Answers to some sample questions

Question Show that $((A \longrightarrow (B \longrightarrow C)) \longrightarrow ((A \wedge B) \longrightarrow C))$ is a valid formula.

Proof We need to show that all (suitable) assignments make the formula, $F = ((A \longrightarrow (B \longrightarrow C)) \longrightarrow ((A \wedge B) \longrightarrow C))$ true. So, let us consider an arbitrary assignment, $\mathcal{A}$. We need to show that $\mathcal{A}(F) = 1$, i.e., if $\mathcal{A}(A \longrightarrow (B \longrightarrow C)) = 1$ then $\mathcal{A}(((A \wedge B) \longrightarrow C)) = 1$.
Let us assume that $\mathcal{A}(A \longrightarrow (B \longrightarrow C)) = 1$ (I). Now we need to show that if $\mathcal{A}(A \wedge B) = 1$ then $\mathcal{A}(C) = 1$.
Let us therefore assume $\mathcal{A}(A \wedge B) = 1$, i.e., $\mathcal{A}(A) = 1$ and $\mathcal{A}(B) = 1$. From $\mathcal{A}(A) = 1$ and (I), we get $\mathcal{A}(B \longrightarrow C) = 1$ (II). From $\mathcal{A}(B) = 1$ and (II), we get $\mathcal{A}(C) = 1$. Thus, we have established if $\mathcal{A}(A \wedge B) = 1$ then $\mathcal{A}(C) = 1$, i.e., $\mathcal{A}(((A \wedge B) \longrightarrow C)) = 1$.
Discharging assumption (I), we have shown if $\mathcal{A}(A \longrightarrow (B \longrightarrow C)) = 1$ then $\mathcal{A}(((A \wedge B) \longrightarrow C)) = 1$, i.e., $\mathcal{A}(F) = 1$. But since the assignment, $\mathcal{A}$ was arbitrarily chosen, we can infer that all assignments make F true. Hence F is valid.

Question If $S_1 \subseteq S_2$ and S_2 is satisfiable then S_1 is satisfiable.

Proof Assume $S_1 \subseteq S_2$ and that S_2 is satisfiable. The latter means that there is an assignment, say $\mathcal{A}$, which is a model of S_2. That is, $\mathcal{A}(F) = 1$ for all $F \in S_2$. Since $S_1 \subseteq S_2$, $\mathcal{A}$ makes all formulae in S_1 true. Hence $\mathcal{A}$ is a model of S_1. Since, we have a model for S_1, we conclude that S_1 is satisfiable as well.

Question If $(F \longrightarrow G)$ is a consequence of S then G is a consequence of $S \bigcup \{F\}$.

Proof Assume the contrary. Then we have $(F \longrightarrow G)$ is a consequence of S but G is not a consequence of $S \bigcup \{F\}$. From the latter, we can say that there is an assignment, say $\mathcal{A}$, such that $\mathcal{A}$ is a model of $S \bigcup \{F\}$ and $\mathcal{A}(G) = 0$. That is, $\mathcal{A}$ is a model of S, $\mathcal{A}(F) = 1$ and $\mathcal{A}(G) = 0$. Now we have $\mathcal{A}$ as a model of S and $\mathcal{A}(F \longrightarrow G) = 0$. This contradicts $(F \longrightarrow G)$ is a consequence of S (as the latter means that every model of S makes $(F \longrightarrow G)$ true).

Question: Show that H is a consequence of $\{F \longrightarrow G, G \longrightarrow H, F\}$.

Proof Consider an arbitrary assignment, say $\mathcal{A}$. Assume $\mathcal{A}$ is a model of $\{F \longrightarrow G, G \longrightarrow H, F\}$, i.e., $\mathcal{A}(F) = 1$ (I), $\mathcal{A}(F \longrightarrow G) = 1$ (II), and $\mathcal{A}(G \longrightarrow H) = 1$ (III). From (I) and (II), we get $\mathcal{A}(G) = 1$. Combining this with (III), we get $\mathcal{A}(H) = 1$.
Since $\mathcal{A}$ was arbitrarily chosen, we have shown that any model of $\{F \longrightarrow G, G \longrightarrow H, F\}$ makes H true. Therefore, H is a consequence of $\{F \longrightarrow G, G \longrightarrow H, F\}$.

Question: Show that H is a consequence of $\{F \longrightarrow (G \vee H), F \longrightarrow \neg G, F\}$.

Proof by Contradiction: Suppose H is not a consequence of $\{F \longrightarrow (G \vee H), F \longrightarrow \neg G, F\}$. Then there is an assignment, say $\mathcal{A}$, such that $\mathcal{A}$ is a model of $\{F \longrightarrow (G \vee H), F \longrightarrow \neg G, F\}$, but $\mathcal{A}(H) = 0$. Since $\mathcal{A}$ is a model of $\{F \longrightarrow (G \vee H), F \longrightarrow \neg G, F\}$, we have $\mathcal{A}(F) = 1$ (I), $\mathcal{A}(F \longrightarrow (G \vee H)) = 1$ (II), and $\mathcal{A}(F \longrightarrow \neg G) = 1$ (III). From (I) and (II), we get $\mathcal{A}(G \vee H) = 1$, and from (I) and (III), we get $\mathcal{A}(G) = 0$. From these two conclusions, we get have $\mathcal{A}(H) = 1$, a contradiction. Thus, our assumption is incorrect and thus we have shown that H is a consequence of $\{F \longrightarrow (G \vee H), F \longrightarrow \neg G, F\}$.

Question: If H is a consequence of $S \bigcup \{F\}$ and H is also a consequence of $S \bigcup \{G\}$ then H is a consequence of $S \bigcup \{(F \vee G)\}$.

Proof by contradiction Assume H is a consequence of both $S \bigcup \{F\}$ as well as $S \bigcup \{G\}$ and that H is not a consequence of $S \bigcup \{(F \vee G)\}$. From the latter, we know there is an assignment, say $\mathcal{A}_1$ such that $\mathcal{A}_1$ is a model of $S \bigcup \{(F \vee G)\}$ but $\mathcal{A}_1(H) = 0$. Therefore $\mathcal{A}_1$ is a model of S and $\mathcal{A}_1(F \vee G) = 1$. If $\mathcal{A}_1(F \vee G) = 1$ then $\mathcal{A}_1(F) = 1$ or $\mathcal{A}_1(G) = 1$.

Case 1: $\mathcal{A}_1(F) = 1$. Now $\mathcal{A}_1$ is a model of $S \bigcup \{F\}$. Since H is a consequence of this set, we have $\mathcal{A}_1(H) = 1$, a contradiction.
Case 2: $\mathcal{A}_1(G) = 1$. We can arrive at a contradiction similarly.

Since we have arrived at a contradiction in all cases, our initial assumption is incorrect. Hence, if H is a consequence of $S \bigcup \{F\}$ and H is also a consequence of $S \bigcup \{G\}$ then H is a consequence of $S \bigcup \{(F \vee G)\}$.