat pass-band and Chebyshev stop-band attenuation. In *Proc. IEEE Int. Symp. Circuits and Systems (ISCAS)*, pages 528–531, 1981.
- [23] P. P. Vaidyanathan. On maximally-flat linear-phase FIR filters. *IEEE Trans. on Circuits and Systems*, 31(9):830–832, September 1984.
- [24] P. P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall, 1992.

Table 1: Notation.

Parameters	
$L + M$	total number of zeros
$L$	number of zeros at $z = -1$
$M$	number of zeros contributing to the passband
$N$	total number of poles
$\omega_o$	half-magnitude frequency
$x_o$	$\frac{1}{2}(1 - \cos \omega_o)$
Flatness	
$L + M + N$	total degrees of flatness
$M + N$	degree of flatness at $\omega = 0$
$L$	degree of flatness at $\omega = \pi$

Table 2: For the choice  $L$ ,  $M$ , and  $N$  shown in the table, the interval of achievable half-magnitude frequencies  $\omega_o$  is given by  $[\omega_{min}, \omega_{max}]$ .  $L + M$  is the numerator degree (number of zeros) and  $N$  is the denominator degree (number of poles).

$L + M$	$L$	$M$	$N$	$\omega_{min}/\pi$	$\omega_{max}/\pi$
4	4	0	4	0	1
5	5	0	4	0	0.5349
	4	1	4	0.5349	1
6	6	0	4	0	0.4620
	5	1	4	0.4620	0.6017
	4	2	4	0.6017	1
7	7	0	4	0	0.4140
	6	1	4	0.4140	0.5299
	5	2	4	0.5299	0.6446
	4	3	4	0.6446	1

Table 3: For the half-magnitude frequency  $\omega_o = 0.6\pi$  and  $L + M + N = 20$ , the table shows the correct values of  $L$  and  $M$ , and the derivative of the magnitude response at  $\omega_o$ . The filter responses are shown in Figure 5.

$L$	$M$	$N$	$ H(e^{j\omega_o}) '$
8	12	0	-1.4366
8	10	2	-2.5410
8	8	4	-3.1869
8	6	6	-3.6882
9	3	8	-3.8012
10	0	10	-3.9430

Table 4: Permissible ranges for  $c$  for the first generalization.

$N$ even	$c \geq 0$
$N$ odd	$c \geq \binom{L-1}{N}$

Table 5: The number and locations of the real roots of  $\mathcal{T}_N\{(1-x)^L\} + cx^N - 4(1-x)^L$  for  $L > N > 0$ .

	$L$ even	$L$ odd
$N$ even	2 real roots:	1 real root:
$c = 0$	$x_1 \in (0, 1), x_2 > 1$	$x_1 \in (0, 1)$
$N$ odd	2 real roots:	3 real roots:
$c = \binom{L-1}{N}$	$x_1 \in (0, 1), x_2 = 1$	$x_1 \in (0, 1), x_2 = 1, x_3 > 1$

Table 6: Auxiliary polynomials. For negative values of  $n$ , the convention [11],  $\binom{n+k-1}{k} = (-1)^k \binom{-n}{k}$  for  $k \geq 0$  is used. In addition, note that  $\binom{n}{k} = 0$  for  $n \geq 0, k < 0$ ; and that  $\binom{n}{k} = 0$  for  $n \geq 0, k > n$ .

$R(x) = \sum_{k=0}^{M-1} \binom{M+N-k-1}{N} \binom{L-N+k-1}{k} x^k$
$T(x) = x \sum_{k=0}^{M-1} \binom{M+N-k-2}{N-1} \binom{L-N+k}{k} x^k$
$S(x) = \sum_{k=0}^M \binom{M+N-k}{N} \binom{L-N+k-1}{k} x^k$
$(1-x)^L R(x) = \sum_{k=0}^{L+M-1} \binom{M+N-k-1}{M-1} \binom{L+M-1}{k} (-x)^k$
$(1-x)^L T(x) = x \sum_{k=0}^{L+M-1} \binom{M+N-k-2}{M-1} \binom{L+M-1}{k} (-x)^k$
$(1-x)^L S(x) = \sum_{k=0}^{L+M} \binom{M+N-k}{M} \binom{L+M}{k} (-x)^k$

Table 7: Permissible ranges for  $c$  for the second generalization.

$N$ even	$-1 \leq c \leq \frac{L - N}{M + N}$
$N$ odd	$\frac{L - N}{N} \leq c$

Table 8: The expression for  $F(x)$  gives the magnitude squared function in the  $x$  domain in terms of a constant  $c$ . When  $c$  is chosen according to the expression given in the table,  $F(x_o)$  equals  $1/4$ .

$F(x)$	$c$	Case
$\frac{(1-x)^L}{(1-x)^L + cx^N}$	$3 \frac{(1-x_o)^L}{x_o^N}$	$L \leq N, M = 0$
$\frac{(1-x)^L}{\mathcal{T}_N\{(1-x)^L\} + cx^N}$	$\frac{4(1-x_o)^L - \mathcal{T}_N\{(1-x)^L\}(x_o)}{x_o^N}$	$L > N, M = 0$
$\frac{(1-x)^L(R(x) + cT(x))}{\mathcal{T}_N\{(1-x)^L(R(x) + cT(x))\}}$	$\frac{4(1-x_o)^L R(x_o) - \mathcal{T}_N\{(1-x)^L R(x)\}(x_o)}{\mathcal{T}_N\{(1-x)^L T(x)\}(x_o) - 4(1-x_o)^L T(x_o)}$	$L > N, M > 0$
$\frac{(1-x)^L S(x)}{\mathcal{T}_N\{(1-x)^L S(x)\}}$	no variation of $\omega_o$	$L > N, M > 0, N$ even

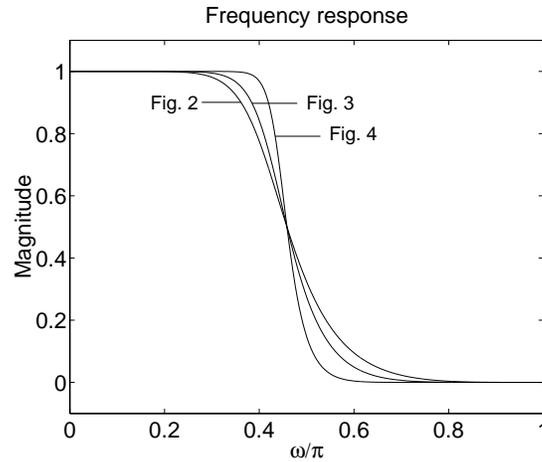


Figure 1: Magnitudes of the three digital IIR filters shown in Figures 2, 3 and 4.

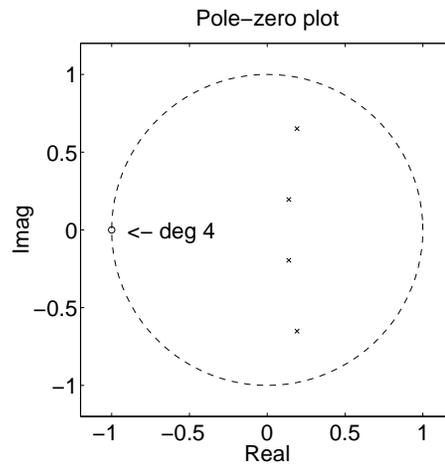


Figure 2:  $L = 4, M = 0, N = 4$ .

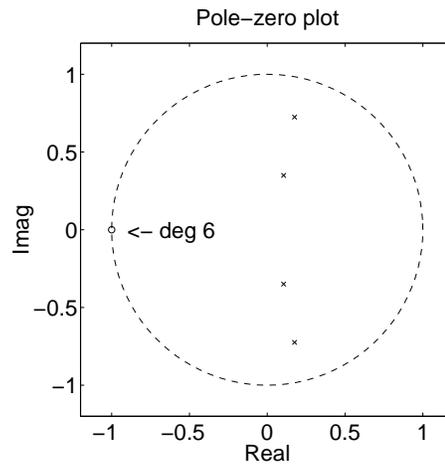


Figure 3:  $L = 6, M = 0, N = 4$ . The poles at the origin are not shown in the figure.

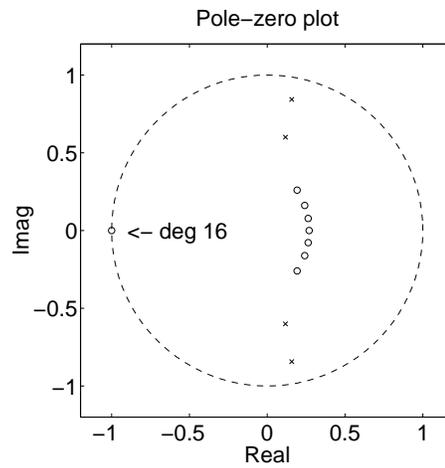


Figure 4:  $L = 16, M = 7, N = 4$ . The poles at the origin are not shown in the figure.

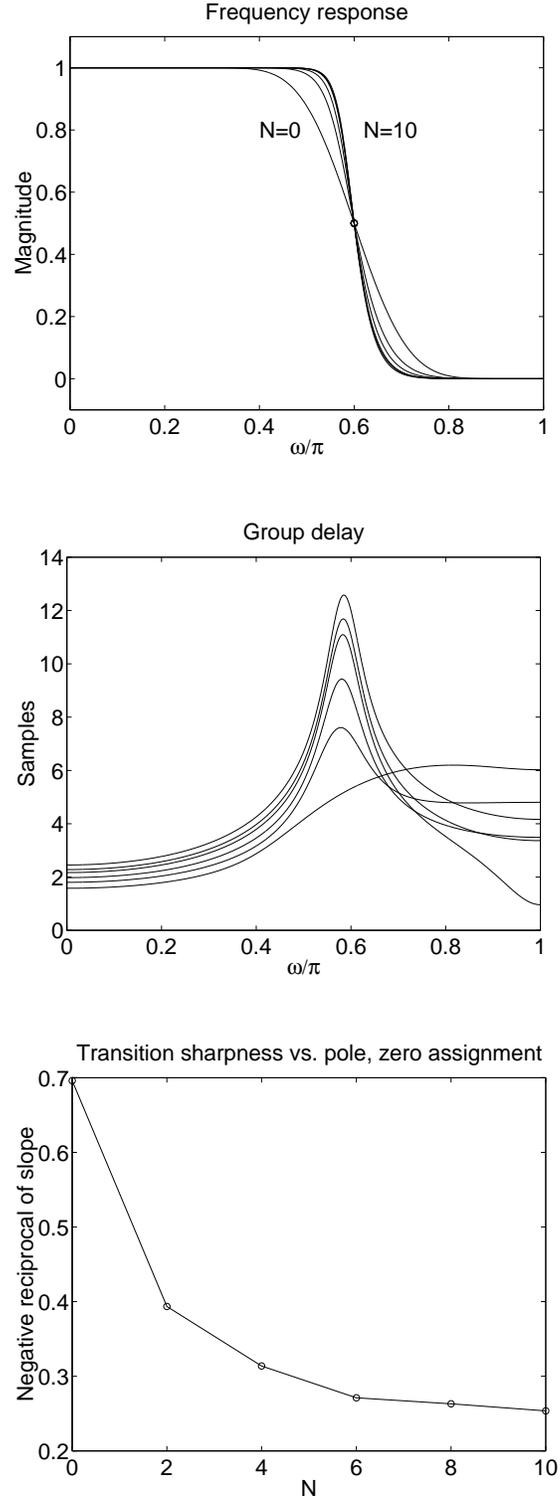


Figure 5: Generalized Butterworth filters.  $L + M + N = 20$ ,  $\omega_o = 0.6\pi$ .  $N$  is varied from 0 to 10 in increments of 2.  $N = 10$  corresponds to the filter having the steepest transition and the most peaked group delay. The values of  $L$ ,  $M$ , and  $N$  are shown in Table 3.

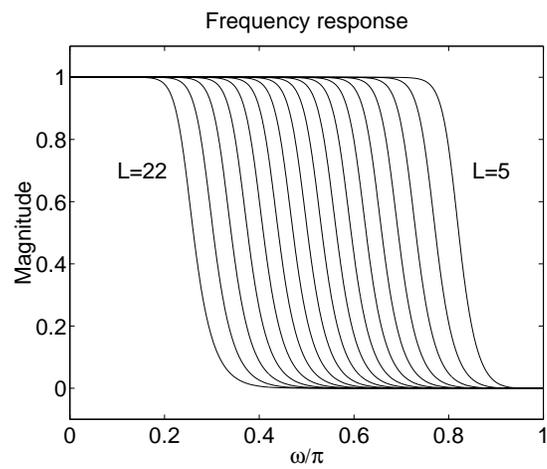


Figure 6: Generalized Butterworth filters for special values of  $\omega_0$ .  $L + M = 22$ ,  $N = 4$ .  $L$  is varied from 5 to 21. The widest band filter corresponds to  $L = 5$ .