CISC 404/604  
Homework 1 Solutions  

1a. \( (B \rightarrow (C \lor D)) \) is valid and \( C \) is unsatisfiable. To prove \( B \rightarrow D \) is valid:  
Suppose bwoc \( B \rightarrow D \) is not valid.  

\[
\therefore \exists \nu_1 \text{ such that } \nu_1(B) = T \quad (1) \\
\& \nu_1(D) = F \text{ [by defns of validity and \( \rightarrow \)].} \quad (2)
\]

Since \( (B \rightarrow (C \lor D)) \) is valid, \( \therefore \nu_1(B \rightarrow (C \lor D)) = T \text{ [by defn of validity].} \)  
Since \( \nu_1(B) = T \text{ [by (1)], } \nu_1(C \lor D) = T \text{ [by defn of \( \lor \)].} \)

Since \( C \) is unsatisfiable, \( \therefore \nu_1(C) = F \text{ [by defn of satisfiable].} \)

\[
\therefore \nu_1(D) = T \text{ [by defn of \( \lor \)], which is a contradiction [cf. (2)].}
\]

\( \therefore (B \rightarrow D) \) is valid [proof by contradiction].

1b. \( \Gamma_1 \models B \), \( \therefore \forall \nu, \text{ if } \nu \text{ satisfies } \Gamma_1, \text{ then } \nu(B) = T \text{ [by defn of } \models]. \)  
\( \Gamma_2 \models C \), \( \therefore \forall \nu, \text{ if } \nu \text{ satisfies } \Gamma_2, \text{ then } \nu(C) = T \text{ [by defn of } \models]. \)  
\( \forall \nu, \text{ if } \nu \text{ satisfies } \Gamma_1 \cup \Gamma_2, \text{ then } \nu(B \lor C) = T \text{ [by defns of satisfies and } \lor]. \)

\( \therefore \forall \nu, \text{ if } \nu \text{ satisfies } \Gamma_1 \cup \Gamma_2, \text{ then } \nu(B \land C) = T \text{ [by defn of } \land]. \)

\( \therefore \Gamma_1 \cup \Gamma_2 \models (B \land C) \text{ [by defn of } \models]. \)

1c. False. Proof by counterexample:  
Suppose \( B \) is A (a statement letter), \( C \) is B, and \( D \) is \( \neg B \).  

\( \therefore (B \rightarrow (C \lor D)) \) is valid [by defns of \( \lor \) & \( \rightarrow \)],  
but neither \( A \rightarrow B \) nor \( A \rightarrow \neg B \) are valid [by defn of validity].

2a. Suppose bwoc \( \{(B \rightarrow (C \rightarrow D)), \neg D\} \not\models (C \rightarrow \neg B) \).  

\( \therefore \exists \nu_1 \text{ such that } \nu_1 \text{ satisfies } \{(B \rightarrow (C \rightarrow D)), \neg D\} \)  
and \( \nu_1(C \rightarrow \neg B) = F \text{ [by defn of } \models]. \)

\( \therefore \nu_1(B \rightarrow (C \rightarrow D)) = T \) and \( \nu_1(\neg D) = T \text{ [by defns of satisfies],} \)  
and \( \nu_1(C) = T \) and \( \nu_1(B) = T \text{ [by defns of } \rightarrow \& \neg]. \)  

Since \( \nu_1(B) = T \) and \( \nu_1(B \rightarrow (C \rightarrow D)) = T \text{ [by (2) & (1)]}, \)

\( \therefore \nu_1(C \rightarrow D) = T \text{ [by defn of } \rightarrow]. \)  

Since \( \nu_1(C) = T \) and \( \nu_1(C \rightarrow D) = T \text{ [by (2) & (3)]}, \)

\( \therefore \nu_1(D) = T, \text{ which is a contradiction [cf. (1)].} \)

\( \therefore \{(B \rightarrow (C \rightarrow D)), \neg D\} \models (C \rightarrow \neg B) \text{ [proof by contradiction].} \)
2b. \( \Gamma \cup \{ \neg B \} \models C \), \( \because \forall \nu, \text{ if } \nu \text{ satisfies } \Gamma \cup \{ \neg B \}, \text{ then } \nu(C) = T \) [by defn of \( \models \)].

\( \Gamma \cup \{ \neg B \} \models \neg C \), \( \because \forall \nu, \text{ if } \nu \text{ satisfies } \Gamma \cup \{ \neg B \}, \text{ then } \nu(\neg C) = T \) [by defn of \( \models \)].

Suppose \( \exists \nu_1 \) such that \( \nu_1 \) satisfies \( \Gamma \cup \{ \neg B \} \).

\( \therefore \nu_1(C) = T \) and \( \nu_1(\neg C) = T \), which is a contradiction.

\( \therefore \exists \nu \) such that \( \nu \) satisfies \( \Gamma \cup \{ \neg B \} \), \( \therefore \Gamma \cup \{ \neg B \} \) is unsatisfiable.

3(i). Assume (I): If \( \Gamma \models B \), then for some finite subset, \( \Gamma_f \subseteq \Gamma \), \( \Gamma_f \models B \).

To show (II) holds, suppose \( \forall \nu \) such that for every finite subset, \( \Gamma_f \subseteq \Gamma \), \( \Gamma_f \) is satisfiable but \( \Gamma \) is unsatisfiable.

\( \therefore \Gamma \models (C \land \neg C) \), for any \( C \) [vacuously true].

\( \therefore \) for some finite subset, \( \Gamma_f \subseteq \Gamma \), \( \Gamma_f \models (C \land \neg C) \) [by (I)].

\( \therefore \Gamma_f \) is unsatisfiable, which is a contradiction [cf. (1)].

\( \therefore \) If for every finite subset, \( \Gamma_f \subseteq \Gamma \), \( \Gamma_f \) is satisfiable, then \( \Gamma \) is satisfiable, \( \therefore \) (II).

3(ii). Assume (II): If every finite subset of \( \Gamma \) is satisfiable, then \( \Gamma \) is satisfiable.

To show (I) holds, suppose \( \forall \nu \) such that for every finite subset, \( \Gamma_f \subseteq \Gamma \), \( \Gamma_f \) is satisfiable but \( \Gamma \) is unsatisfiable.

\( \therefore \forall \Gamma_f \subseteq \Gamma \), \( \Gamma_f \cup \{ \neg B \} \) is satisfiable [by defn of \( \models \)].

\( \therefore \Gamma \cup \{ \neg B \} \) is satisfiable [by (II)].

\( \therefore \Gamma \not\models B \) [since \( \Gamma \models B \iff \Gamma \cup \{ \neg B \} \) is unsatisfiable],

which is a contradiction [by (1)].

\( \therefore \) If \( \Gamma \models B \), then \( \exists \Gamma_f \subseteq \Gamma \) such that \( \Gamma_f \models B \), \( \therefore \) (I).

4a. Assume \( \Gamma \) is maximally satisfiable (max sat) and suppose \( \Gamma \models B \).

Suppose \( \exists \nu \) such that \( \nu \) satisfies \( \Gamma \cup \{ B \} \) is unsatisfiable [by defn of max sat]. (1)

Since \( \Gamma \) is max sat, \( \therefore \) \( \Gamma \) is satisfiable,

\( \therefore \exists \nu, \text{ say } \nu_1 \), that satisfies \( \Gamma \) [by defn of satisfiable].

Since \( \Gamma \models B \), \( \therefore \nu_1 \) satisfies \( \Gamma \cup \{ B \} \), which is a contradiction [cf. (1)].

\( \therefore B \in \Gamma \) [proof by contradiction].

4b. Assume \( \Gamma \) is maximally satisfiable and suppose \( (B \land C) \in \Gamma \).

Suppose \( \exists \nu \) such that \( \nu \) satisfies \( \Gamma \cup \{ B \} \) is unsatisfiable [by defn of max sat]. (1)

Since \( \Gamma \) is max sat, \( \therefore \) \( \Gamma \) is satisfiable,

\( \therefore \exists \nu, \text{ say } \nu_1 \), that satisfies \( \Gamma \) [by defn of satisfiable].

Since \( (B \land C) \in \Gamma \), \( \Gamma \models (B \land C) \) and \( \Gamma \models B \) [by defns of max sat & \( \land \)].

Since \( \Gamma \models B \), \( \therefore \nu_1 \) satisfies \( \Gamma \cup \{ B \} \), which is a contradiction [cf. (1)].

\( \therefore B \in \Gamma \) [proof by contradiction] and similarly for \( C \in \Gamma \).
4c. Assume $\Gamma$ is maximally satisfiable and suppose $B \in \Gamma$.

To show $\neg B \not\in \Gamma$, suppose bwoc otherwise: $\neg B \in \Gamma$.
Since $B \in \Gamma$ and $\neg B \in \Gamma$, $\therefore \Gamma$ is unsatisfiable [by defn of satisfiable],
which is a contradiction, since $\Gamma$ is max sat.
$\therefore \neg B \not\in \Gamma$ [proof by contradiction]. (1)

Assume $\Gamma$ is maximally satisfiable and suppose $\neg B \not\in \Gamma$. (2)
To show $B \in \Gamma$, suppose bwoc otherwise: $B \not\in \Gamma$.
Since $\Gamma$ is max sat, $\therefore \Gamma \cup \{B\}$ is unsatisfiable [by defn of max sat].
Since $\Gamma$ is satisfiable, $\therefore \exists \nu$, say $\nu_1$, that satisfies $\Gamma$ [by defn of satisfiable].
Since $\Gamma \cup \{B\}$ is unsatisfiable, $\therefore \nu_1(B) = F$,
or in other words, $\nu_1(\neg B) = T$. [by defn of $\neg$]
$\therefore \nu_1$ satisfies $\Gamma \cup \{\neg B\}$, but since this set is not unsatisfiable,
it must mean $\neg B \in \Gamma$, which is a contradiction [cf. (2)].
$\therefore B \in \Gamma$ [proof by contradiction]. (3)

$\therefore$ If $\Gamma$ is max sat, then $B \in \Gamma \iff \neg B \not\in \Gamma$ [by (1) & (3)].