CISC 301
Homework 4 Sample Solution

1. (a) The following structure satisfies $F_1$ and $F_2$, but not $F_3$.
\[
U_A = \mathbb{N}
\]
\[
P_A = \{(m, n) \mid m, n \in \mathbb{N}, |m - n| \leq 1\}
\]
It does not satisfy $F_3$ because $P_A$ is not transitive. For example, $(0, 1) \in P_A$ and $(1, 2) \in P_A$ but $(0, 2) \notin P_A$.

(b) The following structure satisfies $F_1$ and $F_3$, but not $F_2$.
\[
U_A = \mathbb{N}
\]
\[
P_A = \{(m, n) \mid m, n \in \mathbb{N}, m \leq n\}
\]
It does not satisfy $F_2$ because $P_A$ is not symmetric. For example, $(0, 1) \in P_A$ holds but $(1, 0) \notin P_A$.

(c) The following structure satisfies $F_3$ and $F_2$, but not $F_1$.
\[
U_A = \mathbb{N}
\]
\[
P_A = \{(m, n) \mid m, n \in \mathbb{N}, m \cdot n > 0\}
\]
It does not satisfy $F_1$ because $P_A$ is not reflexive as $(0, 0) \notin P_A$.

2. (a) Satisfies:
   i. $U_A = \mathbb{N}$
      \[
P_A = \{(m, n) \mid m, n \in \mathbb{N}, m \leq n\}
\]
   ii. $U_A = \mathbb{Q}$ (the rationals)
      \[
P_A = \{(m, n) \mid m, n \in \mathbb{N}, m < n\}
\]
      will also satisfy the given formula.

Falsifies:
\[
U_A = \mathbb{N}
\]
\[
P_A = \{(m, n) \mid m, n \in \mathbb{N}, m < n\}
\]
To see why, consider the assignment of $0$ to $x$ and $1$ to $y$.

(b) Satisfies:
\[
U_A = \mathbb{N}
\]
\[
P_A = \mathbb{N}
\]
\[
Q_A = \mathbb{N}
\]
Falsifies:
\[
U_A = \mathbb{N}
\]
\[
P_A = \{n \mid n \in \mathbb{N}, n \geq 1\}
\]
\[
Q_A = \{n \mid n \in \mathbb{N}, n \geq 2\}
\]
To see why, note that $\forall x[P(x) \rightarrow Q(x)]$ is not satisfied in this structure (consider the assignment of $1$ to $x$). However, $(\forall x P(x) \rightarrow \forall x Q(x))$ is satisfied in this structure because $\forall x P_A(x)$ is not true in it.
(c) Satisfies:

\[
U_A = \mathbb{N} \\
P^A = \{ (\ell, m, n) \mid \ell, m, n \in \mathbb{N}, \ell + m = n \} \\
f^A(x, y) = x + y
\]

Falsifies:

\[
U_A = \mathbb{N} \\
P^A = \{ (\ell, m, n) \mid \ell, m, n \in \mathbb{N}, \ell + m = n \} \\
f^A(x, y) = 0
\]

(d) Any structure which interprets \( P \) as a relation that is not transitive will satisfy \( F_4 \).

Consider \( U_A = \{0, 1, 2, \ldots\} \) and \( P^A = \{ \langle m, n \rangle \mid m < n \} \). This structure will falsify \( F_4 \). Since \(<\) is transitive and irreflexive, it satisfies \( \forall x \forall y \forall z ([P(x, y) \land P(y, z)] \rightarrow P(x, z)) \land \forall x \neg P(x, x) \). Additionally, since every natural number has a number larger than it, this structure satisfies \( \forall x \exists y P(x, y) \). Hence this structure falsifies \( \exists x \forall y \neg P(x, y) \).

(e) A structure that satisfies \( (\forall x P(x) \leftrightarrow \forall x Q(x) ) \) will satisfy \( F_5 \). This can be done by setting \( U_A = P^A = Q^A = \{1\} \).

For a structure \( A \) to falsify \( F_5 \), it must satisfy \( \exists x P(x) \leftrightarrow \exists x Q(x) \) and falsify \( \forall x P(x) \leftrightarrow \forall x Q(x) \). To meet the latter constraint, it must satisfy one of \( \forall x P(x) \) or \( \forall x Q(x) \) but not the other. So let’s say \( A(\forall x P(x)) = 1 \). That is, \( P^A = U_A \). Then we must have \( A(\forall x Q(x)) = 0 \), i.e., \( Q^A = U_A \). Note \( A(\exists x P(x)) = 1 \) since \( A(\forall x P(x)) = 1 \). Hence, in order for \( A \) to satisfy \( \exists x P(x) \leftrightarrow \exists x Q(x) \), we must have \( A(\exists x Q(x)) = 1 \), i.e., \( Q^A = \phi \).

All of these requirements are met by letting \( U_A = \{1, 2\} = P^A \) and \( Q^A = \{1\} \).