Signals refresher.

Fourier Series: for a periodic signal $g(t)$, with period to

$$g(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

where $$a(k) = \frac{1}{T_s}\int_0^{T_s} g(t) e^{-j2\pi k f_0 t} dt$$

for LTI systems only

Fourier Transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

FT properties:

Linearity: $a x(t) + b y(t) \xleftrightarrow{\mathcal{F}} aX(f) + bY(f)$

time delay: $x(t-t_0) \xleftrightarrow{\mathcal{F}} X(f) e^{-j2\pi f t_0}$ ← Linear phase

frequency translation: $e^{j2\pi f_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(f-f_0)$

$x(-t) \xleftrightarrow{\mathcal{F}} X(-f)$

Scaling: $x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$

Convolution $x(t) \circledast y(t) \xleftrightarrow{\mathcal{F}} X(f) \cdot Y(f)$

Multiplication $x(t) \cdot y(t) \xleftrightarrow{\mathcal{F}} X(f) \circledast Y(f)$

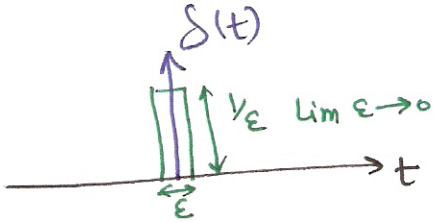
Duality: $X(t) \xleftrightarrow{\mathcal{F}} x(-f)$

Parseval's Theorem (Energy conservation)

$$\int_{-\infty}^{\infty} x(t) x^*(t) dt = \int_{-\infty}^{\infty} X(f) X^*(f) dt$$

Delta function (?)

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



$x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$ ← picks the value at $t=0$

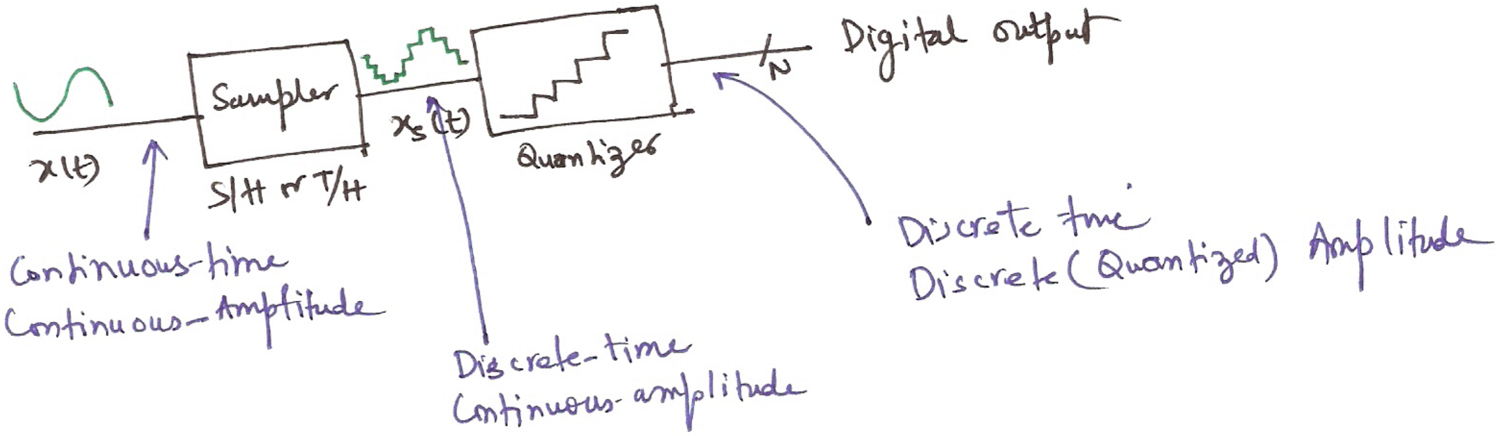
$$x(t) \cdot \delta(t-t_0) = x(t_0)\, \delta(t-t_0)$$

$$x(t) \circledast \delta(t) = x(t)$$

$$x(t) \circledast \delta(t-t_0) = x(t-t_0)$$

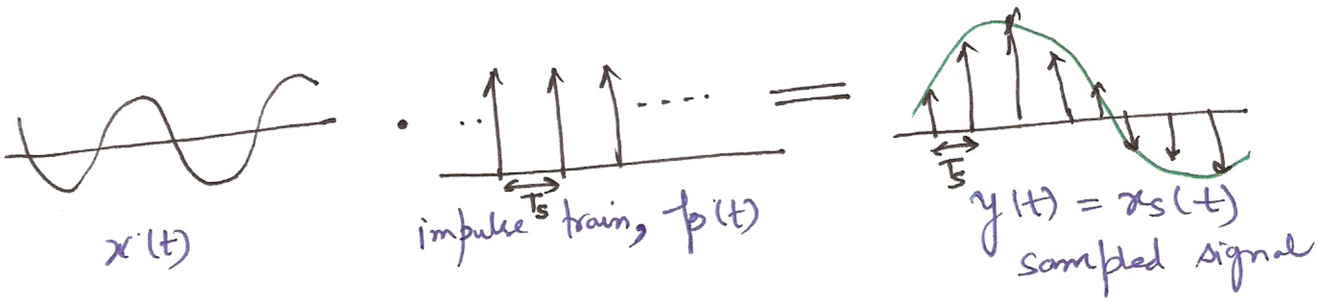
$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

ADC (Analog-to-Digital Converter)



Using continuous-time representation of signals to keep S/H analysis simple. $\Rightarrow y(t) = x_s(t) \Rightarrow x(nT_s)$ is held

Ideal sampling (impulse sampling)



$$y(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = x(t) \cdot p(t)$$

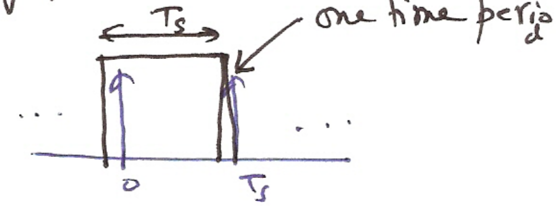
$$\Rightarrow Y(f) = X(f) \circledast P(f)$$

How to find $P(f) = ?$

$p(t) \leftarrow$ periodic function

↳ express as fourier series

↳ easy to find spectrum of $p(t)$



$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_s t}$$

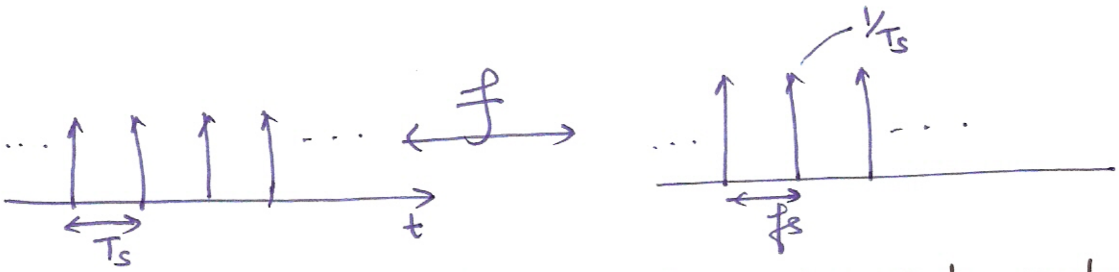
$$\Rightarrow \quad a_k = \frac{1}{T_s}\int_0^{T_s^-} p(t) e^{-j2\pi k f_s t} dt = \frac{1}{T_s}\int_0^{T_s^-} \delta(t) e^{-j2\pi k f_s (0)} dt \qquad \boxed{f_s = \frac{1}{T_s}}$$

$$= \frac{1}{T_s}\int_0^{T_s^-} \delta(t) dt = 1/T_s \quad \leftarrow \text{Same amplitude for all harmonics}$$

$\Rightarrow$ Now, $$p(t) = \sum_{k=-\infty}^{\infty} 1/T_s \, e^{j2\pi k f_s t}.$$

$\Rightarrow$ Taking fourier transform

$$\boxed{P(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \delta(f - kf_s)}$$



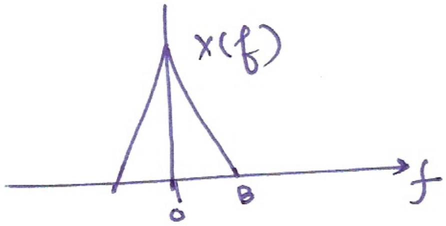
$\Rightarrow$ Impulse train signal is invariant under Fourier Transform.

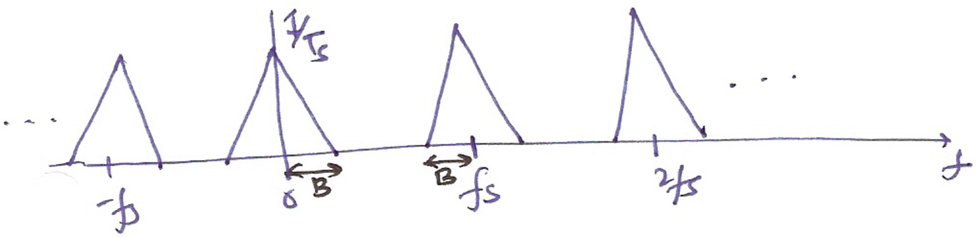
Now,

$$Y(f) = X(f) \circledast P(f)$$

$$= X(f) \circledast \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \delta(f-kf_s)$$

$$\Rightarrow \boxed{Y(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(f-kf_s)}$$



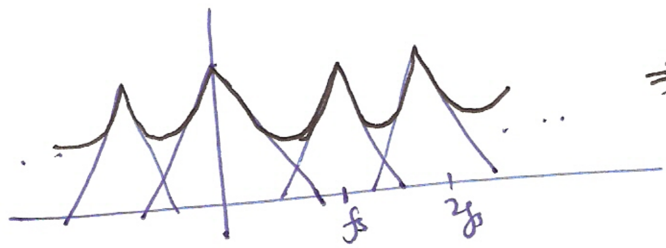


⇒ To avoid aliasing

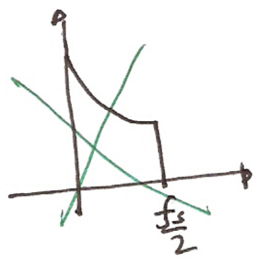
$\boxed{f_s > 2B}$

Nyquist Sampling Theorem

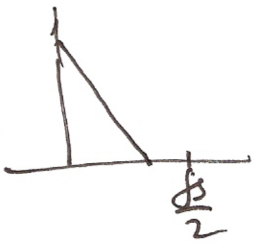
If $f_s < 2B$



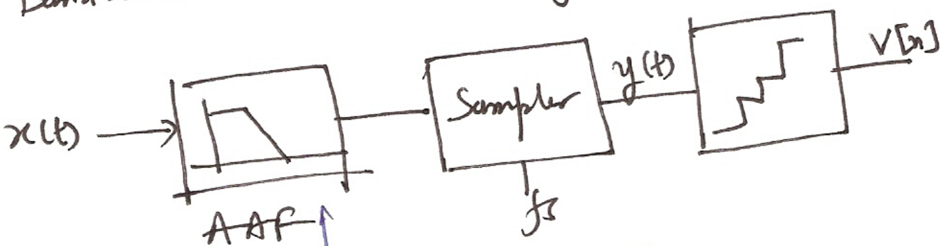
⇒



instead of



⇒ Bandlimit the input signal to the Sampler.

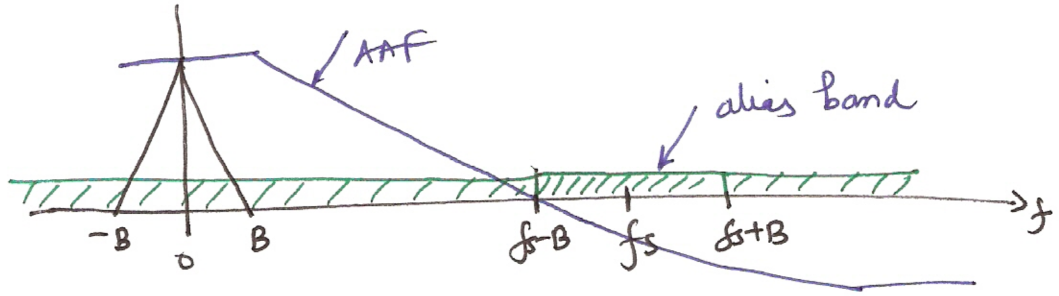


Anti-aliasing filter

↳ Always CT filter

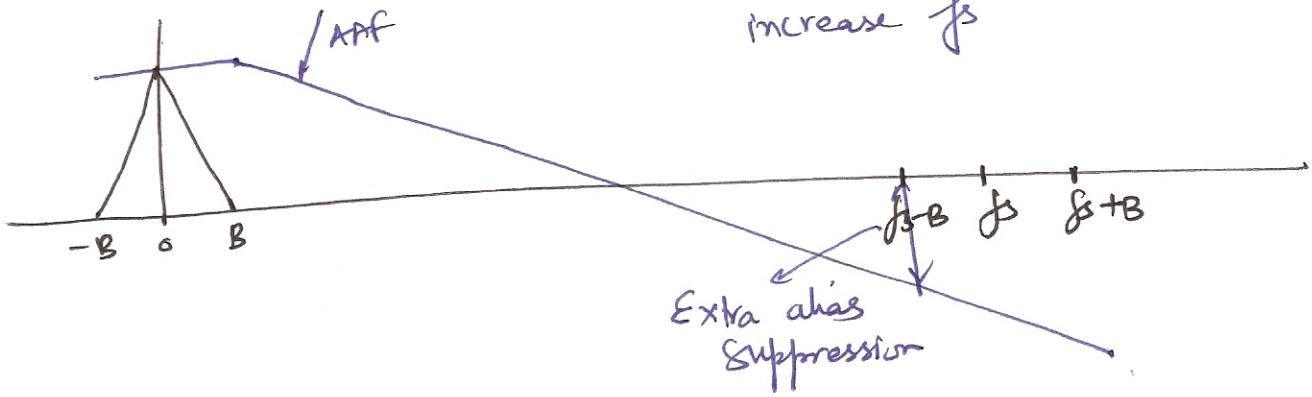
Can't use switched-capacitor filter here!

But what about thermal/wideband noise present at the input of the sampler?



* Even if the signal was bandlimited, the noise will alias from $kf_s + [-B, B]$ to the baseband.
* AAF ~~limits the~~ suppresses the noise in the alias bands.
  ⇒ AAF is always a must before a sampler.

* Ideal brickwall AAF is not realizable



⇒ Oversampling results in
↳ better alias rejection with the same AAF.
↳ lower order AAF for same amount of alias rejection.

⇒ Oversampling relaxes the requirements on AAF.

$$\text{Oversampling ratio} = \frac{f_s}{f_{s,\text{Nyquist}}} = \boxed{\frac{f_s}{2B} = OSR}$$