CISC 401/601
Homework 3

Answers

1. a. Let

\[ H_1(x) = \begin{cases} 
1 & \text{if } \Phi(x, x) \downarrow \\
\uparrow & \text{otherwise}
\end{cases} \]

Show that \( H_1(x) \) is partially computable (p77 Ex 2 (a)).

**Ans.**
Since a macro expansion for the universal program is known, the following \( L \) program, say \( P_1 \), exists.

\( \Phi(x, x) \)

\( y \leftarrow 1 \)

Obviously, \( \Psi_{P_1}(x) = H_1(x) \). Hence, \( H_1(x) \) is known to be partially computable.

b. Show that there is a program \( P \) such that the predicate \( \text{HALT}(x, p) \) is not computable where \( p = \#(P) \).

**Ans.**
Let \( p = \#(P_1) \), where \( P_1 \) is from 1 (a).

**Proof**
Suppose \( \text{HALT}(x, p) \) is computable.
Then, there must be a corresponding \( L \) program, say \( P_2 \), for \( \text{HALT}(x, p) \).
Note that \( \forall x, \Psi_{P_2}(x) \in \{0, 1\} \).

**case 1:** \( \Psi_{P_2}(x) = 1 \)
\( \Psi_{P_2}(x) = 1 \) means \( \text{HALT}(x, p) \) is 1 or true, which is \( \Phi(x, p) \downarrow \) or equivalently \( H_1(x) \downarrow \). Therefore, \( H_1(x) \) must output 1, and the condition “if \( \Phi(x, x) \downarrow \)” in the definition of \( H_1 \) must be satisfied.
Hence, \( \Psi_{P_2}(x) = 1 \iff \Phi(x, x) \downarrow \iff \text{HALT}(x, x) = 1 \).

**case 2:** \( \Psi_{P_2}(x) = 0 \)
\( \Psi_{P_2}(x) = 0 \) means \( \text{HALT}(x, p) \) is 0 or false, which is \( \Phi(x, p) \uparrow \) or equivalently \( H_1(x) \uparrow \). Therefore, the condition “if \( \Phi(x, x) \downarrow \)” in the definition of \( H_1 \) must not be satisfied.
Hence, \( \Psi_{P_2}(x) = 0 \iff \Phi(x, x) \uparrow \iff \text{HALT}(x, x) = 0 \).
There is no other case, and we conclude $\Psi_{\text{p}2}(x) = \text{HALT}(x,x)$.
Therefore, if $\Psi_{\text{p}3}(x)$, namely $\text{HALT}(x,p)$, is computable, then $\text{HALT}(x,x)$
can be a computable function, which is a contradiction.

Therefore, $\text{HALT}(x,p)$ is not computable.

2. Let $f(x_1, x_2, \ldots, x_n)$ be computed by program $P$, and suppose that for
some primitive recursive function $g(x_1, x_2, \ldots, x_n)$,

$$\text{STP}^{(n)}(x_1, x_2, \ldots, x_n, \#(P), g(x_1, x_2, \ldots, x_n))$$

is true for all $x_1, x_2, \ldots, x_n$. Show that $f(x_1, x_2, \ldots, x_n)$ is primitive recursive.

**Ans.**
Clearly, for all $x_1, x_2, \ldots, x_n$

$$f(x_1, x_2, \ldots, x_n) = (r(\text{SNAP}^{(n)}(x_1, x_2, \ldots, x_n, \#(P), g(x_1, x_2, \ldots, x_n))))_1.$$  

Knowing that $g$, SNAP, $r$, $(\ )_1$ are all primitive recursive, $f$, a composition
of primitive recursive functions, is primitive recursive.

3. Show that there is no computable function $f(x)$ such that $f(x) = \Phi(x, x) + 1$
whenever $\Phi(x, x) \downarrow$.

(p86 Ex 8)

**Ans.**
Suppose there is one.
Then, there must be a corresponding $L$ program as well as a corresponding
program number, say $p$.
Note that $\forall x, \Phi(x, p) \downarrow$.
Then, $f(p)$ should be $\Phi(p, p) + 1$ by definition because the condition
$\Phi(p, p) \downarrow$ is satisfied.
$\Phi(p, p)$, however, is equivalent to $f(p)$. Therefore, $f(p) = \Phi(p, p) + 1 = f(p) + 1$ is a contradiction.

Therefore, $f$ does not exist.

4. Let $A = \{ y \mid (\exists t)P(t, y) \}$, where $P$ is a computable predicate. Show that
$A$ is r.e.

**Ans.**
Since $P$ is computable, there exists a corresponding $L$ program, and we
can assume \( V_i \leftarrow P(V_2, V_3) \) as a macro in the following program, say \( Q \).

\[
[A] \quad Z_2 \leftarrow P(Z_1, X) \\
Z_1 \leftarrow Z_1 + 1 \\
\text{IF } Z_2 = 0 \text{ THEN GOTO A}
\]

Clearly, \((\exists t)P(t, y) \iff \Psi_Q(y) \downarrow\).
Thus, \( A = \{ y \mid (\exists t)P(t, y) \} = \{ y \mid \Psi_Q(y) \downarrow \} \).

Since there exists a partially computable function whose domain is \( A \), \( A \) is shown to be r.e.