CISC 301
Homework 4 Sample Solution

1. (a) The following structure satisfies F_1 and F_2, but not F_3.
   $U_{\mathcal{A}} = \mathbb{N}$
   $P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, |m-n| \leq 1\}$
   It does not satisfy F_3 because $P^{\mathcal{A}}$ is not transitive. For example, $(0,1) \in P^{\mathcal{A}}$ and $(1,2) \in P^{\mathcal{A}}$ but $(0,2) \notin P^{\mathcal{A}}$.
   (b) The following structure satisfies F_1 and F_3, but not F_2.
   $U_{\mathcal{A}} = \mathbb{N}$
   $P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, m \leq n\}$
   It does not satisfy F_2 because $P^{\mathcal{A}}$ is not symmetric. For example, $(0,1) \in P^{\mathcal{A}}$ holds but $(1,0) \notin P^{\mathcal{A}}$.
   (c) The following structure satisfies F_3 and F_2, but not F_1.
   $U_{\mathcal{A}} = \mathbb{N}$
   $P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, m \cdot n > 0\}$
   It does not satisfy F_1 because $P^{\mathcal{A}}$ is not reflexive as $(0,0) \notin P^{\mathcal{A}}$.

2. (a) Satisfies:
   i. $U_{\mathcal{A}} = \mathbb{N}$
      $P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, m \leq n\}$
   ii. $U_{\mathcal{A}} = \mathbb{Q}$ (the rationals)
      $P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, m < n\}$
      will also satisfy the given formula.

   Falsifies:
   $U_{\mathcal{A}} = \mathbb{N}$
   $P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, m < n\}$
   To see that this structure falsifies the formula, consider the assignment of 0 to x and 1 to y.
   (b) Satisfies:
   $U_{\mathcal{A}} = \mathbb{N}$
   $P^{\mathcal{A}} = \mathbb{N}$
   $Q^{\mathcal{A}} = \mathbb{N}$
   Falsifies:
   $U_{\mathcal{A}} = \mathbb{N}$
   $P^{\mathcal{A}} = \{n \mid n \in \mathbb{N}, n \geq 1\}$
   $Q^{\mathcal{A}} = \{n \mid n \in \mathbb{N}, n \geq 2\}$
   To see why, note that $\forall x[P(x) \to Q(x)]$ is not satisfied in this structure (consider the assignment of 1 to x). However, $(\forall x P(x) \to \forall x Q(x))$ is satisfied in this structure because $\forall x P^{\mathcal{A}}(x)$ is not true in it.

(c) Satisfies:

$U_{\mathcal{A}} = \mathbb{N}$
$P^{\mathcal{A}} = \{(\ell, m, n) \mid \ell, m, n \in \mathbb{N}, \ell + m = n\}$
$f^{\mathcal{A}}(x, y) = x + y$

Falsifies:

$U_{\mathcal{A}} = \mathbb{N}$
$P^{\mathcal{A}} = \{(\ell, m, n) \mid \ell, m, n \in \mathbb{N}, \ell + m = n\}$
$f^{\mathcal{A}}(x, y) = 0$

(d) Any structure which interprets P as a relation that is not transitive will satisfy F_4.

Consider $U_{\mathcal{A}} = \{0, 1, 2 \ldots\}$ and $P^{\mathcal{A}} = \{\langle m, n\rangle \mid m < n\}$. This structure will falsify F_4. Since $<$ is transitive and irreflexive, it satisfies $[\forall x \forall y \forall z[(P(x, y) \wedge P(y, z)) \rightarrow P(x, z)] \wedge \forall x \neg P(x, x)]$. Additionally, since every natural number has a number larger than it, this structure satisfies $\forall x \exists y P(x, y)$. Hence this structure falsifies $\exists x \forall y \neg P(x, y)$.

(e) A structure that satisfies $(\forall x P(x) \leftrightarrow \forall x Q(x))$ will satisfy F_5. This can be done by setting $U_{\mathcal{A}} = P^{\mathcal{A}} = Q^{\mathcal{A}} = \{1\}$.

For a structure $\mathcal{A}$ to falsify F_5, it must satisfy $\exists x P(x) \leftrightarrow \exists x Q(x)$ and falsify $\forall x P(x) \leftrightarrow \forall x Q(x)$. To meet the latter constraint, it must satisfy one of $\forall x P(x)$ or $\forall x Q(x)$ but not the other. So let's say $\mathcal{A}(\forall x P(x)) = 1$. That is, $P^{\mathcal{A}} = U_{\mathcal{A}}$. Then we must have $\mathcal{A}(\forall x Q(x)) = 0$, i.e., $Q^{\mathcal{A}} \neq U_{\mathcal{A}}$. Note $\mathcal{A}(\exists x P(x)) = 1$ since $\mathcal{A}(\forall x P(x)) = 1$. Hence, in order for $\mathcal{A}$ to satisfy $\exists x P(x) \leftrightarrow \exists x Q(x)$, we must have $\mathcal{A}(\exists x Q(x)) = 1$, i.e., $Q^{\mathcal{A}} \neq \phi$.

All of these requirements are met by letting $U_{\mathcal{A}} = \{1, 2\} = P^{\mathcal{A}}$ and $Q^{\mathcal{A}} = \{1\}$.