Spectral Estimation: Basics

Variants of Fourier transform:

$$x(t) \xrightarrow{nT_s} x(nT_s) \sim x[n]$$

frequency \ time	Continuous	Discrete
Continuous	Fourier Transform (FT) $X(f)=\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$	$\times$
Discrete	DTFT $X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$	DFT (FFT) $X(k)=\sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

$$W_N = e^{-j\frac{2\pi}{N}}$$

DTFT (Discrete Time Fourier Transform)

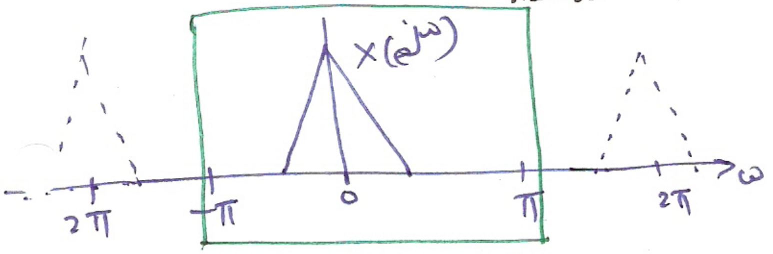
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n},$$

$x[n]$ is absolutely summable or square summable

$$\Rightarrow \sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{or}$$

$$\sum_{n=-\infty}^{\infty} (x[n])^2 < \infty.$$

$\Rightarrow x[n]$ can not be periodic

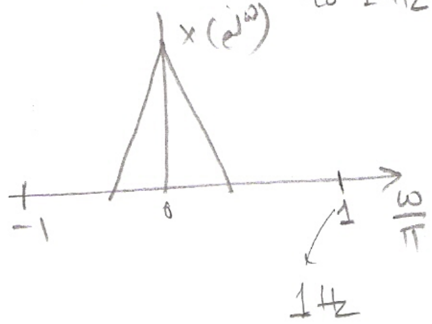


$X(e^{j\omega})$ is periodic with period 2π

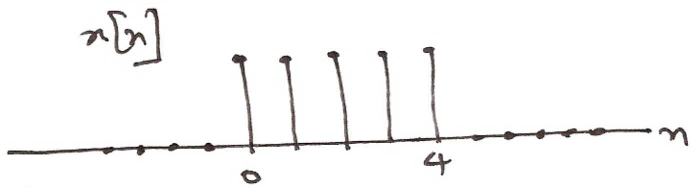
$*$ 2π corresponds to $\omega_s = 2\pi f_s$

$\Rightarrow$ similar to the sampled spectrum

$*$ Sometimes ω_s is normalized to 1 Hz



Example:



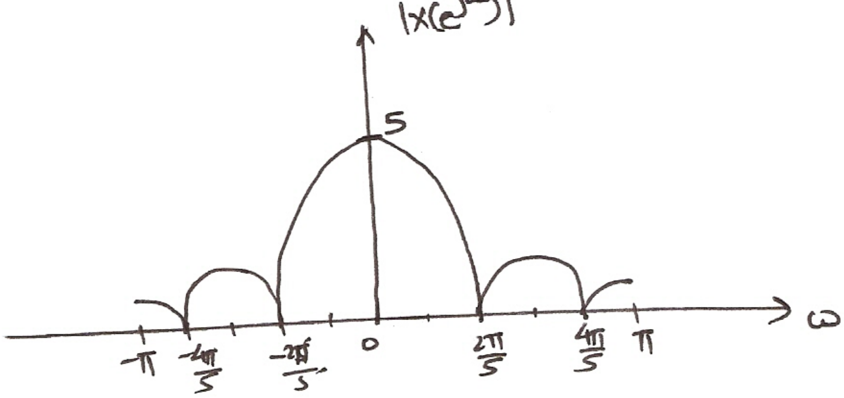
$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n}$$

$$= \frac{e^{-j\frac{5}{2}\omega}\left(e^{j\frac{5}{2}\omega} - e^{-j\frac{5}{2}\omega}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$$= e^{-j2\omega} \cdot \frac{\sin\left(\frac{5\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

average delay = 2

$$\lim_{x\to 0} \frac{\sin\left(\frac{Mx}{2}\right)}{\sin\left(\frac{x}{2}\right)} = M$$

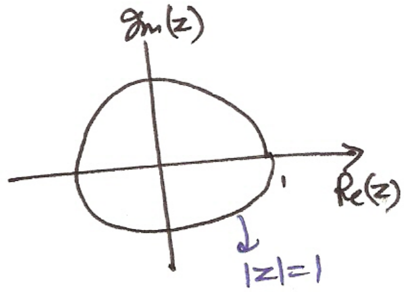


Relation with z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

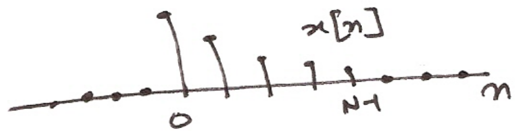
$$\Rightarrow X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$$

$\Rightarrow$ $X(z)$ evaluated along the unit circle.



- But DTFT is continuous in frequency
  ↳ Not good for computations using digital computers
  ↳ Need some transform which is also discretized in frequency axis. (Fourier series is discrete in frequency!)

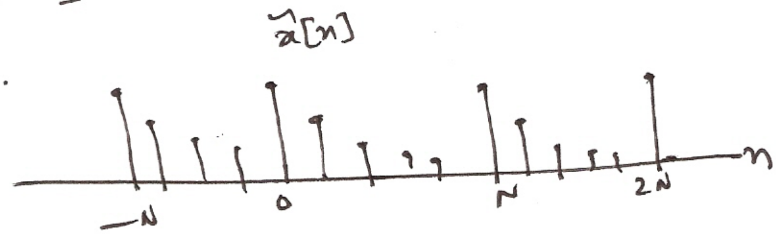
- Consider a finite length sequence, $x[n]$ of length N.
  $\Rightarrow$ $x[n] = 0$ outside $0 \le n \le N-1$



- Now create a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

or $\tilde{x}[n] = x[n \text{ modulo } N]$
$\quad = x[(n)_N]$.



Thus $x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

* Since $\tilde{x}[n]$ is periodic with period N,
  ↳ can be represented as a summation of complex exponentials with a ~~period~~ frequency equal to the integer multiples of the fundamental frequency $\left(\frac{2\pi}{N}\right)$.

* periodic complex exponentials

$$e_k[n] = e^{j\left(\frac{2\pi}{N}\right)kn} = e_k[n+rN].$$

then

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

⟸ Discrete fourier Series representation (DFS)

think of

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi n f_0 t}$$

with additional factor $N \to$ period.

<u>Test</u>:

$$e_{k+lN}[n] = e^{j\frac{2\pi}{N}(k+lN)}$$
$$= e^{\left(j\frac{2\pi k}{N} + j2\pi l\right)}$$
$$= e^{j\frac{2\pi k}{N}} \cdot \left(e^{j2\pi}\right)^{l}$$
$$= e^{j\frac{2\pi k}{N}} = e_k \leftarrow \text{periodic with } N$$

Thus we need only $e_0[n]$ to $e_{N-1}[n]$. to represent $\tilde{x}[n]$.

$$\Rightarrow \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

↳ DFS coefficient

only N frequency components

where

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

think of

$$a_k = \frac{1}{T_s}\int_{T_s} x(t)\, e^{-j2\pi n f_0 t}$$

Since $x[n]$ and $e^{-j\frac{2\pi}{N}kn}$ are both periodic with N

$\Rightarrow \tilde{X}[k]$ is also periodic with period N.

for convenience we use $W_N = e^{-j\frac{2\pi}{N}}$

$\Rightarrow$ Analysis Eq^n: $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk}$

DFS: Synthesis Eq^n $\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-nk}$

both $\tilde{x}[n]$ and $\tilde{X}[k]$ are periodic with N.

* for properties of DFS, ~~seed~~ refer to Oppenheim & Schafer DSP book.

Now, define

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

'k' are the frequency domain indices

↳ N-point sequence

↳ Discrete-fourier Transform (DFT)

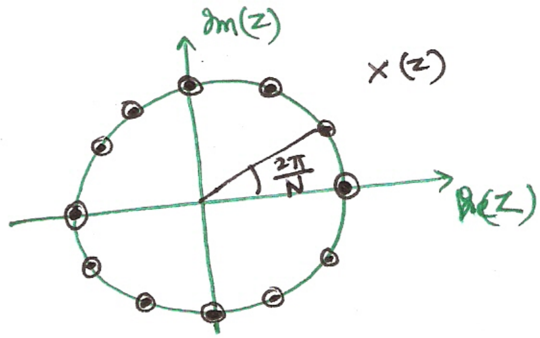
↳ fast algorithm for computation → FFT (matlab has fft functions).

Also, we can show that

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}}$$

↓

$X[k]$ for $k \in [0, N-1]$



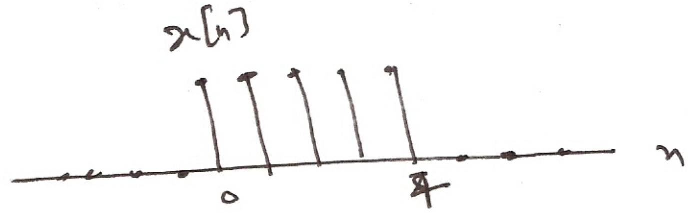
DFT is DTFT sampled in frequency domain as $\omega = \frac{2\pi k}{N}$.

DFT computation

1. make periodic extensions of $x[n]$ to obtain $\tilde{x}[n]$ with period N.
2. Find DFS coefficients $\tilde{X}[k]$ of $\tilde{x}[n]$
3. DFT $X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$

   → N-point FFT, usually $N = 2^L$.

Example:



N=5

we had:

$$X(e^{j\omega}) = e^{-j2\omega} \cdot \frac{\sin\left(\frac{5\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

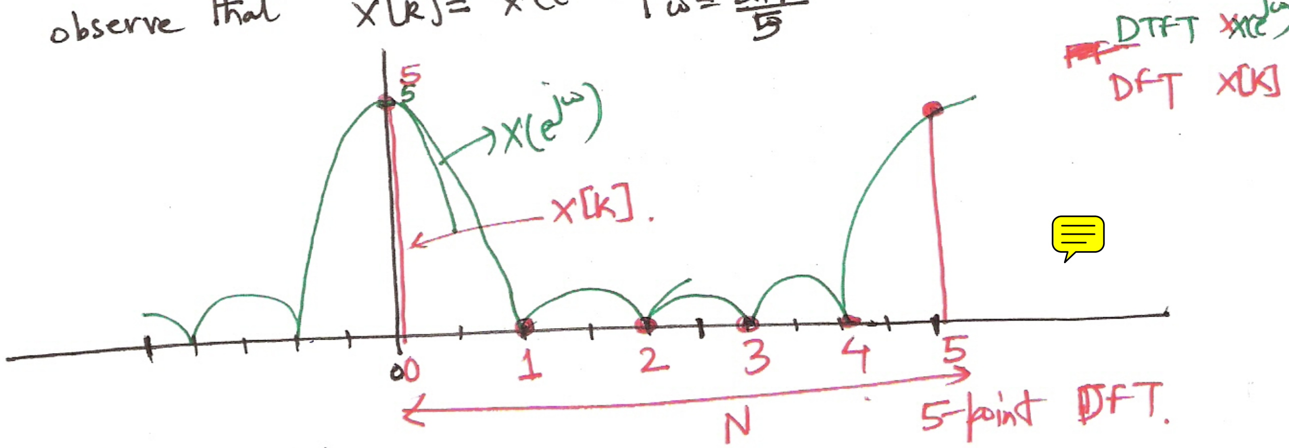
Now.

$$X[k] = \sum_{n=0}^{4} W_5^{nk} = \sum_{n=0}^{4} \left(e^{-j\frac{2\pi}{5}}\right)^{nk} = \sum_{n=0}^{4} \left(e^{-j\frac{2\pi k}{5}}\right)^{n}$$

$$= \frac{1 - e^{-j2\pi k}}{1 - e^{-j\frac{2\pi k}{5}}}$$

$$= \frac{e^{-j\frac{2\pi k}{2}} \left(e^{j\pi k} - e^{-j\pi k}\right)}{e^{-j\frac{2\pi k}{10}} \left(e^{+j\frac{2\pi k}{10}} - e^{-j\frac{2\pi k}{10}}\right)} = e^{-j\left(\frac{4\pi k}{10}\right)} \cdot \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$

observe that $X[k] \triangleq X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{5}}$

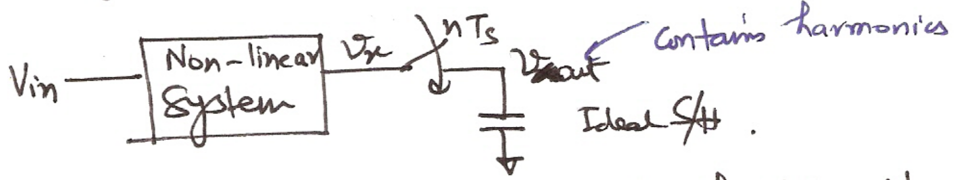


Red → 'k' indices.

⇒ DFT is a discrete representation of $X(e^{j\omega})$ sampled at $\omega = \frac{2\pi k}{N}$.

Signal estimation using DSP / Matlab :

Example : Characterizing the distortion of a S/H using a single tone input.



Use a discrete time periodic sequence with period N.

$v[n] = v[n+N]$.

Then $v[n]$ can be repsented as DFS.

$$v[n] = \sum_{k=0}^{N-1} V[k] \, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$\to$ complex DFS coefficients

$V[k] = |V[k]| \cdot e^{j \angle V[k]}$
$= B_m \, e^{j\phi_m}$

$V[k]$ are easily computed using FFT.

- N $\to$ ~~record length~~ ~~or~~ FFT size.

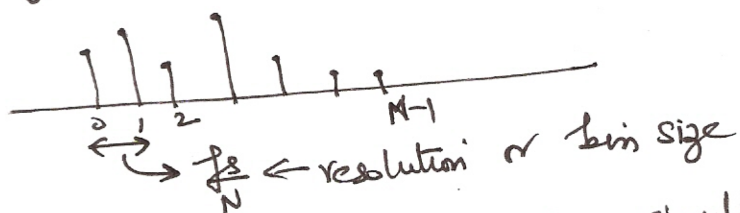
frequencies :

$\left\{0, \frac{2\pi}{N}, \frac{2\pi}{N}\cdot 2, \ldots, \left(\frac{2\pi}{N}\right)(N-1)\right\}$.

$\frac{N}{f_s} \to$ time in continuous time axis

$\frac{f_s}{N} \to$ resolution of the FFT.

$\hookrightarrow$ spacing between the tones.



P.S.

- M $\to$ record length = size of the data collected from simulation or measurement
- If the FFT is taken over the whole record length then FFT size = M. i.e N=M.