1. A Horn clause is a clause with at most one positive literal. We can consider four types of Horn clauses: (i) with zero positive literals: this has two subtypes (i.a) with no negative literals i.e., the empty clause or (i.b) with one or more negative literals (such clauses are also called query clauses in Prolog) and (ii) with exactly one positive literal: the two subtypes are (ii.a) with no negative literals (called facts in Prolog terms) and (ii.b) having one or more negative literals (rules in Prolog terms).

a. Show that result of resolving two Horn clauses is another Horn clause.
b. Let $C_1, C_2, C_3, C_4$ be Horn clauses (i.e., having at most one positive literal). Consider the following resolution derivation: $C_5$ is obtained by resolving $C_1$ and $C_2$; $C_6$ is obtained by resolving $C_3$ and $C_4$; and finally, $C_7$ is obtained by resolving $C_5$ and $C_6$.

Show that there is a (input) resolution derivation of $C_7$ from $S = \{C_1, C_2, C_3, C_4\}$ where in each step at least one of the two resolved clauses is from the set $S$.

Hint. Note at each step of the original derivation, some literal is involved. WLOG, assume that the positive literal (which is resolved) in the 3 original resolutions belongs to $C_1$, $C_3$, and $C_5$ respectively. Then reorder the derivation using the fact that we are only considering Horn clauses. Also, you may assume that each of the $C_i's$ do not contain both a statement-letter, $A$, and its negation, $\neg A$.

2. Show

a. $\vdash_{\text{res}} ((A \Rightarrow B) \land (B \Rightarrow C)) \Rightarrow ((\neg C \land A) \Rightarrow B)$.
b. $\{((A \land B) \lor (C \Rightarrow D))\} \vdash_{\text{res}} ((A \lor (C \Rightarrow D)) \land (B \lor (C \Rightarrow D)))$
c. $(B \Rightarrow C)$ is a consequence of the set $\{\neg((\neg C \Rightarrow B) \Rightarrow B), (\neg C \Rightarrow \neg A), ((B \land \neg C) \Rightarrow A)\}$.

3. a. Let $\Gamma$ be a set of statement forms. $\Gamma$ is said to be inconsistent if for some statement form, $\alpha$, $\Gamma \vdash_L \alpha$ as well as $\Gamma \vdash_L \neg \alpha$. Show that if $\Gamma$ is inconsistent then for any statement form $\beta$, $\Gamma \vdash_L \beta$.
b. Show $\vdash_L (A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$.

Hint. The proposition and lemma proved in pages 37 through 39 of the text can be assumed/used in answering both parts of this question.
c. Show that if $\Gamma \cup \{\alpha\} \vdash_L \beta$ then $\Gamma \cup \{\neg \beta\} \vdash_L \neg \alpha$. 