ECE 517 - Lecture 4

Note Title 1/26/2017

DTFT

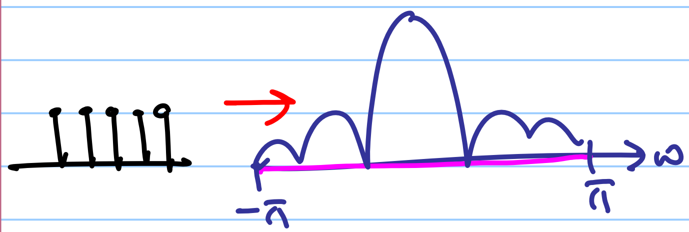
$x(t) \longrightarrow X(e^{j\omega})$ $\xrightarrow{FT} X(j\omega)$

$|z|=1$

$X(z)$

$e^{j(\omega+2\pi)} = e^{j\omega}e^{j2\pi}$

$= e^{j\omega}$

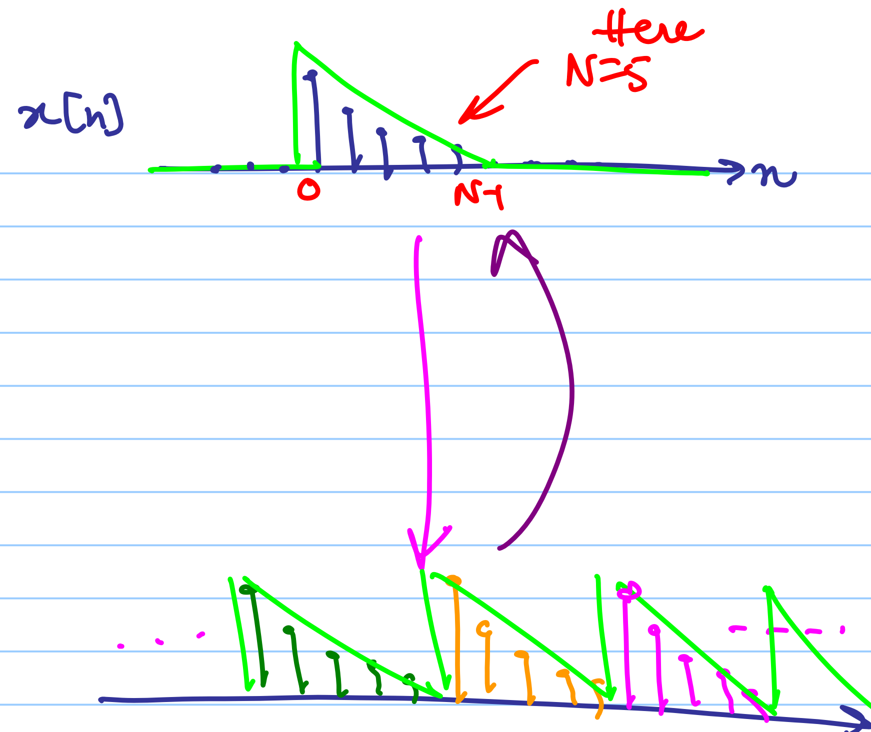


$\omega \in [-\pi, \pi]$

FFT (DFT)

↳ frequency is also discretized

finite length sequence



periodic sequence

Create a periodic sequence

$$\boxed{\tilde{x}[n]} = \sum_{r=-\infty}^{\infty} x[n - rN]$$

$$= x[n \text{ modulo } N]$$

$$= x[(n)_N]$$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$\tilde{x}[n] \rightarrow$ periodic with period 'N'

$\hookrightarrow$ can represent as a summation of "complex exponentials" with fundamental frequency $\omega_0 = \frac{2\pi}{N}$

$\omega_0, 2\omega_0, 3\omega_0, \ldots \infty$

* periodic complex exponentials

$$e_k[n] = \underline{e}^{j\left(\frac{2\pi}{N}\right)kn} = e_k[n + rN]$$

\# harmonic (k)   time (n)

$$e_k[n+rN] = e^{j\frac{2\pi}{N}(n+rN)k} = e^{\frac{j2\pi kn}{N}} \cdot e^{j2\pi rk}$$

$$= e^{\frac{j2\pi kn}{N}} \cdot \cancel{\left(e^{j2\pi}\right)^{rk}}^{\,1} = e_k[n]$$

$$\left\{\begin{aligned} \tilde{x}[n] &= \frac{1}{N}\sum_{k=-\infty}^{\infty} \tilde{X}[k]\, \underbrace{e^{j\left(\frac{2\pi}{N}\right)kn}}_{e_k[n]} \\ \tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \end{aligned}\right.$$

← Discrete Fourier Series (DFS)

think

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi f_0 n t}$$

$k = -\infty$ to ∞

$$e_k[n]$$

$$e_k[n+N] = e_k[n]$$

"frequency domain"

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=-\infty}^{\infty} \tilde{X}[k]\, \underbrace{e_k[n]}_{\text{periodic with } N} \triangleq \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e_k[n]$$

$k = 0$ to $N-1$

$x[n]$ $\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k] \rightarrow X[k]$

$n \in [0, N-1]$

DFT

$k = 0, 1, \ldots (N-1)$

only N frequency components!

Define $X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$

→ N-point sequence

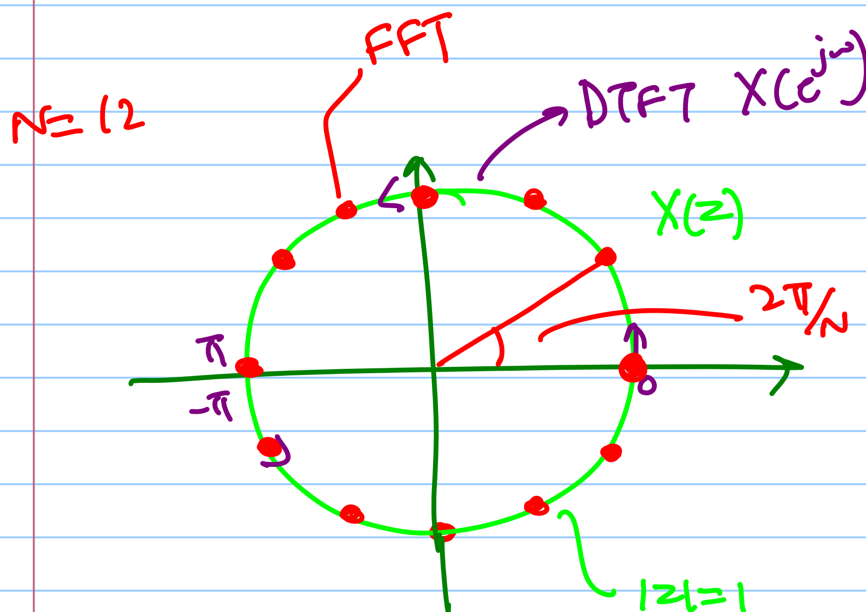
→ Discrete Fourier Transform (DFT)

↳ "fast-algorithm" for computing DFT

↳ "FFT"

Fast Fourier Transform

Matlab fft()



DFT is DTFT sampled in the frequency domain as $\omega = \frac{2\pi}{N} \cdot k$

FFT

$$W_N = e^{-j\frac{2\pi}{N}}$$

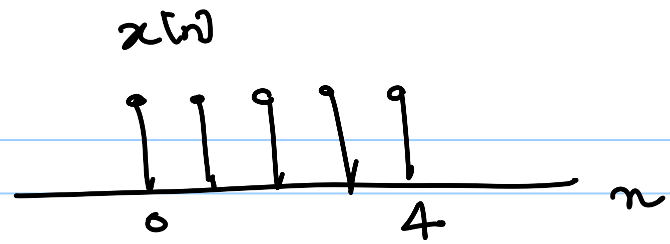
FFT coefficients

$$\underline{\underline{X[k]}} = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$ ← FFT → Analysis Eq^n

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \underline{\underline{X[k]}}\, W_N^{-nk}$$ ← IFFT → Synthesis Eq^n

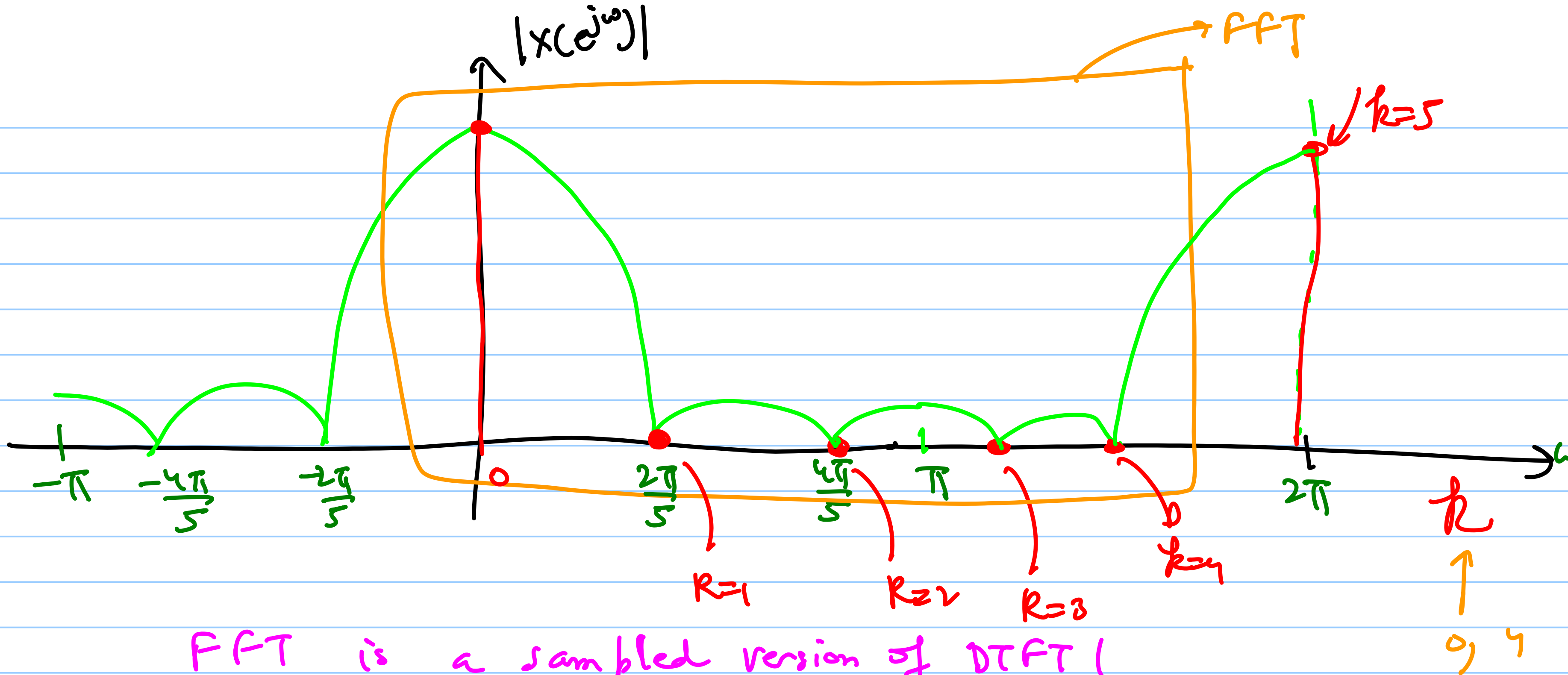
inverse FFT

Example:



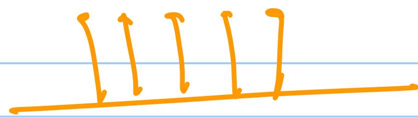
N=5

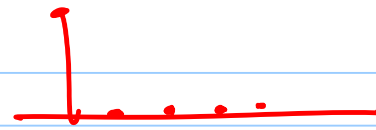
DFT:
$$X[k] = \left| e^{-j\left(\frac{4\pi k}{10}\right)} \right| \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$



FFT is a sampled version of DTFT!

↳ $X(e^{j\omega})$ is sampled at $\omega = \frac{2\pi}{N} k$

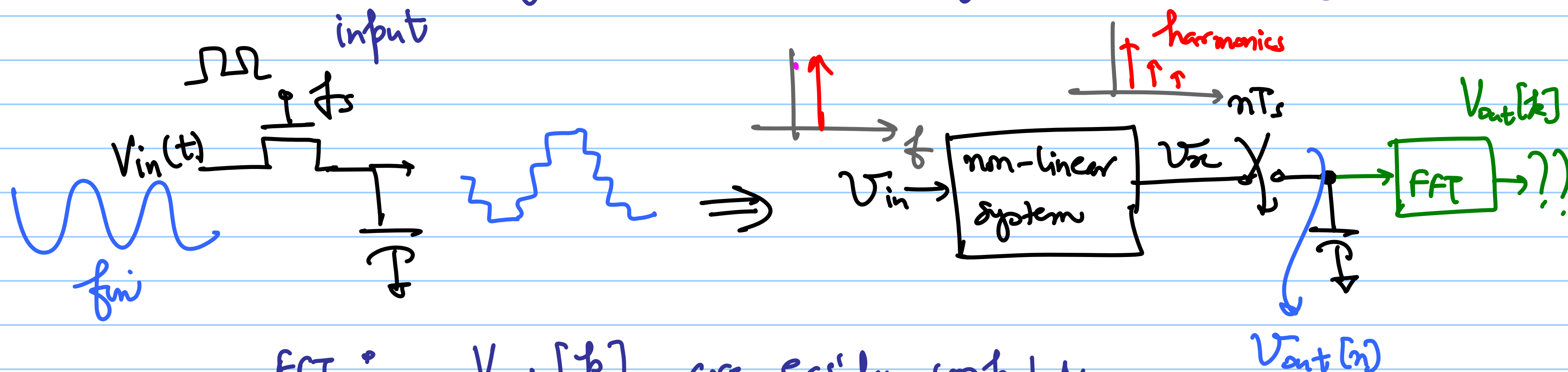




5-point FFT

Spectral Estimation using FFT

Ex. Characterizing the distortion of a S/H using a single-tone input



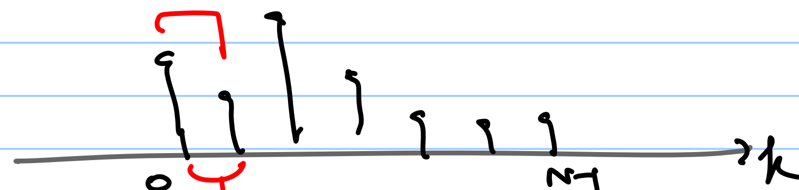
FFT: $V_{out}[k]$ are easily computed using FFT frequencies $\{\overset{0}{0}, \frac{2\pi}{N}, \frac{2\pi}{N}\cdot 2 \; \ldots\ldots \; \overset{f_s}{\frac{2\pi}{N}(N-1)}\}$

$k = 0 \quad 1 \quad 2 \quad \ldots \quad N-1$

$N \rightarrow$ sets the resolution of FFT

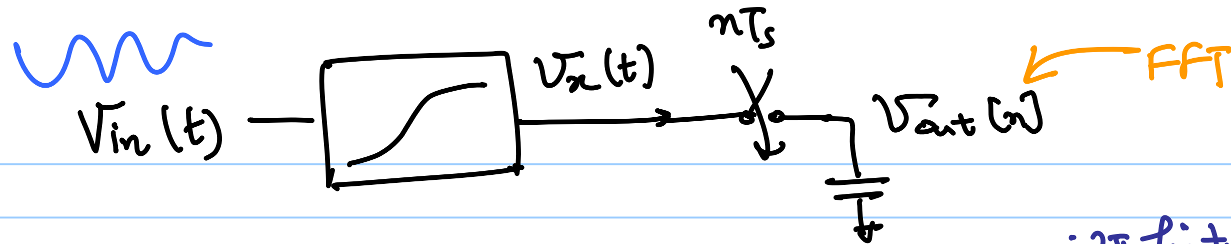
Sample rate $\rightarrow \frac{f_s}{N} \leftarrow$ resolution of the FFT

f_s/N



0 $\quad\quad$ f_s

$\rightarrow$ spacing between the tones $\Rightarrow$ resolution or "bin size"



$$v_{in}(t) = A\sin(2\pi f_{in} t) = \mathrm{Im}\left[A e^{j2\pi f_{in} t}\right]$$

1 tone at f_{in}

$$v_x = \sum_k a_k e^{j2\pi k f_{in} t} \qquad t = nT_s = \frac{n}{f_s}$$

$$v_{out} = \sum_k a_k e^{j2\pi k f_{in}\left(\frac{n}{f_s}\right)}$$

should be able to create $\tilde{V}_{out}[n]$ and it should be periodic

$\tilde{V}_{out}[n]$ is periodic only when

$$e^{j2\pi k \frac{f_{in}(n+N)}{f_s}} = e^{j2\pi k \frac{f_{in} n}{f_s}}$$

$$\Rightarrow \quad 2\pi \frac{f_{in}}{f_s} \cdot N = 2m\pi$$

$$\Rightarrow \quad \boxed{\frac{f_{in}}{f_s} = \frac{m}{N}} \qquad m, N \in I$$

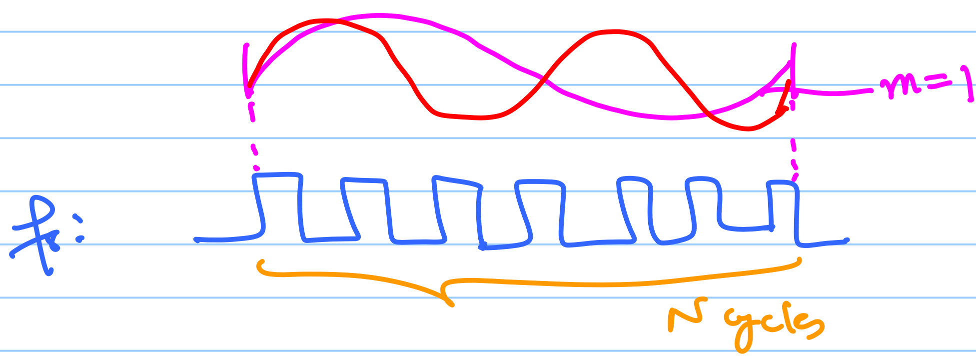
only if this condition is satisfied

$\hookrightarrow$ $\tilde{V}_{out}[n]$ a periodic sequence

DFS is valid

$$N \cdot f_{in} = m f_s$$

m cycles of f_s should fit within N cycles of f_{in}



$$\boxed{f_{in} = \frac{m}{N} f_s}$$