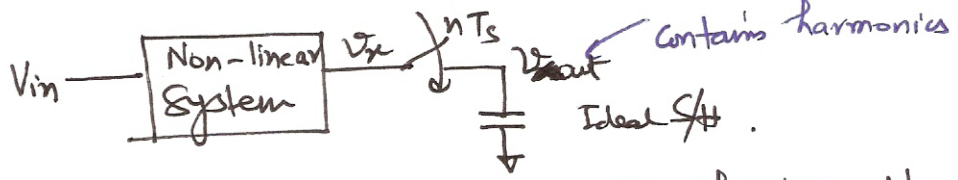
Signal estimation using DSP / Matlab:

Example: characterizing the distortion of a S/H using a single tone input.



Use a discrete time periodic sequence with period N.

$v[n] = v[n+N]$.

Then $v[n]$ can be repsented as DFS.

$$v[n] = \sum_{k=0}^{N-1} V[k]\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$V[k]$ → complex DFS coefficients

$$V[k] = |V[k]| \cdot e^{j \angle V[k]} = B_m\, e^{j\phi_m}$$

$V[k]$ are easily computed using FFT.

- $N \rightarrow$ ~~record length~~ FFT size.

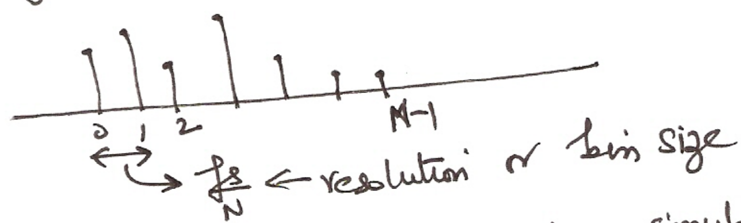
frequencies:

$$\left\{0, \frac{2\pi}{N}, \frac{2\pi}{N}\cdot 2, \ldots, \left(\frac{2\pi}{N}\right)(N-1)\right\}.$$

$\frac{N}{f_s} \rightarrow$ time in continuous-time axis

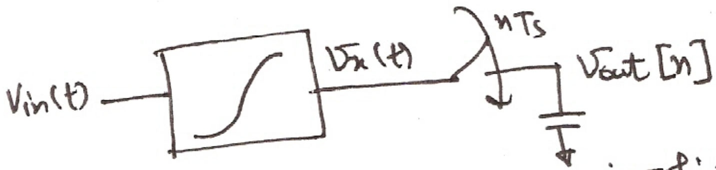
$\frac{f_s}{N} \rightarrow$ resolution of the FFT.

$\quad \rightarrow$ spacing between the tones.



P.S.

- $M \rightarrow$ record length = size of the data collected from simulation or measurement
- If the FFT is taken over the whole record length then FFT size = M. i.e N=M.



$$V_{in} = A \sin(2\pi f_{in} t) = \mathrm{Im}\{A e^{+j2\pi f_{in} t}\}$$

$$V_x = \sum_k a_k e^{j2\pi k f_{in} t}$$

$a_1 \rightarrow$ fundamental
$a_{2,3} \ldots$ harmonics

$$\Rightarrow V_{out}[n] = V_{out}\left(\frac{n}{f_s}\right) = \sum_k a_k e^{j2\pi k f_{in} \cdot \frac{n}{f_s}}$$

$V_{out}[n]$ is periodic only when

$$e^{j2\pi k f_{in} \frac{(n+N)}{f_s}} = e^{j2\pi k \frac{f_{in}}{f_s} n}$$

$$\Rightarrow 2\pi \frac{f_{in}}{f_s} \cdot N = 2m\pi, \quad m \in I$$

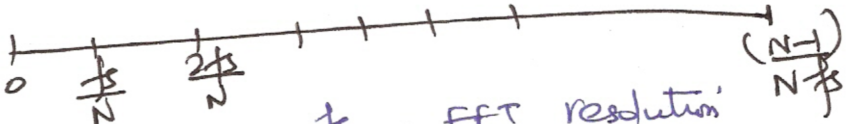
$$\Rightarrow \boxed{\frac{f_{in}}{f_s} = \frac{m}{N}}$$

← only then $V_{out}[n]$ is a periodic sequence and DFS is valid. → (1)

In time-domain

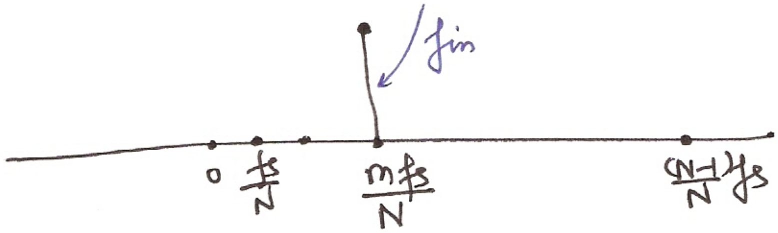
$$\frac{m}{f_{in}} = \frac{N}{f_s} \Rightarrow \text{'m' cycles of } f_{in} = \text{'N' cycles of } f_s$$

If (1) is satisfied Vout[n] can be expressed as discrete-~~time~~ Fourier Series.

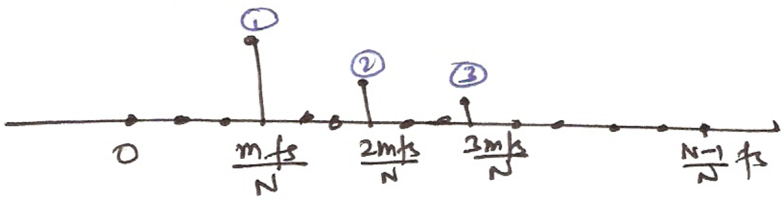


- $\frac{f_s}{N}$ = FFT resolution
- Each tone is called a "bin"
- 'N' bins for an N-point FFT.

for $f_{in} = \frac{m}{N} f_s$



with distortion → components at multiples of $\frac{m f_s}{N}$



- choose f_{in} and f_s carefully else the sampled sequence will not be periodic and the FFT will not show the correct results.

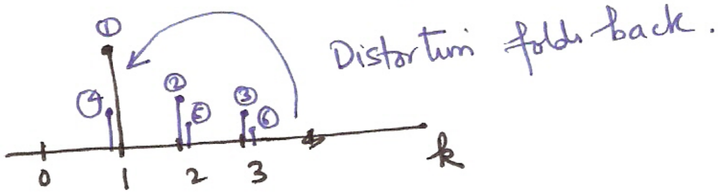
Example: select $f_{in} = \frac{m}{N} f_s = \frac{f_s}{4}$.

harmonic components → $0, \frac{f_s}{4}, \frac{2 f_s}{4}, \frac{3 f_s}{4}, f_s, \ldots$

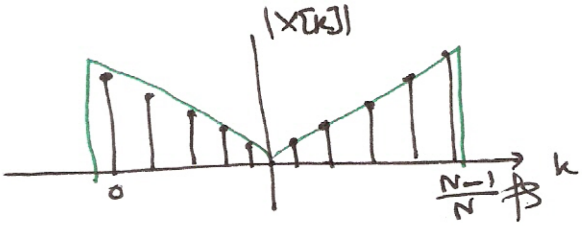
record length $= M = 4$.

for $\frac{f_s}{4} = \frac{2 f_s}{8}$, is record length equal to 8 ??

↳ make sure that m & $N (= M)$ are relatively prime.



DFT (or FFT)



conjugate symmetry

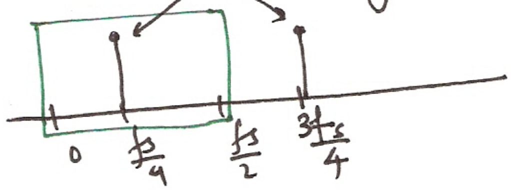
$$X^*[k] = X[N-k]$$

$$\Rightarrow |X[k]| = |X[N-k]|$$

and $\angle X[k] = -\angle X[N-k]$

for $f_{in} = f_s/4$

not distinguishable



↳ will not yield correct distortion analysis.

N=4.

$\Rightarrow \frac{f_s}{4}, \frac{3f_s}{4}, \ldots (2m+1)\frac{f_s}{4}$ are indistinguishable from each other

↳ information is contained in a single bin

$\Rightarrow$ Not just enough that if $\frac{f_{in}}{f_s} = \frac{m}{N}$, necessary but not sufficient

$\Rightarrow$ Use $f_{in} = \frac{m}{N} f_s$, m is prime wrt N (and N is large compared to m), so that the harmonics don't alias back to the fundamental

Also choose $N = 2^p$ for FFT computation

Eg. $f_{in} = \frac{m}{2^{10}} f_s = \frac{m}{1024} f_s$.

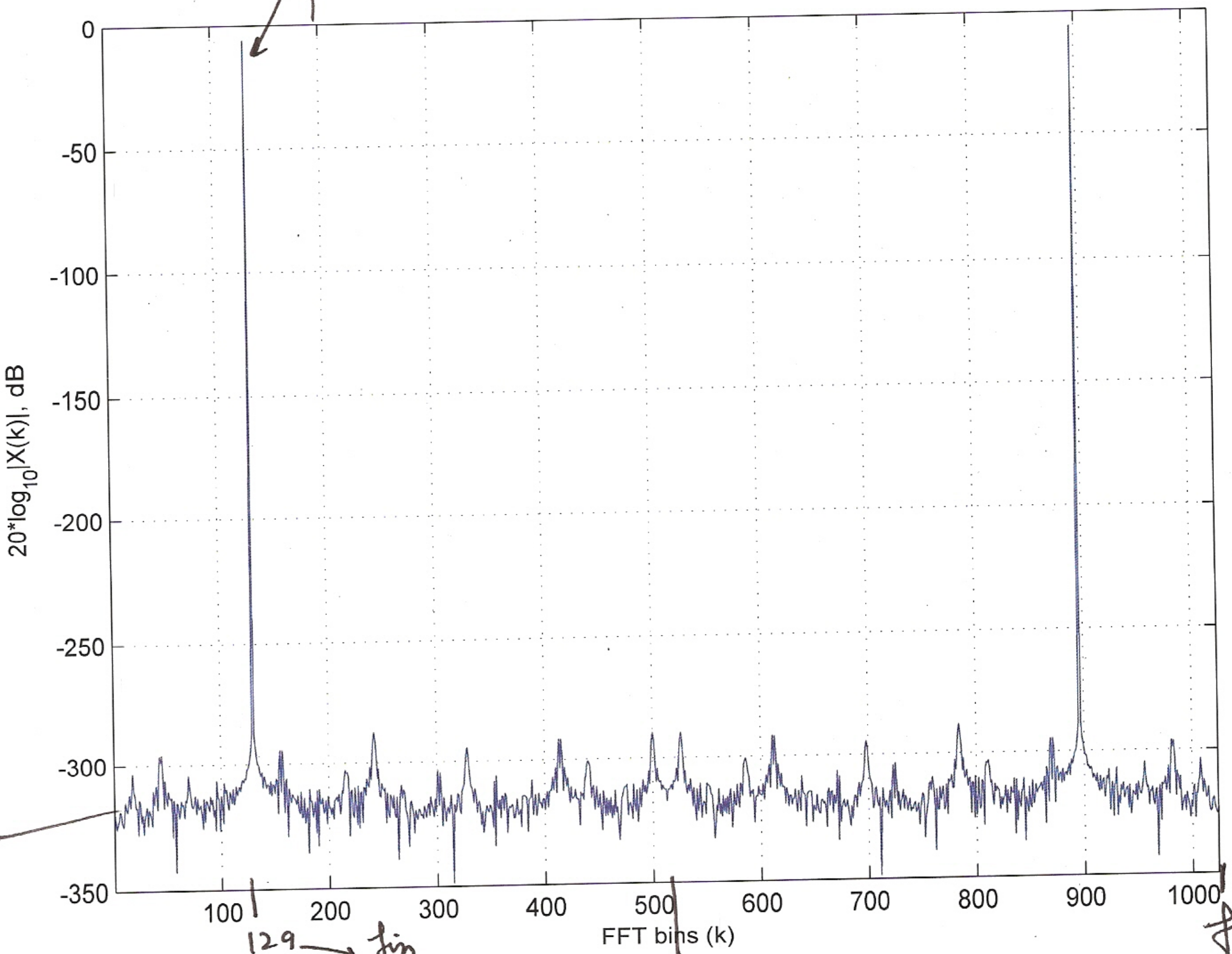
See "FFTDemo1.m"

Here $f_{in} = \frac{129}{1024} f_s$.

"FFT Demo1.m"

- Absolute value of the FFT doesn't matter due to the normalizing factors.

fundamental tone



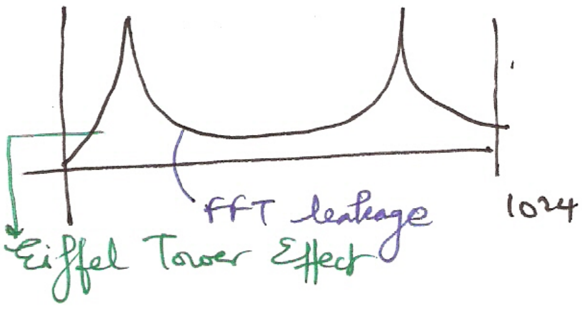
129 → fin

$\frac{fs}{2}$

fs
1024

Noise floor due to the limited precision of the computer.

- What happens when $f_{in} = \frac{129.01}{1024} f_s$.



- Bins which are supposed to be zero are now filled up, and are only 80/90 dB lower,

$\Rightarrow$ Signal is not periodic

$\Rightarrow \frac{m}{f_{in}} \neq \frac{N}{f_s}$

(FFT Recap)

Consider a finite length sequence $x[n]$ with length N.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n} \longrightarrow \text{DTFT}$$

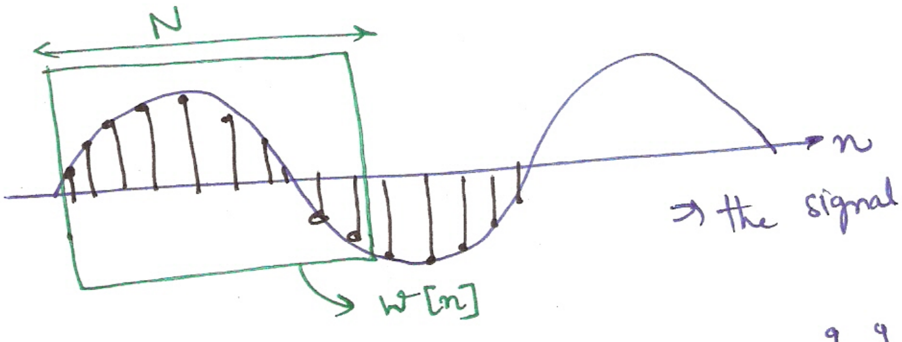
$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN].$$

$\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k] \leftarrow$ periodic

$\Rightarrow x[n] \xleftrightarrow{DFT} X[K] \leftarrow$ fixed length, N'

DFT is samped DTFT with $\omega = \frac{2\pi k}{N}$.

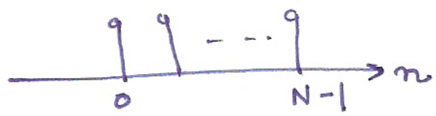
But $x[n] = \sin(2\pi \frac{f_{in}}{f_s} n)$ is a periodic signal of infinite length and we restrict it to a length N.



$\Rightarrow$ the signal $x[n]$ is windowed.

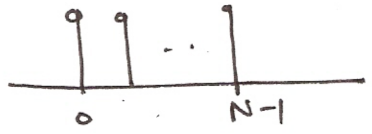
$\Rightarrow \quad x[n] \cdot W[n]$

rectangular window

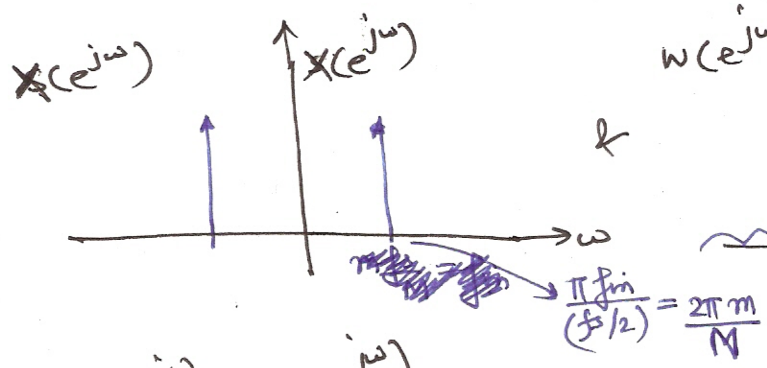


$$p[n] = x[n]\cdot w[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \circledast W(e^{j\omega})$$

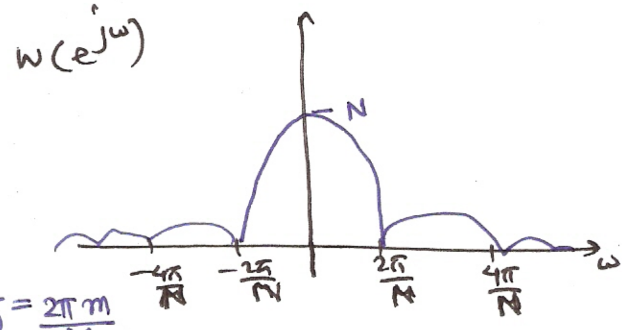
avg delay of $\frac{(N-1)}{2}$ samples



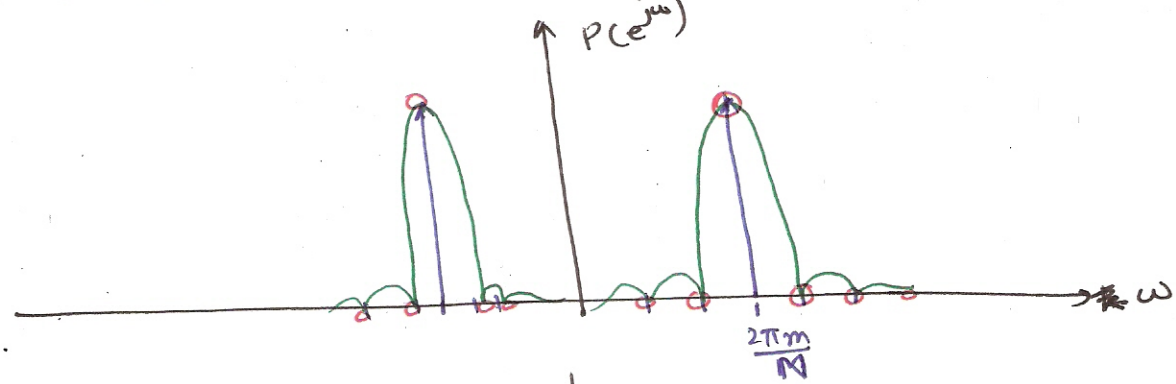
$$\Rightarrow e^{-j\omega\frac{(N-1)}{2}} \cdot \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

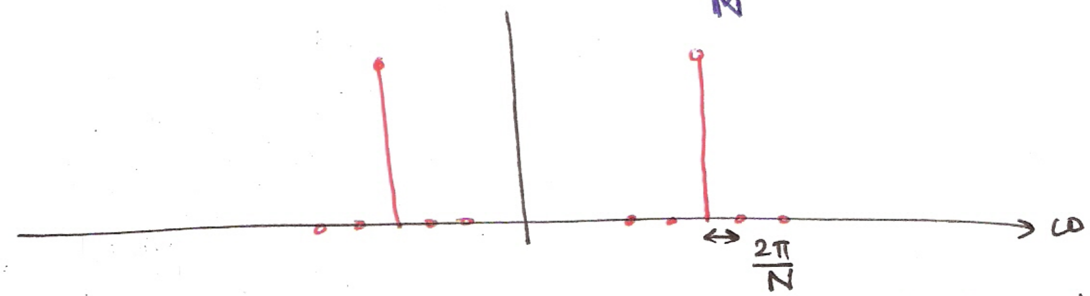


&



$$P(e^{j\omega}) = X(e^{j\omega}) \circledast W(e^{j\omega})$$

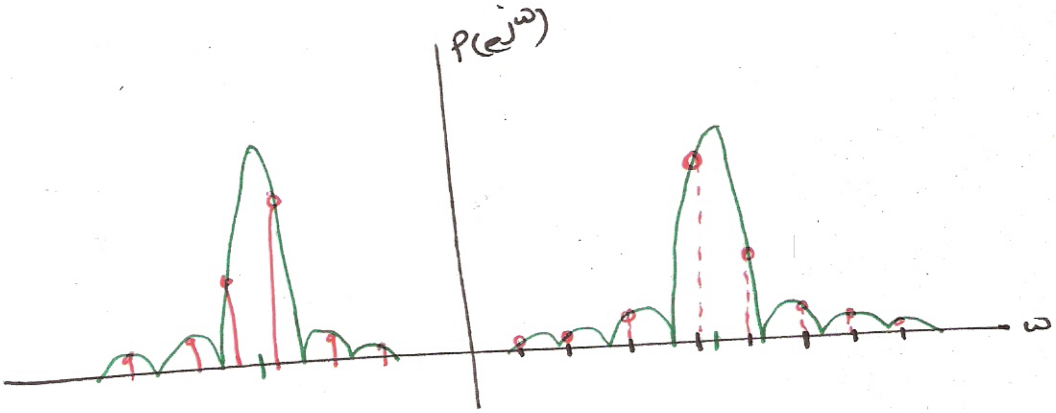


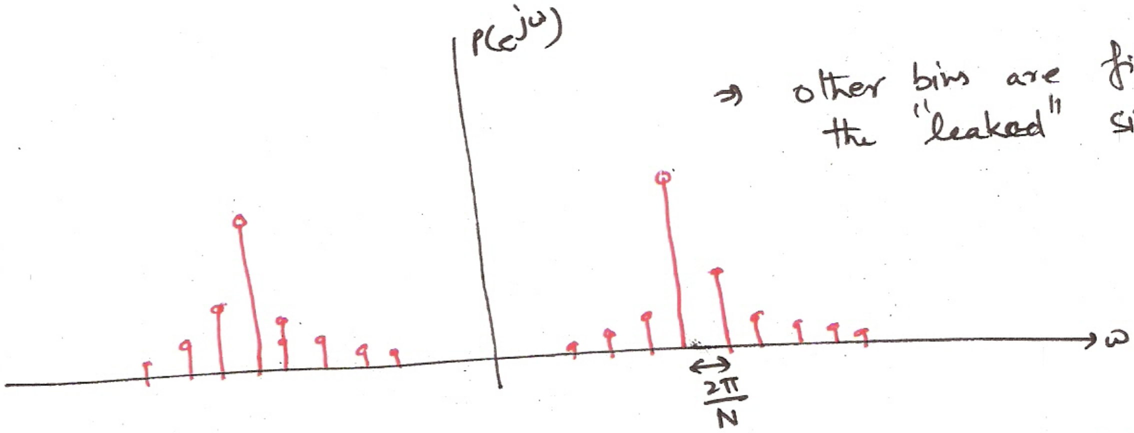


$\Rightarrow$ If $f_{in} = \frac{m}{N} f_s$, i.e. the input is rationally related to the clock frequency, we get a single peak and other bins are zero.

$\Rightarrow$ DFT computes DFS properly.

What if $\frac{f_{in}}{f_s} \neq \frac{m}{M}$





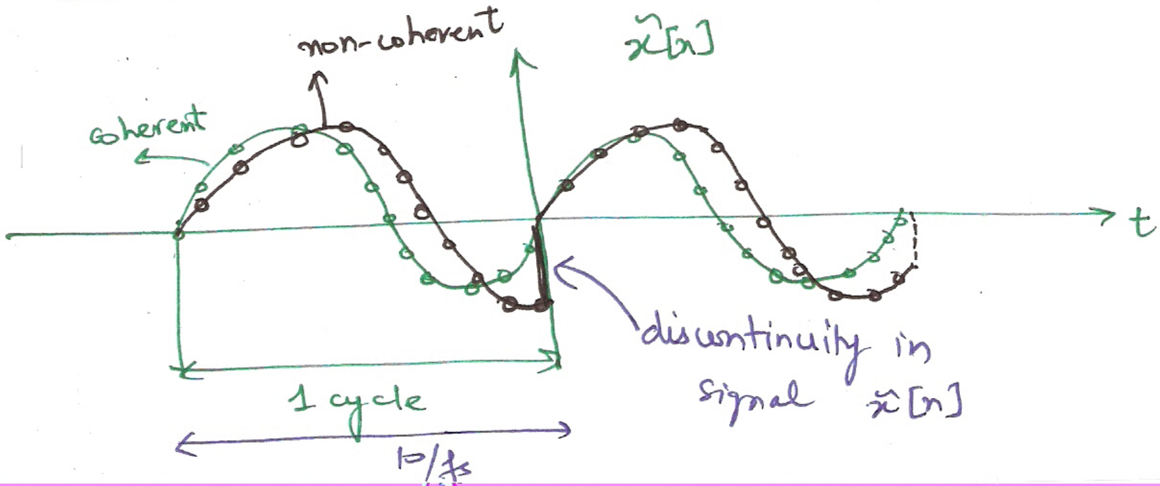
⇒ other bins are filled up with the "leaked" sine wave.

The closer $\frac{f_{in}}{f_s} \to \frac{m}{N}$, the Eiffel tower becomes taller.

[see Matlab code FFTdemo2].

- Synchronous sampling
  or Coherent sampling.

<u>Time domain understanding</u>.

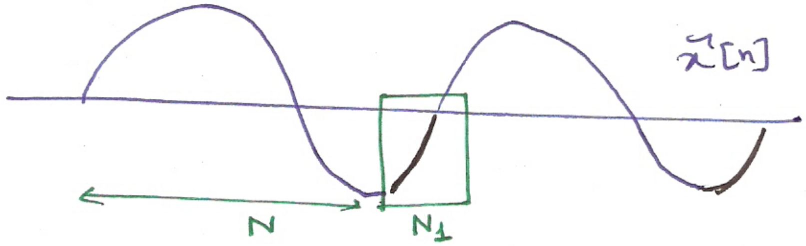


Temporal discontinuity in $\tilde{x}[n]$ while taking DFS causes FFT leakage

$\because$ discontinuity $\rightarrow$ step function $\rightarrow$ has all frequency components.

How to subdue the effect of the discontinuity in ~~inexact~~ non-coherent sampling?

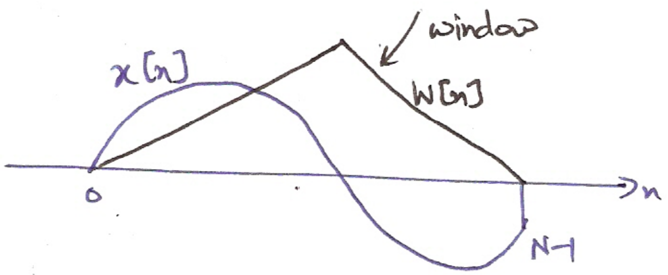
① Signal reconstruction



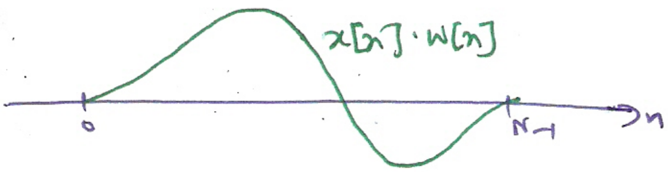
Adjust 'N' such that it has full cycles of $x[n]$

↳ impractical to implement.

② Attenuate the discontinuity.



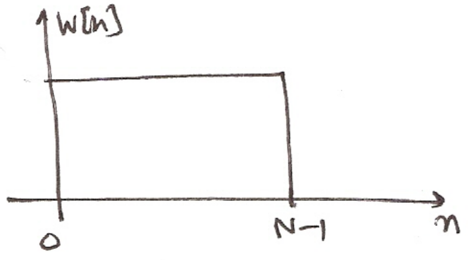
Use custom window instead of rectangular window.



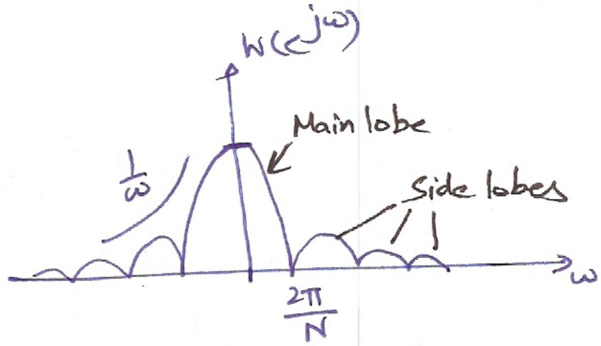
↳ give lesser emphasis on the ends and more importance to the signal in the middle.

↳ A large number of windows are reported in literature and are available in Matlab.

Rectangular window

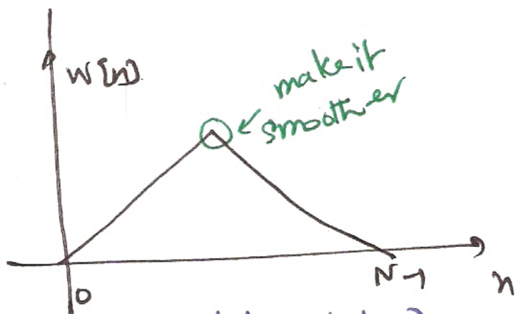


$$\propto \frac{\sin\left(\frac{N\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$



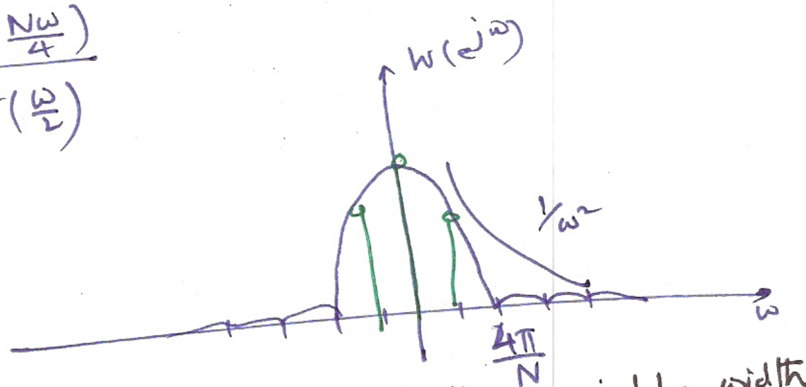
no. of signal bins, $nb = 1$

Triangular window



(Bartlet window)

$$\propto \frac{\sin^2\left(\frac{N\omega}{4}\right)}{\sin^2\left(\frac{\omega}{2}\right)}$$



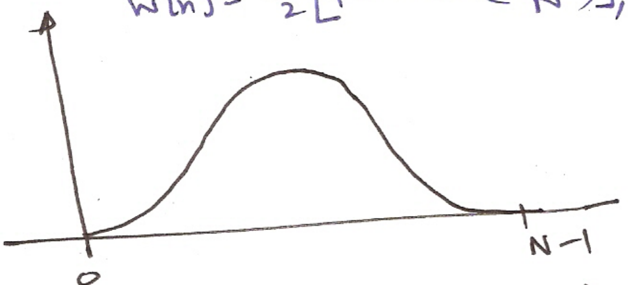
Double the main-lobe width

larger side-lobe suppression

number of signal bins, $= nb = 3$

Raised-cosine window (Hann, Hanning.)

$$w[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N}\right)\right], \quad 0 \le n \le N-1$$



for $x[n] = A \sin\left(\frac{2\pi}{1024} \cdot 129n\right)$.

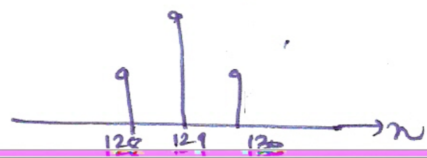
$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$p[n] = A\sin\left(\frac{2\pi}{1024} \cdot 129n\right) \cdot \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{1024}\right)\right].$$

$$= A\sin\left(\frac{2\pi}{1024} \cdot 129n\right) - \frac{A}{4}\sin\left(\frac{2\pi}{1024} \cdot 128n\right) - \frac{A}{4}\sin\left(\frac{2\pi}{1024} \cdot 130n\right)$$

Signal bins $\Rightarrow$ 128, 129, 130

$\Rightarrow nb = 3$



Using Hann window,

- If the sampling is not coherent, the energy of the tone disperses into the side bins
- But with Hann window, the FFT leakage is a lot smaller than ~~as~~ with the rectangular window.

$\boxed{\text{check with Matlab}}$:

for $f_{in} = \frac{129.01}{1024} f_s$

⇒ Also the harmonics don't leak and smear into each other as much as with rect window.

other windows

Blackmann-Harris Window :

$$W[n] = a_0 + a_1 \cos\left(\frac{2\pi}{N} n\right) + a_2 \cos\left(\frac{2\pi}{N} 2n\right) + a_3 \cos\left(\frac{2\pi}{N} 3n\right).$$

$-\frac{N}{2} \leq n \leq \frac{N}{2}$,

length $L = N+1$

$a_0 = 0.35875$

$a_1 = 0.48829$

$a_2 = 0.14128$

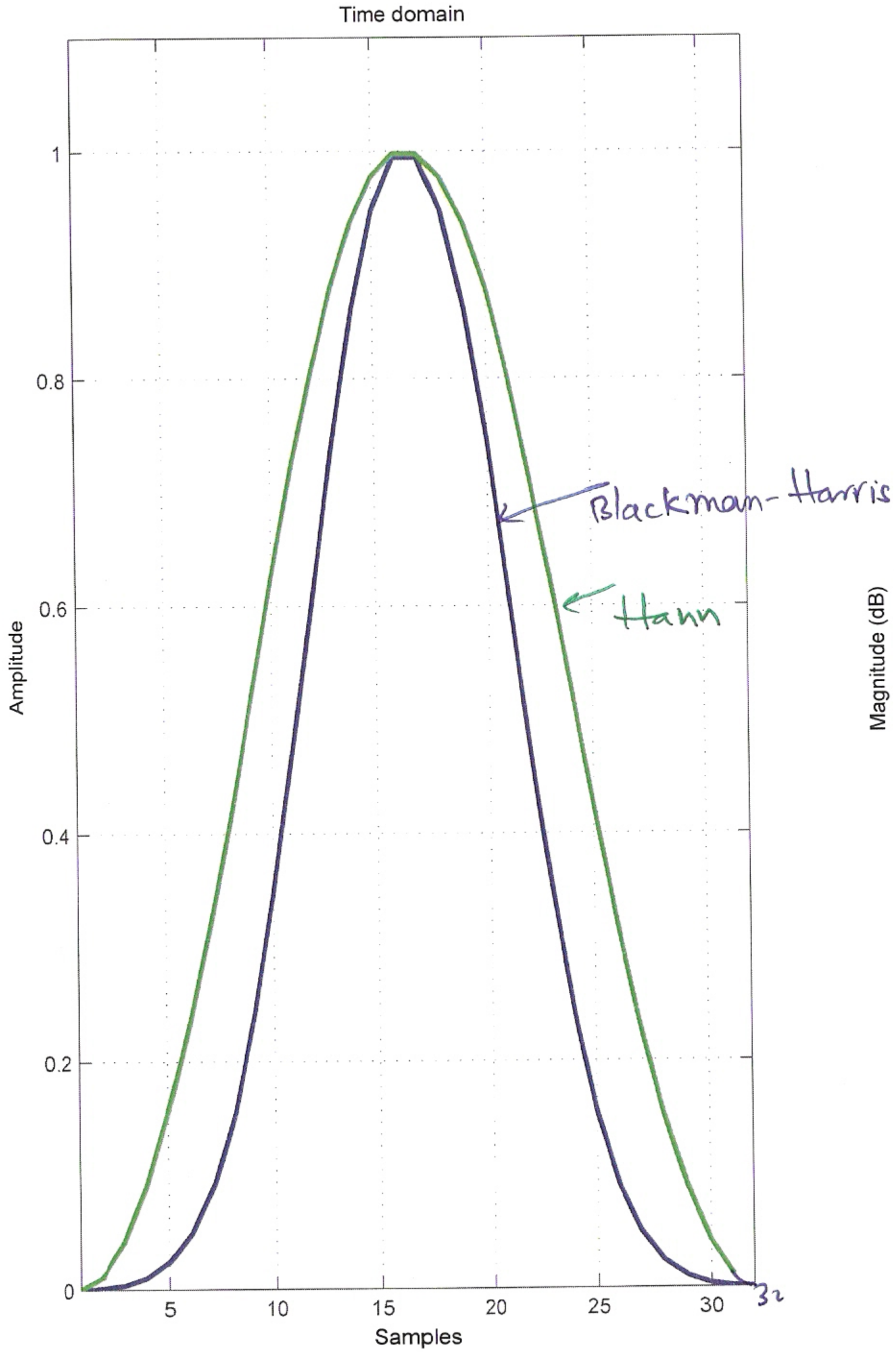
$a_3 = 0.01168$

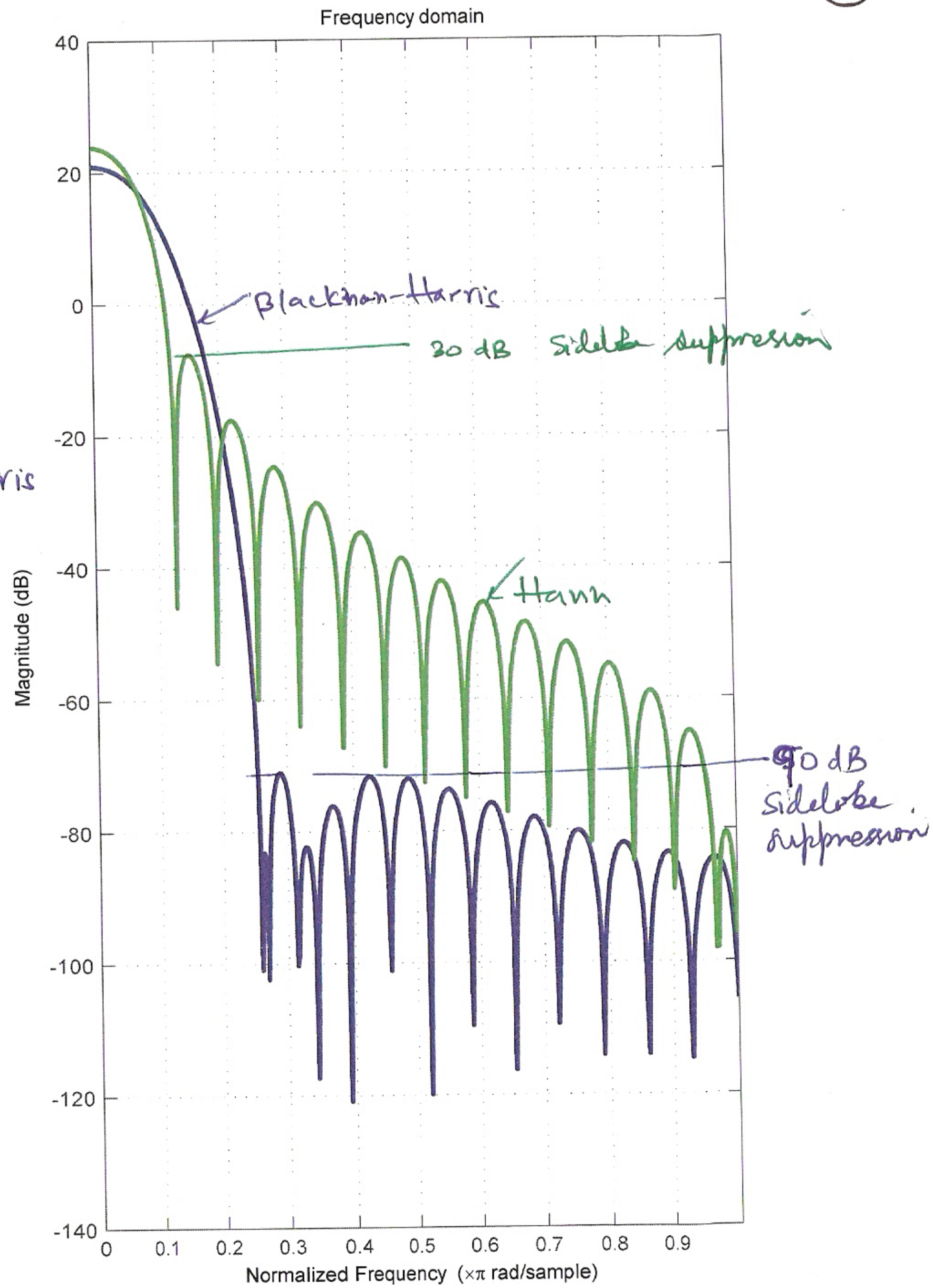
Look up in Matlab

⇒ Maximum Sidelobe suppression (but main lobe width is ~~much~~ larger)

Matlab : L=32;

wvtool(blackmannharris(L)).


Time domain
Amplitude
Samples
0
0.2
0.4
0.6
0.8
1
5
10
15
20
25
30
32
Blackman-Harris
Hann


Frequency domain
Magnitude (dB)
Normalized Frequency (×π rad/sample)
40
20
0
-20
-40
-60
-80
-100
-120
-140
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Blackman-Harris
30 dB Sidelobe suppression
Hann
90 dB Sidelobe suppression