THE SPECTRAL SEQUENCE OF A SPLIT EXTENSION AND THE COHOMOLOGY OF AN EXTRASPECIAL GROUP OF ORDER p^3 AND EXPONENT p

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Abstract. Let (E_r, d_r) be the LHS spectral sequence associated to a split extension $1 \to H \to G \to G/H \to 1$ of finite groups with coefficients in a field k. We prove a version of a theorem of Charlap and Vasquez which gives an explicit formula for d_2. We then apply this to the case where p is an odd prime, k has characteristic p, G is extraspecial of order p^3 and exponent p, and H is elementary abelian of order p^2. We calculate the terms of the spectral sequence in this case and prove $E_3 = E_\infty$ (and if $p = 3$, $E_2 = E_\infty$).

1. Introduction

The Lyndon-Hochschild-Serre spectral sequence associates to any group extension

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1 \tag{1}$$

and field k a sequence (E_r, d_r) $(r \geq 0)$ of differential bigraded k-algebras such that $E_{r+1} = H(E_r, d_r)$. One has $E_2 \cong H^*(Q, H^*(H, k))$, and the sequence converges to the graded object associated to a certain filtration of $H^*(G, k)$ (Hochschild-Serre [7]). The most common case considered in the literature is where (1) is a central extension, for then Q acts trivially on $H^*(H, k)$ and the universal coefficient theorem implies $E_2 \cong H^*(Q, k) \otimes H^*(H, k)$. But if the extension is not central then Q does not necessarily act trivially, and even E_2 can be quite complicated. Nevertheless, the non-central case is often quite interesting and can sometimes offer computational advantage. In particular if the extension is split and H is abelian then the few examples which have been calculated tend to suggest that the spectral sequence in this case converges quite rapidly. But in general, very little is known about the spectral sequence of a split extension.

The mod-2 cohomology rings of the extraspecial 2-groups were calculated long ago by Quillen [14], but the corresponding problem for p

Date: January 17, 1995.

odd remains largely unsolved. Here we analyze the mod-p LHS spectral sequence arising from the extraspecial group of order p^3 and exponent p presented as a split extension (p an odd prime). The integral cohomology of this group was calculated by Lewis [11], and the mod-p cohomology ring, which is quite complicated, was calculated by Leary [8, 9, 10]. Benson and Carlson [3] have also produced an algebraic version of Leary's proof. However, both proofs involve spectral sequences which converge at approximately the E_{2p} page. In contrast, we show here that for the split extension, $E_3 = E_\infty$ (and if $p = 3$, $E_2 = E_\infty$).

The key to our calculation is a theorem of Charlap and Vasquez [4, 5] which allows one to calculate d_2 in a variety of settings. We state a slightly refined version of their theorem which can be easily applied to our case, and give a simplified proof. We next determine the E_2 term for the split extension described above, and calculate the differentials d_2 using the Charlap-Vasquez theorem. Finally, we prove that $E_3 = E_\infty$ (and if $p = 3$, $E_2 = E_\infty$) by considering automorphisms of the group extension.

2. A theorem of Charlap and Vasquez

Let p be a prime and k a field of characteristic p. Assume (1) is a split extension of finite groups, and fix a splitting once and for all, so that we may identify Q with a subgroup of G, and $G = HQ$. Let M be a kG-module (all modules are assumed to be finitely generated left modules) such that M is semisimple as a kH-module. We construct the LHS spectral sequence as follows. Let $X \to k$ be a kG-projective resolution, $Y \to k$ the kQ-bar resolution. Let

$$E_0 = \mathrm{Hom}_{kQ}(Y, \mathrm{Hom}_{kH}(X, M))$$

and define differentials $d' : E_0^{r,s} \to E_0^{r+1,s}$ and $d'' : E_0^{r,s} \to E_0^{r,s+1}$ by $d'(f) = f \circ \partial$ and $d''(f)(y) = (-1)^r f(y) \circ \partial$, and let $\{(E_r, d_r) \mid r \geq 0\}$ be the spectral sequence arising from the double complex (E_0, d'', d').

Let $P \to k$ be the minimal kH-projective resolution. Recall that the image of the differential on P is contained in $\mathrm{rad}_{kH}(P)$, so since $\mathrm{rad}_{kH}(M) = 0$, we may identify $H^*(H, M)$ with $\mathrm{Hom}_{kH}(P, M)$, and then identify

$$E_1 = \mathrm{Hom}_{kQ}(Y, \mathrm{Hom}_{kH}(P, M)).$$

How does Q act on $\mathrm{Hom}_{kH}(P, M)$ under these identifications? Even though Q does not necessarily act on P in a manner consistent with the action of H, it does "act up to homotopy". To make this precise, for each $\sigma \in Q$, the Comparison Theorem (cf. Theorem 2.4.2 and the remark following it in Benson [2]) guarantees the existence of a kH-chain map $A(\sigma) : P \to P^\sigma$ which commutes with the augmentation.

Here, $P^\sigma = P$ as k-complexes but has H-action $h.x = \sigma h \sigma^{-1} x$ ($h \in H, x \in P$). It is not to difficult to see that the action of Q is given by

$$\sigma.F = \sigma \circ F \circ A(\sigma^{-1}), \qquad \sigma \in Q,\ F \in \mathrm{Hom}_{kH}(P, M),$$

where, by a slight abuse of notation, we let σ also denote the automorphism induced by σ on M.

For $\sigma, \tau \in Q$, the uniqueness part of the Comparison Theorem implies that $A(\sigma\tau)$ is chain homotopic to $A(\sigma)A(\tau)$, i.e. there exists $U(\sigma,\tau) \in \mathrm{Hom}_{kH}(P, P^{\sigma\tau})_1$ such that

$$\partial U(\sigma,\tau) + U(\sigma,\tau)\partial = A(\sigma\tau) - A(\sigma)A(\tau). \tag{2}$$

The pair (A, U) is called a *Q-system for the extension* (1). Charlap and Vasquez's theorem reduces the problem of calculating d_2 to the calculation of a Q-system. To see how this is done, first define a set of maps $D_2 : E_1^{r,s} \to E_1^{r+2,s-1}$ by

$$D_2(f)[\sigma_1|\cdots|\sigma_{r+2}] = \sigma_1\sigma_2 \circ f[\sigma_3|\cdots|\sigma_{r+2}] \circ U(\sigma_2^{-1}, \sigma_1^{-1}).$$

Theorem 1. *(Charlap-Vasquez) Suppose $\zeta \in E_2^{r,s}$ ($r \geq 0, s \geq 1$) is represented by the Q-cocycle $f \in E_1^{r,s}$. Then $d_2(\zeta)$ is represented by $(-1)^r D_2(f)$.*

Proof. We first show that $D_2(f)$ is a Q-cocycle (i.e., is in the kernel of d_1). Indeed, since f is a Q-cocycle,

$$D_2(f) \circ \partial([\sigma_1|\cdots|\sigma_{r+3}]) = \sigma_1\sigma_2\sigma_3 \circ f[\sigma_4|\cdots|\sigma_{r+3}] \circ \phi(\sigma_3^{-1}, \sigma_2^{-1}, \sigma_1^{-1}), \tag{3}$$

where for $\rho, \sigma, \tau \in Q$,

$$\phi(\rho,\sigma,\tau) = U(\rho,\sigma)A(\tau) - U(\rho,\sigma\tau) + U(\rho\sigma,\tau) - A(\rho)U(\sigma,\tau).$$

Using (2), one checks that $\phi(\rho,\sigma,\tau) : P \to P^{\rho\sigma\tau}[-1]$ is a chain map (if C is a complex then $C[n]$ denotes the complex with $C[n]_i = C_{i-n}$ and with differentials given by multiplyling the differentials in C by $(-1)^n$.) The Comparison Theorem then implies that it is chain homotopic to zero; hence its image is contained in $\mathrm{rad}_{kH}(P^{\rho\sigma\tau}) = \mathrm{rad}_{kH}(P)$, and (3) is zero, as required.

An equally straightforward argument shows that the element of $E_2^{r+2,s-1}$ defined by $D_2(f)$ is independent of the choice of Q-system (A, U). We can therefore prove the theorem using any Q-system we want.

Since P is the minimal resolution, there exist kH-chain maps $\iota : P \to X$ and $\pi : X \to P$, commuting with the augmentations, such that $\pi\iota = 1$. Let $W = \mathrm{Ker}(\pi)$, so $X = \iota(P) \oplus W$. W is a kH-projective resolution of 0, hence is kH-contractible. Take any contracting homotopy of W and extend it to a map $s \in \mathrm{Hom}_{kH}(X, X)_1$ by setting $s(x) = 0$ for

$x \in \iota(P)$. Then $\partial s + s\partial = 1 - \iota\pi$ and $s\iota = \pi s = 0$. Now define $A(\sigma) = \pi\sigma\iota$ and $U(\sigma, \tau) = \pi\sigma s\tau\iota$ $(\sigma, \tau \in Q)$. Since

$$\partial U(\sigma,\tau)+U(\sigma,\tau)\partial = \pi\sigma(\partial s+s\partial)\tau\iota = \pi\sigma(1-\iota\pi)\tau\iota = A(\sigma\tau)-A(\sigma)A(\tau),$$

(A, U) is a Q-system.

We now calculate $d_2(\zeta)$. Clearly, ζ is represented on $E_0^{r,s}$ by the map

$$x^{r,s} : [\sigma_1|\cdots|\sigma_r] \mapsto f[\sigma_1|\cdots|\sigma_r] \circ \pi$$

(but note: this defines $x^{r,s}$ on a kQ-basis of Y_r; it is *not* necessarily the case that $x^{r,s}(y) = f(y) \circ \pi$ for all $y \in Y_r$). Hence

$$\begin{aligned} d'(x^{r,s})[\sigma_1|\cdots|\sigma_{r+1}] &= \sigma_1 \circ f[\sigma_2|\cdots|\sigma_{r+1}] \circ \pi\sigma_1^{-1} \\ &\qquad +(-f[\sigma_1\sigma_2|\cdots|\sigma_{r+1}] + \cdots \pm f[\sigma_1|\cdots|\sigma_r]) \circ \pi. \end{aligned}$$

Notice that in general, if $F : X \to M$ is a coboundary, then $F \circ \iota = 0$, hence

$$F = F \circ (1 - \iota\pi) = F \circ (\partial s + s\partial) = (Fs) \circ \partial.$$

It follows that if we define $x^{r+1,s-1} \in E_0^{r+1,s-1}$ by

$$\begin{aligned} x^{r+1,s-1}[\sigma_1|\cdots|\sigma_{r+1}] &= (-1)^r d'(x^{r,s})[\sigma_1|\cdots|\sigma_{r+1}] \circ s \\ &= (-1)^r \sigma_1 \circ f[\sigma_2|\cdots|\sigma_{r+1}] \circ \pi\sigma_1^{-1}s \end{aligned}$$

(using $\pi s = 0$), then $d'(x^{r,s}) + d''(x^{r+1,s-1}) = 0$. By definition, $d_2(\zeta)$ is represented on E_0 by $z = d'(x^{r+1,s-1})$, and

$$z[\sigma_1|\cdots|\sigma_{r+2}] = (-1)^r\sigma_1\sigma_2 \circ f[\sigma_3|\cdots|\sigma_{r+2}] \circ \pi\sigma_2^{-1}s\sigma_1^{-1} + (\cdots) \circ s.$$

Using the fact that $s\iota = 0$, we see that the image of z in E_1 is the map

$$[\sigma_1|\cdots|\sigma_{r+2}] \mapsto (-1)^r\sigma_1\sigma_2 \circ f[\sigma_3|\cdots|\sigma_{r+2}] \circ \pi\sigma_2^{-1}s\sigma_1^{-1}\iota,$$

completing the proof. □

3. The cyclic case

It is unfortunate that Theorem 1 is formulated using the bar resolution Y, since any kQ-projective resolution can be used to construct the LHS spectral sequence. However, at least in a special case, we can find a similar expression for d_2 using the minimal resolution in place of Y.

As before, let p be a prime and k a field of characteristic p, but now let

$$1 \longrightarrow H \longrightarrow G = H \rtimes B \longrightarrow B \longrightarrow 1$$

be a split extension of finite groups with $B = \langle b \mid b^p = 1\rangle$. Let $Z \to k$ be the minimal kB-resolution; say $Z_n = kBe_n$ $(n \geq 0)$, $\partial(e_n) = N(b)e_{n-1}$

(if $n > 0$ is even), where $N(b) = \sum_{i=1}^{p} b^i$, and $\partial(e_n) = (b-1)e_{n-1}$ (if n is odd). It follows that for any kB-module V,

$$H^r(B,V) \cong \begin{cases} \mathrm{soc}_{kB}(V) & \text{if } r = 0 \\ \mathrm{soc}^{p-1}_{kB}(V)/\mathrm{rad}_{kB}(V) & \text{if } r \text{ is odd} \\ \mathrm{soc}_{kB}(V)/\mathrm{rad}^{p-1}_{kB}(V) & \text{if } r > 0 \text{ is even.} \end{cases} \tag{4}$$

For the general definitions of "soc" and "rad" see Alperin [1]. In this case, $\mathrm{soc}_{kB}(V)$ is just V^B, the set of invariant elements of V under the action of B, $\mathrm{soc}^{p-1}_{kB}(V)$ is the kernel of $N(b)$ on V, $\mathrm{rad}_{kB}(V)$ is the image of $b-1$, and $\mathrm{rad}^{p-1}_{kB}(V)$ is the image of $N(b)$.

Let M be a kG-module on which H acts trivially, $X \to k$ a kG-resolution, $P \to k$ the minimal kH-resolution, and $\{(E_r, d_r)\}$ the spectral sequence of the double complex $E_0 = \mathrm{Hom}_{kB}(Z, \mathrm{Hom}_{kH}(X, M))$.

Corollary 2. *Let $\alpha : P \to P^{b^{-1}}$ be a kH-chain map commuting with the augmentation, and $\upsilon \in \mathrm{Hom}_{kH}(P,P)_1$ a map satisfying $\partial\upsilon + \upsilon\partial = 1 - \alpha^p$. Suppose $\zeta \in E_2^{r,s}$ ($r \geq 0$, $s \geq 1$) is represented by $f \in \mathrm{Hom}_{kH}(P_s, M)$. Then $d_2(\zeta)$ is represented by $(-1)^r f \circ \upsilon$.*

Proof. Define, for $0 \leq i, j \leq p-1$,

$$A(b^{-i}) = \alpha^i, \qquad U(b^{-i}, b^{-j}) = \begin{cases} 0 & \text{if } i + j < p \\ \upsilon\alpha^{i+j-p} & \text{otherwise.} \end{cases}$$

It follows that (A, U) is a B-system.

We next construct explicit comparisons between the minimal resolution Z and the bar resolution Y. Define a kB-chain map $\theta : Y \to Z$ by setting, for $n \geq 0$ and $0 \leq i_1, \ldots, i_{2n+1} \leq p-1$,

$$\begin{aligned} \theta[b^{i_1}|b^{i_2}|\cdots|b^{i_{2n}}] &= \begin{cases} e_{2n} & \text{if } i_{2j-1} + i_{2j} \geq p \text{ for all } 1 \leq j \leq n \\ 0 & \text{otherwise} \end{cases} \\ \theta[b^{i_1}|b^{i_2}|\cdots|b^{i_{2n+1}}] &= \begin{cases} \sum_{i=0}^{i_1-1} b^i e_{2n+1} & \text{if } i_{2j} + i_{2j+1} \geq p \text{ for all } 1 \leq j \leq n \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{5}$$

(the empty sum is understood to be 0). Define a kB-chain map $\phi : Z \to Y$ by $\phi(e_0) = []$, $\phi(e_1) = [b]$, and for $n \geq 1$,

$$\begin{aligned} \phi(e_{2n}) &= \sum_{0 \leq i_1, \ldots, i_n \leq p-1} [b^{i_1}|b|\cdots|b^{i_n}|b] \\ \phi(e_{2n+1}) &= \sum_{0 \leq i_1, \ldots, i_n \leq p-1} [b|b^{i_1}|\cdots|b|b^{i_n}|b]. \end{aligned} \tag{6}$$

One can check that both (5) and (6) actually define chain maps.

Assume $n \geq 0$ and $r = 2n$ (the case where r is odd is handled similarly). By (5), ζ is represented via the bar resolution by

$$[b^{i_1}|\cdots|b^{i_{2n}}] \mapsto \begin{cases} f & \text{if } i_{2j-1}+i_{2j} \geq p \text{ for all } 1 \leq j \leq n \\ 0 & \text{otherwise,} \end{cases}$$

so by Theorem 1, $d_2(\zeta)$ is represented by

$$\gamma : [b^{i_1}|\cdots|b^{i_{2n+2}}] \mapsto \begin{cases} b^{i_1+i_2} \circ f \circ U(b^{-i_2}, b^{-i_1}) & \text{if } i_{2j-1}+i_{2j} \geq p \text{ for all } 2 \leq j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

And

$$\gamma(\phi(e_{2n+2})) = \sum_{i=0}^{p-1} \gamma[b^i|b|b^{p-1}|b|\cdots|b^{p-1}|b] = \sum_{i=0}^{p-1} b^{i+1} \circ f \circ U(b^{-1}, b^{-i}) = f \circ \upsilon,$$

completing the proof. □

Of course if the map α (or υ) is defined only on degrees 0 through n (for some $n > 0$), then it can always be extended to a map on all of P. Hence to calculate d_2 on $E_2^{r,s}$ for $s \leq n$, one really only needs to know α and υ on degrees $0, \ldots, n$.

4. The calculation of E_2

Let p be an odd prime, k a field of characteristic p, and

$$G = \langle a, b, c \mid c = [a, b],\ a^p = b^p = [a, c] = [b, c] = 1 \rangle,$$

where $[g, h] = g^{-1}h^{-1}gh$. G is the (unique, up to isomorphism) extraspecial group of order p^3 and exponent p. Let $H = \langle a, c \rangle$ and $B = \langle b \rangle$. We consider the LHS spectral sequence (E_r, d_r) arising from the split extension $1 \to H \to G \to B \to 1$.

Our first task is to analyze $E_2 \cong H^*(B, H^*(H, k))$. Our approach is to determine the kB-module structure of $H^*(H, k)$. Recall that by Jordan canonical form, there are, up to isomorphism, p indecomposable kB-modules (cf. [1]). We denote these by J_i $(1 \leq i \leq p)$, where J_i has dimension i.

Define $\alpha_1, \beta_1 \in H^1(H, k) \cong \mathrm{Hom}(H, k)$ by $\alpha_1(a^i c^j) = i$, and $\beta_1(a^i c^j) = j$. Let $\alpha_2 = \delta(\alpha_1)$ and $\beta_2 = \delta(\beta_1)$, where δ, as always, denotes the appropriate Bockstein homomorphism. Then $H^*(H, k)$ is the graded-commutative ring[1] $k[\alpha_1, \beta_1, \alpha_2, \beta_2]$, and the action of B on $H^*(H, k)$ is given by $b(\alpha_i) = \alpha_i$ and $b(\beta_i) = \alpha_i + \beta_i$ $(i = 1, 2)$.

[1]All graded rings presented with generators and relations will be assumed to be *graded-commutative*, i.e. $xy = (-1)^{\deg(x)\deg(y)}yx$, for all homogeneous elements x, y.

The product on E_2 can be calculated using the identifications of (4) in the following way: if $\chi \in H^s(H,k)$ represents $x \in E_2^{r,s}$ and $\zeta \in H^{s'}(H,k)$ represents $y \in E_2^{r',s'}$, then $(-1)^{r's}xy \in E_2^{r+r',s+s'}$ is represented by

$$\begin{cases} \chi\zeta & \text{if } r \text{ or } r' \text{ is even} \\ \displaystyle\sum_{0\le i<j<p} b^i(\chi)b^j(\zeta) & \text{if } r \text{ and } r' \text{ are odd.} \end{cases} \tag{7}$$

Now let W be the graded k-submodule of $H^*(H,k)$ spanned by $\{\alpha_2^i\beta_2^j \mid i \ge 0, 0 \le j < p\}$. W is a kB-submodule of $H^*(H,k)$, and multiplication by $\alpha_2 : W^n \to W^{n+2}$ is an injective kB-homomorphism for $n \ge 0$ and an isomorphism for $n \ge 2(p-1)$. Furthermore, $W^{2i} \cong J_{i+1}$ $(i \le p-1)$ and $\mathrm{rad}_{kB}(W)$ is spanned by $\{\alpha_2^i\beta_2^j \mid i \ge 1, 0 \le j < p\}$ (cf. [1, pp. 14–16]).

Set $N = k[\alpha_1,\beta_1] \otimes W$, and $x_{2p} = \prod_{i=0}^{p-1}(\beta_2 + i\alpha_2)$. It is easy to see that x_{2p} is B-invariant and that $k[\alpha_2,\beta_2] = k[x_{2p}] \otimes W$. It follows that

$$H^*(H,k) = k[\alpha_1,\beta_1] \otimes k[\alpha_2,\beta_2] = k[x_{2p}] \otimes N.$$

Moreover, since $k[x_{2p}]$ is a trivial kB-module,

$$\mathrm{soc}^i_{kB}(H^*(H,k)) = k[x_{2p}] \otimes \mathrm{soc}^i_{kB}(N), \qquad i \ge 0, \tag{8}$$

and (8) holds with "soc" replaced by "rad" as well. In particular, the identifications given in (4) yield isomorphisms of graded k-modules

$$E_2^{r,*} \cong \begin{cases} k[x_{2p}] \otimes \mathrm{soc}_{kB}(N) & \text{if } r = 0 \\ k[x_{2p}] \otimes \mathrm{soc}^{p-1}_{kB}(N)/\mathrm{rad}_{kB}(N) & \text{if } r \text{ is odd} \\ k[x_{2p}] \otimes \mathrm{soc}_{kB}(N)/\mathrm{rad}^{p-1}_{kB}(N) & \text{if } r > 0 \text{ is even.} \end{cases} \tag{9}$$

It remains to analyze the structure of N. The referee has pointed out that the structure of $\mathrm{soc}_{kB}(N)$ was determined by Minh [12, Theorem 2.4] (though the generating set given there is not minimal) and Mùi [13, Theorem 5.6]; we compute this structure directly, using representation theory.

Proposition 3. (i) $\mathrm{soc}_{kB}(N) = k[x_1,x_2,y_2,x_3]/(y_2^2, x_1y_2, y_2x_3, x_1x_3 + x_2y_2)$,
where $x_1 = \alpha_1$, $x_2 = \alpha_2$, $y_2 = \alpha_1\beta_1$, *and* $x_3 = \alpha_1\beta_2 - \beta_1\alpha_2$.

(ii) $\mathrm{rad}_{kB}(N^1) = k\alpha_1$, *and for* $1 \le i < p$,

$$\{\alpha_2^j\beta_2^{i-j} \mid 1 \le j \le i\} \cup \{\alpha_1\beta_1\alpha_2^j\beta_2^{i-j-1} \mid 1 \le j < i\}$$

is a basis for $\mathrm{rad}_{kB}(N^{2i})$, *and*

$$\{\alpha_1\alpha_2^j\beta_2^{i-j} \mid 1 \le j \le i\} \cup \{\beta_1\alpha_2^j\beta_2^{i-j} \mid 2 \le j \le i\} \cup \{\alpha_1\beta_2^i + i\beta_1\alpha_2\beta_2^{i-1}\}$$

is a basis for $\mathrm{rad}_{kB}(N^{2i+1})$.

(iii) *For each $s \geq 0$, the following chart gives the kB-module structure of N^s, a basis for $\mathrm{soc}_{kB}^{p-1}(N^s)/\mathrm{rad}_{kB}(N^s)$, and a basis for $\mathrm{soc}_{kB}(N^s)/\mathrm{rad}_{kB}^{p-1}(N^s)$:*

	N^s	$\mathrm{soc}^{p-1}/\mathrm{rad}$	$\mathrm{soc}/\mathrm{rad}^{p-1}$
$s \geq 2p-1$	$J_p \oplus J_p$	$\emptyset$	$\emptyset$
$s = 2p-2$	$J_{p-1} \oplus J_p$	$[\alpha_1\beta_1\beta_2^{p-2}]$	$[x_2^{p-2}y_2]$
$s = 2p-3$	$J_{p-2} \oplus J_p$	$[\alpha_1\beta_2^{p-2}]$	$[x_2^{p-3}x_3]$
$s = 2i+1$ *	$J_i \oplus J_{i+2}$	$[\alpha_1\beta_2^i]$, $[\beta_1\beta_2^i]$	$x_1x_2^i$, $x_2^{i-1}x_3$
$s = 2i$ **	$J_i \oplus J_{i+1}$	$[\alpha_1\beta_1\beta_2^{i-1}]$, $[\beta_2^i]$	x_2^i, $x_2^{i-1}y_2$
$s = 1$	J_2	$[\beta_1]$	x_1
$s = 0$	J_1	1	1

* $1 \leq i \leq p-3$ ** $1 \leq i \leq p-2$

Proof. For $1 < i < p$, $J_2 \otimes J_i \cong J_{i-1} \oplus J_{i+1}$ (cf. [1, p. 50]) and $J_2 \otimes J_p \cong J_p \oplus J_p$ since J_p is projective. This, together with the known structure of W, gives the kB-module structure of N.

Next, we prove (i). One checks that x_1, x_2, y_2, and x_3 are fixed by b, satisfy the relations, and that the subring they generate is contained in N. On the other hand, the graded-commutative ring S described abstractly by these generators and relations satisfies $S^1 = kx_1$, and, for $i \geq 1$,

$$\begin{aligned} S^{2i} &= kx_2^i \oplus kx_2^{i-1}y_2 \\ S^{2i+1} &= kx_1x_2^i \oplus kx_2^{i-1}x_3. \end{aligned} \tag{10}$$

By the known structure of N, S has the same dimension as $\mathrm{soc}_{kB}(N)$ in each degree. Hence $S \cong \mathrm{soc}_{kB}(N)$, and we have established (i).

The expression for $\mathrm{rad}_{kB}(N)$ in even degress follows from the basis for $\mathrm{rad}_{kB}(W)$, since B fixes $\alpha_1\beta_1$. To obtain the expression for odd degrees, one computes $(b-1)N$ explicitly; the last term comes from $(b-1)\beta_1\beta_2^i$.

To fill in the chart for $s = 2p-3$, observe that $N(b)\alpha_1\beta_2^{p-2} = 0$ and $x_1x_2^{p-2} = -N(b)\beta_1\beta_2^{p-2}$. For $s = 2p-2$, $N(b)\alpha_1\beta_1\beta_2^{p-2} = 0$ and $x_2^{p-1} = -N(b)\beta_2^{p-1}$. The proof of the rest of (iii) is routine. □

Now define $\gamma_1 \in H^1(B,k) \cong E_2^{1,0}$ by $\gamma_1(b^i) = i$, and let $\gamma_2 = \delta(\gamma_1)$. Let

$$z_{2j} = [\beta_1\beta_2^{j-1}] \in E_2^{1,2j-1}, \quad z_{2j+1} = [\beta_2^j] \in E_2^{1,2j}, \qquad 1 \leq j \leq p-2,$$

set $w_n = -x_1z_{n-1}$ $(3 \leq n \leq 2p-2)$ and $w_{2p-1} = y_2z_{2p-3}$. Notice that w_{2i} is represented by $\alpha_1\beta_2^{i-1}$ and w_{2i+1} by $\alpha_1\beta_1\beta_2^{i-1}$.

Corollary 4. (i) *Multiplication by $x_{2p} : E_2^{r,s} \to E_2^{r,s+2p}$ is an isomorphism for $r \geq 1, s \geq 0$.*

(ii) *Multiplication by* $\gamma_2 : E_2^{r,s} \to E_2^{r+2,s}$ *is surjective for all* $r, s \geq 0$ *and is an isomorphism if* $r \geq 1$ *or* $s \leq 2p-4$.
(iii) $E_2^{0,*} \cong k[x_{2p}] \otimes k[x_1, x_2, y_2, x_3,]/(y_2^2, x_1y_2, y_2x_3, x_1x_3 + x_2y_2)$
(iv) *We have the following bases for* $E_2^{r,s}$*:*

s	0	1	$2i$	$2i+1$
$E_2^{0,s}$	1	x_1	$x_2^i, x_2^{i-1}y_2$	$x_1x_2^i, x_2^{i-1}x_3$

$(1 \leq i \leq p-1)$

s	0	1	$2 \leq s \leq 2p-4$	$2p-3$	$2p-2$	$2p-1$
$E_2^{1,s}$	γ_1	z_2	z_{s+1}, w_{s+1}	w_{2p-2}	w_{2p-1}	$\emptyset$

s	$2p-3$	$2p-2$	$2p-1$
$E_2^{2,s}$	$\gamma_2x_2^{p-3}x_3$	$\gamma_2x_2^{p-2}y_2$	$\emptyset$

(v) E_2 *is generated by* x_1, γ_1, x_2, y_2, γ_2, x_3, x_{2p}, z_2, z_3, $\ldots$, z_{2p-3}.

Proof. As in (9), we may write $E_2^{r,*} \cong k[x_{2p}] \otimes C$, for a graded kB-module C. By Proposition 3(iii), if $r \geq 1$ then C is concentrated in degrees 0 through $2p-1$, proving (i). Now for any kB-module V, multiplication by $\gamma_2 : H^r(B, V) \to H^{r+2}(B, V)$ is just the natural map under the identifications in (4). This is a surjection for $r \geq 0$ and an isomorphism for $r \geq 1$. If $r = 0$ and $s \leq 2p-4$ then Proposition 3(iii) shows that $\mathrm{rad}_{kB}^{p-1}(H^s(H, k)) = 0$, hence the natural map is an isomorphism in this case too, proving (ii). Statements (iii) and (iv) follow directly from Proposition 3 and (9). Now (v) is clear since one can get a basis for $E_2^{r,s}$ for any r, s by starting with one of the bases in (iv) and multiplying by appropriate powers of x_{2p} and γ_{2p}, according to (i) and (ii). $\square$

5. Differentials

We now determine all the differentials in the spectral sequence. The data is presented diagrammatically in Figure 1; the integer to the upper right of each point there indicates k-dimension.

Theorem 5. (i) $x_1, \gamma_1, x_2, y_2, z_2, \gamma_2, x_3, z_3, x_{2p}$ *all live to* E_∞. *In particular, if* $p = 3$ *then* $E_2 = E_\infty$.
(ii) *If* $p > 3$ *then in addition*

$$\begin{array}{rclcll} d_2(z_{2i+2}) & = & i\gamma_2 w_{2i+1} & = & -ix_1\gamma_2 z_{2i}, & 1 \leq i \leq p-3, \\ d_2(z_{2i+1}) & = & i\gamma_2 w_{2i} & = & -ix_1\gamma_2 z_{2i-1}, & 2 \leq i \leq p-2. \end{array}$$

Proof. Let $Q' \to k$ be the minimal $k\langle a\rangle$-resolution, where $Q'_n = k\langle a\rangle e'_n$, and $Q'' \to k$ the minimal $k\langle c\rangle$-resolution, with $Q''_n = k\langle c\rangle e''_n$. Let $P = Q' \otimes Q''$, so $P \to k$ is the minimal kH-resolution. For $n, k \in \mathbb{Z}$, set $e_k^n = e'_{n-k} \otimes e''_k$ if $0 \leq k \leq n$ and 0 otherwise. Hence $e_0^n, e_1^n, \ldots, e_n^n$ is a kH-basis for P_n $(n \geq 0)$. Also, write $\bar{e}_k^n$ for the image of e_k^n

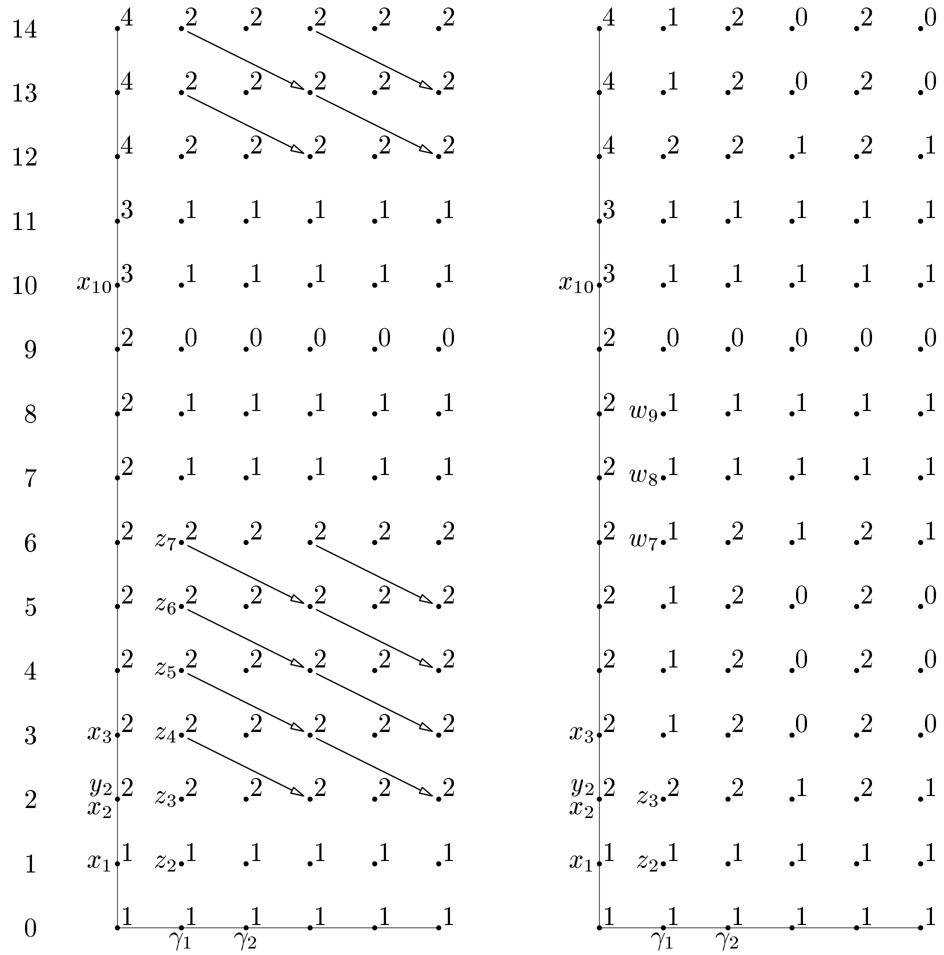

FIGURE 1. E_2 and $E_3 = E_\infty$ for extraspecial 5^3, exponent 5.

in $P_n/\mathrm{rad}_{kH}(P_n) \cong H_n(H,k)$. It is not hard to see that the duality between $H^*(H,k)$ and $H_*(H,k)$ is given by

$$(\alpha_1^{i_1}\beta_1^{j_1}\alpha_2^{i_2}\beta_2^{j_2})^* = \bar{e}_{j_1+2j_2}^{i_1+j_1+2i_2+2j_2}, \qquad 0 \le i_1, j_1 \le 1,\ 0 \le i_2, j_2. \tag{11}$$

Define $\rho, \kappa \in kH$ by $\rho = \sum_{0 \le j \le i < p} a^i c^j$, and $\kappa = \sum_{i=0}^{p-1} i a^i$. Define maps $\alpha \in \mathrm{Hom}_{kH}(P, P^{b^{-1}})_0$ and $\upsilon \in \mathrm{Hom}_{kH}(P,P)_1$ by

$$\begin{array}{rclrcl}
\alpha e^{2i}_{2j} & = & \displaystyle\sum_{k\ge j} \binom{k}{j} (e^{2i}_{2k} - \rho e^{2i}_{2k+1}), & \upsilon e^{2i}_{2j} & = & -(j+1)\kappa e^{2i+1}_{2j+2}, \\
\alpha e^{2i}_{2j+1} & = & \displaystyle\sum_{k\ge j} \binom{k}{j} c e^{2i}_{2k+1}, & \upsilon e^{2i}_{2j+1} & = & -(j+1) e^{2i+1}_{2j+3}, \\
\alpha e^{2i+1}_{2j} & = & \displaystyle\sum_{k\ge j} \binom{k}{j} (c e^{2i+1}_{2k} + e^{2i+1}_{2k+1}), & \upsilon e^{2i+1}_{2j} & = & -(j+1) e^{2i+2}_{2j+2}, \\
\alpha e^{2i+1}_{2j+1} & = & \displaystyle\sum_{k\ge j} \binom{k}{j} e^{2i+1}_{2k+1}, & \upsilon e^{2i+1}_{2j+1} & = & -(j+1)\kappa e^{2i+2}_{2j+3},
\end{array} \tag{12}$$

for $0 \le j \le i < p$ (of course the sums are finite since $e^n_m = 0$ for $m > n$). It is straightforward, though a bit tedious, to check that α is a chain map, and that $\partial\upsilon + \upsilon\partial = 1 - \alpha^p$. (The reader who checks this will find the following relations in kH useful: $\rho(c-1) = N(a) - cN(ac)$, $\rho(a-1) = N(ac) - N(c)$, and $\kappa(1-a) = N(a)$.) Hence we can apply Corollary 2.

We first calculate d_2 on $z_{2i+2} = [\beta_1\beta_2^i]$ $(1 \le i \le p-3)$. According to (11), the map $f : P_{2i+1} \to k$, defined by $f(e^{2i+1}_n) = 1$ if $n = 2i+1$ and 0 otherwise, corresponds to $\beta_1\beta_2^i$. Using (12) we see that $f \circ \upsilon : P_{2i} \to k$ sends e^{2i}_n to $-i$ if $n = 2i-1$ and 0 otherwise. Hence $-f\circ\upsilon$ corresponds to $i(\bar{e}^{2i}_{2i-1})^* = i\alpha_1\beta_1\beta_2^{i-1}$, which represents $i\gamma_2 w_{2i+1}$.

Similarly, z_{2i+1} $(1 \le i \le p-2)$ is represented by $f = (\bar{e}^{2i}_{2i})^*$, and $-f \circ \upsilon = i(\bar{e}^{2i-1}_{2i-2})^* = i\alpha_1\beta_2^{i-1}$. If $i \ge 2$ this shows $d_2(z_{2i+1}) = i\gamma_2 w_{2i}$. If $i = 1$, Proposition 3(ii) says $\alpha_1 \in \mathrm{rad}_{kB}(H^1(H,k))$, so $d_2(z_3) = 0$. Finally $y_2 = [\alpha_1\beta_1]$ is represented by $f = (\bar{e}^2_1)^*$, and $f \circ \upsilon = 0$, so $d_2(y_2) = 0$.

Now for a split extension, all differentials into the horizontal edge vanish (cf. [6, Proposition 7.3.2]), so x_1, y_2, z_2, and z_3 live to E_∞. Therefore $\delta(x_1) = x_2$ and

$$\delta(y_2) = \delta(\alpha_1\beta_1) = \delta(\alpha_1)\beta_1 - \alpha_1\delta(\beta_1) = \alpha_2\beta_1 - \alpha_1\beta_2 = -x_3$$

live to E_∞ as well. Since

$$x_{2p} = \prod_{\sigma \in G/H} \sigma^*(\beta_2) = \mathrm{res}_{G\to H} N_{H\to G}(\beta_2),$$

where $N_{H\to G}$ denotes the Evens norm map (cf. Theorem 6.1.1 of [6]), x_{2p} lives to E_∞. Of course we could have used the Charlap-Vasquez theorem for these cases as well, but then we would not get the additional information about the vanishing of the higher differentials. □

Corollary 6. *If $p > 3$ then:*

(i) *Multiplication by $x_{2p} : E_3^{r,s} \to E_3^{r,s+2p}$ is an isomorphism for $r \geq 1, s \geq 0$.*
(ii) *Multiplication by $\gamma_2 : E_3^{r,s} \to E_3^{r+2,s}$ is surjective for $r, s \geq 0$, and is an isomorphism for $r \geq 2, s \geq 0$.*
(iii) *For r even, $E_3^{r,*} = E_2^{r,*}$. For $r = 1, 3$ we have the following k-bases for $E_3^{r,s}$ $(0 \leq s \leq 2p-1)$:*

s	0	1	2	3	4	5	$6 \leq s \leq 2p-2$	$2p-1$
$E_3^{1,s}$	γ_1	z_2	$z_3, \gamma_1 y_2$	$x_1 z_3$	$z_2 x_3$	$x_3 z_3$	w_{s+1}	$\emptyset$

s	0	1	2	$3 \leq s \leq 2p-5$	$2p-4 \leq s \leq 2p-2$	$2p-1$
$E_3^{3,s}$	$\gamma_1\gamma_2$	$\gamma_2 z_2$	$\gamma_2 z_3$	$\emptyset$	$\gamma_2 w_{s+1}$	$\emptyset$

(iv) *E_3 is generated by x_1, γ_1, x_2, y_2, z_2, γ_2, x_3, z_3, x_{2p}, and $w_7, \ldots, w_{2p-1}$.*

Proof. To see that multiplication by γ_2 is surjective, argue in E_2 as follows: suppose $r, s \geq 0$, $y \in E_2^{r+2,s}$ and $d_2(y) = 0$. By Corollary 4(ii), there exists $x \in E_2^{r,s}$ such that $\gamma_2 x = y$. Hence $\gamma_2 d_2(x) = 0$. Since $\gamma_2 : E_2^{r+2,s-1} \to E_2^{r+4,s-1}$ is injective, $d_2(x) = 0$, i.e., x represents an element of E_3. To get injectivity on E_3, suppose $x \in E_2^{r,s}$ $(r \geq 2)$, $d_2(x) = 0$, and $\gamma_2 x = d_2(w)$, for some $w \in E_2^{r,s+1}$. Then $w = \gamma_2 v$, for some $v \in E_2^{r-2,s+1}$. Hence $\gamma_2 x = \gamma_2 d_2(v)$, and since $\gamma_2 : E_2^{r,s} \to E_2^{r+2,s}$ is injective, $x = d_2(v)$, i.e., x represents 0 in E_3, and we have proved (ii). The proof of (i) is entirely similar.

It follows from Theorem 5 that $E_3^{1,s} = kw_{s+1}$ for $3 \leq s \leq 2p-2$. But $w_4 = -x_1 z_3$ and

$$z_2 x_3 = [\beta_1][\alpha_1\beta_2 - \beta_1\alpha_2] = -[\alpha_1\beta_1\beta_2] = -w_5.$$

For $s = 5$, $z_3 x_3 = [\beta_2][\alpha_1\beta_2 - \beta_1\alpha_2] = [\alpha_1\beta_2^2 - \beta_1\alpha_2\beta_2]$. The cosets are in $M/\mathrm{rad}_{kB}(M)$, where $M = H^5(H, k)$. But according to Proposition 3(ii), $\alpha_1\beta_2^2 + 2\beta_1\alpha_2\beta_2$ lies in $\mathrm{rad}_{kB}(M)$, hence $z_2 x_3 = \frac{3}{2}[\alpha_1\beta_2^2] = \frac{3}{2}w_6$, and this is non-zero since $p > 3$. Now the rest of (iii) is obvious, and (iv) follows from (i), (ii), and (iii). ☐

Theorem 7. *If $p > 3$ then $E_3 = E_\infty$. Hence for $p \geq 3$, the Poincaré series for $H^*(G, k)$ is*

$$p_G(t) = \frac{1 + t + 2t^2 + 2t^3 + \sum_{i=4}^{2p-1} t^i}{(1-t)(1-t^{2p})}.$$

Proof. We need to show that $w_7, \ldots, w_{2p-1}$ live to E_∞. Let m be a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$. Hence $m^{p-1} = 1$ but $m^i \neq 1$ for $1 \leq i \leq p-2$. We may also consider m to be in k through the embedding $\mathbb{Z}/p \hookrightarrow k$. Define automorphisms ϕ and ψ of G by $\phi(a^i b^j c^k) = a^i b^{mj} c^{mk}$

and $\psi(a^i b^j c^k) = a^{mi} b^j c^{mk}$. Since these automorphisms preserve H, they induce automorphisms of $H^*(H,k)$ and E_n $(n \geq 2)$, and one can easily check that these induced automorphisms simply multiply each generator by the scalar given in the following chart:

	α_ϵ	β_ϵ	x_1	γ_ϵ	x_2	y_2	x_3	z_{2i}	z_{2i+1}	w_{2i}	w_{2i+1}
ϕ	1	m	1	m	1	m	m	m^{i+1}	m^{i+1}	m^i	m^{i+1}
ψ	m	m	m	1	m	m^2	m^2	m^i	m^i	m^i	m^{i+1}

Now let $n \geq 3$ and assume $d_r = 0$ for $3 \leq r < n$; we show $d_n = 0$. We first consider the case where $n = 2j+1$ is odd $(j \geq 1)$.

Consider $d_n(w_{2i}) \in E_3^{2(j+1),2(i-j)-1}$ $(4 \leq i \leq p-1)$. If $i - j \geq 2$ then employing the basis for E_3 given in Corollary 6, we have

$$d_n(w_{2i}) = \lambda_1 \gamma_2^{j+1} x_1 x_2^{i-j-1} + \lambda_2 \gamma_2^{j+1} x_2^{i-j-2} x_3, \tag{13}$$

for unique $\lambda_1, \lambda_2 \in k$. Applying ϕ to (13) yields

$$\lambda_1(1 - m^{i-j-1}) = 0 = \lambda_2(1 - m^{i-j-2}).$$

Since $1 \leq i-j-1 \leq p-2$, $m^{i-j-1} \neq 1$; hence $\lambda_1 = 0$. If $i - j > 2$ then $\lambda_2 = 0$ as well. If $i - j = 2$, we apply ψ to (13) and obtain $\lambda_2(1 - m^{i-2}) = 0$. Since $2 \leq i-2 \leq p-3$, we have $\lambda_2 = 0$. If $i-j = 1$ then $d_n(w_{2i}) = \lambda \gamma_2^{j+1} x_1$ for a unique $\lambda \in k$. Applying ψ yields $\lambda(1 - m^{i-1}) = 0$, forcing $\lambda = 0$.

Now consider $d_n(w_{2i+1}) \in E_3^{2(j+1),2(i-j)}$ $(3 \leq i \leq p-1)$. Since all differentials into the horizontal edge vanish, we can assume $i - j \geq 1$, so that

$$d_n(w_{2i+1}) = \lambda_1 \gamma_2^{j+1} x_2^{i-j} + \lambda_2 \gamma_2^{j+1} x_2^{i-j-1} y_2, \tag{14}$$

for some $\lambda_1, \lambda_2 \in k$. Applying ψ yields $\lambda_1(1 - m^{j+1}) = 0 = \lambda_2(1 - m^j)$. Since $1 \leq j \leq i-1 \leq p-2$, we have $\lambda_2 = 0$. If $j < p-2$ then $\lambda_1 = 0$ as well. If $j = i-1 = p-2$ then we apply ϕ to (14) and obtain $\lambda_1(1-m) = 0$.

We now consider the case where $n = 2j$ is even $(j \geq 2)$. Since $E_3^{2i+1,s} = 0$ for $i \geq 1$ and $3 \leq s \leq 2p-5$, the only possible nonzero differentials in this case are those that map into $E_3^{*,2}$ or $E_3^{*,1}$. In the former case, we must consider $d_n(w_{2(j+1)}) = \lambda \gamma_2^j z_3 \in E_3^{2j+1,2}$, in the latter, $d_n(w_{2j+1}) = \lambda \gamma_2^j z_2 \in E_3^{2j+1,1}$. In either case, applying ϕ yields $\lambda(1-m) = 0$, hence $\lambda = 0$, completing the inductive step.

To get $p_G(t)$, let L be the subring of E_∞ generated by all of the generators other than x_{2p}. One then has $L^{r,s} = 0$ for $r \geq 1, s \geq 2p$,

$L^{r,s} = E_\infty^{r,s}$ for $s \le 2p-1$, and $E_\infty = k[x_{2p}] \otimes L$. It follows that

$$\sum_{r+s=n} \dim_k(L^{r,s}) = \begin{cases} n+1 & n = 0,1 \\ 4 & n = 2 \\ n+3 & 3 \le n \le 2p-1 \\ 2p+2 & n \ge 2p, \end{cases}$$

so $1-t$ times the Poincaré series for L is $1+t+2t^2+2t^3+\sum_{i=4}^{2p-1} t^i$. $\square$

6. Remarks

(1) One could probably go on from this point to deduce the full ring structure of $H^*(G,k)$, much as in [8], but it is not clear that this yields any new information or insight into the calculation. However, it is reassuring to note that the Poincaré series, as well as the number and the degrees of the generators, agree with the results in [8]!

(2) It would be interesting to know if there is a different way to calculate d_2 in this spectral sequence. We know of no way other than the Charlap-Vasquez technique outlined here.

(3) It is natural to ask if the Charlap-Vasquez formula can be generalized to a formula for the higher differentials d_r $(r \ge 2)$. Indeed, this can be done (cf. [15]), and will be the subject of a future paper.

Acknowledgements

This paper is based on part of the author's Ph.D. dissertation. Funding for this work was provided through an Office of Naval Research Graduate Fellowship and a Sloan Foundation Dissertation Fellowship. I am grateful to my advisor, Jon Alperin, for the advice and encouragement he provided during my time at the University of Chicago. Thanks are also due to Len Evens for many helpful conversations related to this field, and to the referee for several helpful comments.

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