INTEGRAL COHOMOLOGY RING OF $U_3(5)$ AND HS AT THE PRIME 5

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Abstract. In this paper we calculate the localization at the prime 5 of the integral cohomology ring of the 3 dimensional Unitary group over the field of twenty-five elements, and for the Higman-Sims sporadic simple group. For both of these groups the Sylow subgroup is isomorphic to the extraspecial group of height 5 and order 125.

1. Introduction

In this paper we present the 5-part of the integral cohomology ring of the three dimensional unitary group over the field of twenty five elements, $U_3(5)$. The even part of the cohomology has already been computed by Tezuka and Yagita [TY96, Thm 5.9]. In their paper Tezuka and Yagita describe the elements of the even dimensional cohomology for $U_3(5)$ and HS, as well as many other groups. They also describe the relationship between the 5-part of the integral cohomology of $U_3(5)$ and the 5-part of the integral cohomology of several sporadic simple groups including HS, Mcl, Co_3 and Co_2.

In this paper we describe the entire cohomology ring. Also, our description of the ring is much more specific than that of Tezuka and Yagita. We give Poincaré series and module structure for both rings, and from our calculation it is easy to see how each of these cohomology rings relates to the other, as well as to the cohomology ring of their Sylow 5-subgroup.

The Sylow 5-subgroups of both $U_3(5)$ and HS are isomorphic to the extraspecial group of height 5 and order 125, which we will denote 5_+^{1+2}. This group is probably most easily represented as the group of strictly upper triangular matrices inside $SL_3(5)$.

We list our main results below.

Theorem 1.1. *The ring $H^*(U_3(5),\mathbb{Z})_{(5)}$ is generated by:*

Generator	a,b	x,y,z	m,n	p	q	c	r	θ
Degree	*5*	*8*	*14*	*16*	*24*	*25*	*34*	*40*
Additive Order	*5*	*5*	*5*	*5*	*5*	*5*	*5*	*25*

Relations

$$\{a,b,c\}\cdot\{a,b,c\}=0 \qquad \{a,b,c,m,n,p,q,r,x,y,z\}\cdot\{p,q\}=0$$

$$ay=3bz=-bx \qquad by=az+ax \qquad y^2=-x^2-xz$$

$$yz=3xy \qquad z^2=3x^2+2xz$$

$$bm=an \qquad ym=-nx \qquad yn=mx+mz \qquad zn=3nx$$

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$$am^2 = 2cz + cx \qquad an^2 = 3cz + 3cx \qquad bn^2 = bm^2 = 3cy$$

$$cm = ar \qquad cn = br \qquad m^3 = xr + 2zr$$

$$m^2n = 3yr \qquad mn^2 = 3zr - 3xr \qquad n^3 = 3yr$$

$$mr = z\theta \qquad nr = 2y\theta \qquad cr = am\theta + 3bn\theta$$

$$r^2 = m^2\theta + 3n^2\theta$$

This ring has the following module structure:

$$H^*(U_3(5), \mathbb{Z})_{(5)} \cong$$

$$\{\tilde{\mathbb{Z}}_5[x]\langle 1, a, b, y, z, az, m, n, mz, am, an, bn, c, m^2, mn, n^2, cy, cz, r, ar, br, yr, zr, azr\rangle$$

$$\oplus \mathbb{F}_5\langle p, q\rangle\} \otimes \tilde{\mathbb{Z}}_{25}[\theta]$$

and Poincaré series

$$p_{U_3(5)} = \frac{1+2t^5+2t^8+t^{13}+2t^{14}+t^{16}+3t^{19}+t^{22}+t^{25}+3t^{28}-t^{32}+2t^{33}+t^{34}+2t^{39}+2t^{42}+t^{47}}{(1-t^8)(1-t^{40})}$$

Theorem 1.2. *The ring $H^*(HS, \mathbb{Z})_{(5)}$ is generated by:*

Generator	b	x, z	n	p	e	q	f	g	θ
Degree	5	8	14	16	19	24	28	39	40
Additive Order	5	5	5	5	5	5	5	5	25

Relations

$$\{b, e, g\} \cdot \{b, e, g\} = 0 \qquad \{b, e, f, g, n, p, q, x, z\} \cdot \{p, q\} = 0$$

$$bz = 3bx \qquad z^2 = 3x^2 + 2xz \qquad zn = 3nx$$

$$zn^3 = 3xn^3 \qquad n^4 = -x^2\theta - xz\theta$$

$$bn^3 = 3gz + 3gx \qquad gn = 3bx \qquad f^2 = x^2\theta$$

$$ef = gx + 2gz$$

$$bf = bn^2 \qquad ez = gf = -ex - bnx \qquad fz = -fx - n^2x$$

$$en = bn^2 \qquad nf = n^3$$

The ring has the following module structure:

$$H^*(HS, \mathbb{Z})_{(5)} \cong \{\tilde{\mathbb{Z}}_5[x]\langle 1, b, z, n, e, bn, f, n^2, bn^2, g, n^3, gz\rangle \oplus \mathbb{F}_5\langle p, q\rangle\} \otimes \tilde{\mathbb{Z}}_{25}[\theta]$$

and Poincaré series

$$p_{HS} = \frac{1 + t^5 + t^8 + t^{14} + 2t^{19} + t^{26} + 2t^{28} - t^{32} + t^{33} + t^{39} + t^{42} + t^{47}}{(1 - t^8)(1 - t^{40})}$$

2. Preliminaries

For integral cohomology it is convenient to define a functor which is a variation on the symmetric algebra. This is because most elements in our cohomology ring have finite additive order with the exception of the identity element. Let q be a positive integer and $V = \bigoplus_{j\in\mathbb{Z}} V^j$ a graded-free $\mathbb{Z}$-module, that is, a graded $\mathbb{Z}$-module in which each homogeneous component V^j is free. The tensor algebra

$$T(V) = \mathbb{Z} \oplus V \oplus V^{\otimes 2} \oplus \cdots$$

is then a graded ring, in which the homogeneous element $x_1 \otimes \cdots \otimes x_n$, where $x_i \in V^{d(i)}$, has degree $\sum_{i=1}^n d(i)$. The ideal I generated by all qx and $x\otimes y-(-1)^{ij}y\otimes x$ ($x \in V^i$, $y \in V^j$) is homogeneous. Let $S_q(V) = T(V)/I$. Clearly S_q is a functor from the category of graded-free $\mathbb{Z}$-modules to the category of graded-commutative rings.

Define

$$\tilde{\mathbb{Z}}_q[x_1, \ldots, x_n] = S_q(\bigoplus_i \mathbb{Z}x_i).$$

Given a group G and a subgroup H, the *G-stable* elements $z \in H^*(H,\mathbb{Z})_{(p)}$ are those such that

$$Res^H_{H\cap gHg^{-1}}(z) = g \cdot Res^H_{g^{-1}Hg}(z)$$

for all $g \in G$. If H is a Sylow p-subgroup of G, it is well know [Bro82, III.10.3] that the restriction map

$$Res^* : H^*(G,\mathbb{Z})_{(p)} \to H^*(H,\mathbb{Z})_{(p)}$$

is an isomorphism onto the G-stable elements of $H^*(H,\mathbb{Z})_{(p)}$. Note that at a minimum, this implies that the image of the restriction map lies in the invariant subring

$$H^*(H,\mathbb{Z})_{(p)}^{W_G(H)}$$

where $W_G(H) := N_G(H)/H$ as H acts trivially on $H^*(H,\mathbb{Z})_{(p)}$. We call $W_G(H)$ the *Weyl group* of H in G

Proposition 2.1. *The image of the restriction map from the cohomology of $U_3(5)$ to the cohomology of 5_+^{1+2} is exactly*

$$H^*(5_+^{1+2},\mathbb{Z})_{(5)}^{W_{U_3(5)}(5_+^{1+2})}.$$

Proof. We take $G = U_3(5)$ and $H \cong 5_+^{1+2}$, which can be identified as the strictly upper triangular matrices in G.

In order to recognize the image of the restriction map, we must analyze the restrictions from H to intersections with its conjugates. However, in this case if $g \notin N_G(H)$ then $H \cap gHg^{-1}$ is trivial. To see this first note that G acts 2-transitively on conjugates of H [C$^+$85, p. 34], so there are only two orbits of conjugates, namely $\{H\}$ and the set of its non-trivial conjugates. Thus

$$H \cap gHg^{-1} = H \cap g'Hg'^{-1} \quad \forall g, g' \notin N_G(H).$$

Now take the lower triangular matrices in G to be a non-trivial conjugate of H, call this subgroup H'. Since $H \cap H'$ is the trivial subgroup, this completes the proof. $\square$

Thus it remains to calculate the invariants in $H^*(H, \mathbb{Z})_{(5)}$ under the action of $W_G(H) \cong \mathbb{Z}/8$.

3. The ring $H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$

We start with the cohomology of $H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$ as presented in [Lea91, Thm. 3].

Generator	α, β	μ, ν	χ_2	χ_3	χ_4	ζ
Degree	2	3	4	6	8	10
Additive Order	5	5	5	5	5	25

with relations

$$(A_1)\ \alpha\mu = \beta\nu \qquad (A_2)\ \alpha^5\beta = \beta^5\alpha \qquad (A_3)\ \alpha^5\mu = \beta^5\nu$$

$$(B)\ \alpha\chi_2 = \alpha\chi_3 = \beta\chi_2 = \beta\chi_3 = 0$$
$$\mu\chi_2 = \mu\chi_3 = \nu\chi_2 = \nu\chi_3 = 0$$

$$(C)\ \alpha\chi_4 = -\alpha^5, \beta\chi_4 = -\beta^5,\ \mu\chi_4 = -\beta^4\mu,\ \nu\chi_4 = -\alpha^4\nu$$

$$(D_1)\ \chi_i\chi_j = 0 \text{ unless } i = j = 4 \qquad (D_2)\ \chi_4^2 = \alpha^8 + \beta^8 - \alpha^4\beta^4$$
$$(E)\ \mu\nu = \lambda\chi_3\ \lambda \in \mathbb{F}_5^*$$

Note that $(B), (C)$ above allow us to express every element as a sum of polynomials in ζ, χ_i and polynomials in $\zeta, \alpha, \beta, \mu, \nu$.

Lemma 3.1. *A module representation of $H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$ is given by:*

$$H^*(5_+^{1+2}, \mathbb{Z})_{(5)} = \{\tilde{\mathbb{Z}}_5[\alpha]\langle 1, \beta, \beta^2, \beta^3, \beta^4, \nu, \mu, \mu\beta, \mu\beta^2, \mu\beta^3\rangle$$
$$\oplus\tilde{\mathbb{Z}}_5[\beta]\langle \beta^5, \beta^4\mu\rangle \oplus \mathbb{F}_5\langle \chi_2, \chi_3, \chi_4\rangle\} \otimes \tilde{\mathbb{Z}}_{25}[\zeta]$$

Proof. As ζ is not involved in any of the relations the tensor product is clear, therefore we can restrict our attention to monomials without ζ. Note also that with the exception of relation (D_2), all relations are equivalences between monomials. Therefore we deal with χ_4 first. Relation (D_2) shows that χ_4^2 is in the span of the first two direct summands of the module structure. The relations (C) show that multiplying α, β, μ, ν by χ_4 is contained in

$$\tilde{\mathbb{Z}}_5[\alpha](1, \beta, \beta^2, \beta^3, \beta^4, \nu, \mu, \mu\beta, \mu\beta^2, \mu\beta^3) \oplus \tilde{\mathbb{Z}}_5[\beta](\beta^5, \beta^4\mu).$$

Note that this also shows that χ_4^i is in the module structure. Relations (D_1) and (B) show that any multiple of χ_2 and χ_3 with any-other generator are zero. Therefore we have accounted for every polynomial containing any of the χ_i in our module structure.

We now focus on the subring generated by α, β, μ, ν. As stated above we have the following relations derived from the relations given:

$$(A_2) \Rightarrow \quad \alpha^m\beta^n = \alpha^{m+4k}\beta^{n-4k} \text{ where } 1 \le n - 4k \le 4$$

Therefore we have every monomial of the form $\alpha^m\beta^n$.

We have every monomial of the form $\beta^m\nu$ and $\alpha^m\beta^n\mu$ from the relations:

$$(A_1),(A_2) \Rightarrow \quad \beta^m\nu = \beta^{m-1}\alpha\mu = \alpha^{1+4k}\beta^{m-1-4k}\mu,\ k \text{ as above}$$

The remaining monomials are $\alpha^m\nu$ and $\beta^m\mu$ and these are clearly contained in our module structure.

Furthermore, the Poincaré series based on this decomposition agrees with the Poincaré series for $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$ (this may be determined from [Sie96, Thm. 7]):

$$p_{5_+^{1+2}}(t) = \frac{1-t+2t^2+2t^4-t^5+2t^6-t^7+2t^8-t^9+t^{10}}{(1-t)(1-t^{10})}$$

□

Using this module representation, we will determine the invariants under the Weyl group action.

4. Calculation of Invariants

The action under a fixed generator, z, of the Weyl group, $W_{U_3(5)}(5_+^{1+2})$ on $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$ may be determined from [Lea91, Thm. 3]:

$$\begin{array}{ll} \alpha \longmapsto \beta & \beta \longmapsto 2\alpha \\ \mu \longmapsto \nu & \nu \longmapsto 3\mu \\ \chi_2 \longmapsto 4\chi_2 & \chi_3 \longmapsto 2\chi_3 \\ \chi_4 \longmapsto \chi_4 & \zeta \longmapsto -7\zeta \end{array}$$

As the Weyl group is a cyclic group, its characters are all linear. We will write its 5-modular characters as follows: let $\sigma_i(z) = 2^{i/2}$, where z is the distinguished generator described above. As 2 is not a square in $\mathbb{F}_5$ it is clear that this generates a group of order 8 in $\mathbb{F}_{25}^*$.

Let $\mathcal{H}$ be the subring of $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$ generated by everything but ζ. Since the action on ζ is -7ζ (or as 3 when multiplied by something which has additive order 5), and -7 is one of the fourth roots of unity modulo 5, invariants will be of the form

$$\eta_i\zeta,\ \gamma_j^2\zeta^2,\ \delta_k\zeta^3,\ \epsilon_l\zeta^4$$

where $\eta_i,\gamma_j,\delta_k,\epsilon_l \in \mathcal{H}$ and

$$\eta_i \longrightarrow 2\eta_i \qquad \gamma_j \longrightarrow -\gamma_j \qquad \delta_k \longrightarrow 3\delta_k \qquad \epsilon_l \longrightarrow \epsilon_l$$

under the action of of z. Let M_i be the subset of $\mathcal{H}$ generated by all elements, x, such that the action of z is given by

$$x^z = \sigma_i(z)x$$

. From the above we have the following decomposition of $H^*(5_+^{1+2}\!:\!8,\mathbb{Z})_{(5)}$:

$$H^*(5_+^{1+2}\!:\!8,\mathbb{Z})_{(5)} = \{M_0 \oplus M_2\zeta \oplus M_4\zeta^2 \oplus M_6\zeta^3\} \otimes \tilde{\mathbb{Z}}_{25}[\zeta^4]. \tag{4.1}$$

So now we focus our attention on $\mathcal{H}$. From the structure above, we can begin by finding the M_i's. Note that M_0 is a subring of $\mathcal{H}$. In Table 1, we list a basis and the character for the Weyl group acting on $\mathcal{H}$ (up to degree 18).

Lemma 4.2. *For degree greater than nine $H^*(5_+^{1+2},\mathbb{Z})_{(5)} \cong H^{*+8}(5_+^{1+2},\mathbb{Z})_(5)$ as $\mathbb{Z}/8$ modules.*

TABLE 1. Characters up to Degree 18

Degree	Dimension	Basis	Character
0	1	1	σ_0
1	0		
2	2	α, β	$\sigma_1 + \sigma_5$
3	2	μ, ν	$\sigma_3 + \sigma_7$
4	4	$\alpha^2, \alpha\beta, \beta^2, \chi_2$	$2\sigma_2 + \sigma_4 + \sigma_6$
5	3	$\alpha\mu, \beta\mu, \alpha\nu$	$2\sigma_0 + \sigma_4$
6	5	$\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3, \chi_3$	$\sigma_2 + 2\sigma_3 + 2\sigma_7$
7	4	$\alpha^2\mu, \alpha\beta\mu, \beta^2\mu, \alpha^2\nu$	$2\sigma_1 + 2\sigma_5$
8	6	$\alpha^4, \alpha^3\beta, \alpha^2\beta^2, \alpha\beta^3, \beta^4, \chi_4$	$3\sigma_0 + 3\sigma_4$
9	5	$\alpha^3\mu, \alpha^2\beta\mu, \alpha\beta^2\mu, \beta^3\mu, \alpha^3\nu$	$3\sigma_2 + 2\sigma_6$
10	6	$\alpha^5, \alpha^4\beta, \alpha^3\beta^2, \alpha^2\beta^3, \alpha\beta^4, \beta^5$	$3\sigma_1 + 3\sigma_5$
11	6	$\alpha^4\mu, \alpha^3\beta\mu, \alpha^2\beta^2\mu, \alpha\beta^3\mu, \alpha^4\nu, \beta^4\mu$	$3\sigma_3 + 3\sigma_7$
12	6	$\alpha^6, \alpha^5\beta, \alpha^4\beta^2, \alpha^3\beta^3, \alpha^2\beta^4, \beta^6$	$3\sigma_2 + 3\sigma_6$
13	6	$\alpha^5\mu, \alpha^4\beta\mu, \alpha^3\beta^2\mu, \alpha^2\beta^3\mu, \alpha^5\nu, \beta^5\mu$	$3\sigma_0 + 3\sigma_4$
14	6	$\alpha^7, \alpha^6\beta, \alpha^5\beta^2, \alpha^4\beta^3, \alpha^3\beta^4, \beta^7$	$3\sigma_3 + 3\sigma_7$
15	6	$\alpha^6\mu, \alpha^5\beta\mu, \alpha^4\beta^2\mu, \alpha^3\beta^3\mu, \alpha^6\nu, \beta^6\mu$	$3\sigma_1 + 3\sigma_5$
16	6	$\alpha^8, \alpha^7\beta, \alpha^6\beta^2, \alpha^5\beta^3, \alpha^4\beta^4, \beta^8$	$3\sigma_0 + 3\sigma_4$
17	6	$\alpha^7\mu, \alpha^6\beta\mu, \alpha^5\beta^2\mu, \alpha^4\beta^3\mu, \alpha^7\nu, \beta^7\mu$	$3\sigma_2 + 3\sigma_6$
18	6	$\alpha^9, \alpha^8\beta, \alpha^7\beta^2, \alpha^6\beta^3, \alpha^5\beta^4, \beta^9$	$3\sigma_1 + 3\sigma_5$

Proof. Note that as χ_4 is invariant under the action of the Weyl group, multiplication by χ_4 is a module homomorphism. The action of multiplication by χ_4 is given by the relations $\chi_4\alpha = -\alpha^5$ and $\chi_4\beta = -\beta^5$. Therefore by the module structure given in Lemma 3.1 this homomorphism is an isomorphism. $\square$

Notice that using the above result and Table 1 we are able to compute the Poincaré series for the M_i and $H^*(U_3(5), \mathbb{Z})_{(5)}$.

Lemma 4.3. *The Poincaré series for M_0, M_2, M_4, M_6 and $H^*(U_3(5), \mathbb{Z})_{(5)}$ are, respectively:*

$$p_{M_0}(t) = \frac{1 + 2t^5 + 2t^8 + t^{13}}{(1 - t^8)}$$

$$p_{M_2}(t) = \frac{2t^4 + t^6 + 3t^9 + t^{12} - t^{14}}{1 - t^8} = \frac{2t^4 + 3t^9 + t^{12}}{1 - t^8} + t^6$$

$$p_{M_4}(t) = \frac{t^4 + t^5 + 3t^8 - t^{12} + 2t^{13}}{1 - t^8} = \frac{t^5 + 3t^8 + 2t^{13}}{1 - t^8} + t^4$$

$$p_{M_6}(t) = \frac{t^4 + 2t^9 + 2t^{12} + t^{17}}{1 - t^8} = t^4 \cdot p_{M_0}(t)$$

$$p_{U_3(5)} = \frac{1+2t^5+2t^8+t^{13}+2t^{14}+t^{16}+3t^{19}+t^{22}+t^{25}+3t^{28}-t^{32}+2t^{33}+t^{34}+2t^{39}+2t^{42}+t^{47}}{(1-t^8)(1-t^{40})}$$

Proof. The Poincaré series for the M_i can be easily determined from Table 1 and Lemma 4.2. For $p_{U_3(5)}$ we have by Equation 4.1:

$$p_{U_3(5)} = (p_{M_0}(t) + p_{M_2}(t)t^{10} + p_{M_4}(t)t^{20} + p_{M_6}(t)t^{30})/(1 - t^{40})$$

$\square$

We determine generators and a module structure for each of the M_i. We start with M_0.

Lemma 4.4. *The subring $M_0 \subset \mathcal{H} \subset H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$ is generated by:*

Name	A	B	X	Y	Z
In $H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$	$\alpha\nu + 3\beta\mu$	$\alpha\mu$	χ_4	$\alpha\beta^3 + 3\alpha^3\beta$	$\alpha^4 + \beta^4$
Degree	*5*	*5*	*8*	*8*	*8*
Additive Order	*5*	*5*	*5*	*5*	*5*

This ring has module structure:

$$M_0 = \tilde{\mathbb{Z}}_5[X]\langle 1, A, B, Z, Y, AZ\rangle$$

Proof. From Lemma 4.2 we know that inside of $\mathcal{H}$ we have the following decomposition of M_0:

$$M_0 = M_0 \cap \mathcal{H}^{1-9} \oplus \{M_0 \cap \mathcal{H}^{10-17}\} \otimes \tilde{\mathbb{Z}}_5[\chi_4]$$

From the Poincaré series we know that we need only find the eigenvectors of eigenvalue 1 in degrees 0 to 17.

Eigenvectors for σ_0

Degree	Generators
0	1
5	$A = \alpha\nu + 3\beta\mu$ $B = \alpha\mu$
8	$X = \chi_4$ $Y = \alpha\beta^3 + 3\alpha^3\beta$ $Z = \alpha^4 + \beta^4$
13	$AX = -\alpha^5\nu - 3\beta^4\mu$ $BX = -\alpha^5\mu$ $AZ = \alpha^5\nu + 3\beta^5\mu + \alpha^2\beta^3\mu + 3\alpha^4\beta\mu$
16	$X^2 = \alpha^8 + \beta^8 - \alpha^4\beta^4$ $XY = -\alpha^5\beta^3 - 3\alpha^7\beta$ $XZ = -\alpha^8 - \beta^8$

These are linearly independent by the module structure given in Lemma 3.1. □

Note that M_0 is a subring of $\mathcal{H}$ and additionally that M_0, M_2, M_4 and M_6 are all M_0-modules.

Lemma 4.5. *The modules $M_2, M_4, M_6 \subset \mathcal{H} \subset H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$ have the following structure as M_0-modules:*

Generators for M_2

Name	M	N	P
In $H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$	$\alpha^2 + 3\beta^2$	$\alpha\beta$	χ_3
Degree	*4*	*4*	*3*
Additive Order	*5*	*5*	*5*

Module Structure for M_2

$$M_2 = \tilde{\mathbb{Z}}_5[X]\langle M, N, AM, AN, BN, MZ\rangle \oplus \mathbb{F}_5\langle P\rangle$$

Generators for M_4

Name	Q	C	M^2	N^2	MN
In $H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$	χ_2	$\alpha\nu + 2\beta\mu$	$\alpha^4 - \beta^4 + \alpha^2\beta^2$	$\alpha^2\beta^2$	$\alpha^3\beta + 3\alpha\beta^3$
Degree	*4*	*5*	*8*	*8*	*8*
Additive Order	*5*	*5*	*5*	*5*	*5*

Module Structure for M_4

$$M_4 = \tilde{\mathbb{Z}}_5[X]\langle C, M^2, N^2, MN, CY, CZ\rangle \oplus \mathbb{F}_5\langle Q\rangle$$

Generators for M_6

Name	R
In $H^*(5_+^{1+2}, \mathbb{Z})_{(5)}$	$\alpha^2 + 2\beta^2$
Degree	*4*
Additive Order	*5*

Module Structure for M_6

$$M_6 = \tilde{\mathbb{Z}}_5[X]\langle R, AR, BR, YR, ZR, AZR\rangle$$

Proof. From Lemma 4.2 we know that inside of $\mathcal{H}$ that we have the following decomposition for each of the M_i, $i \in \{2, 4, 6\}$:

$$M_i = M_i \cap \mathcal{H}^{1-9} \oplus \{M_i \cap \mathcal{H}^{10-17}\} \otimes \tilde{\mathbb{Z}}_5[\chi_4]$$

From the Poincaré series we need only find the eigenvectors of proper eigenvalue in degree's zero to 17.

Eigenvectors for σ_2

Degree	Generators
4	$M = \alpha^2 + 3\beta^2$ $N = \alpha\beta$
6	$P = \chi_3$
9	$AM = \alpha^3\nu - \beta^3\mu + \alpha^2\beta\mu$ $AN = \alpha^3\mu + 3\alpha\beta^2\mu$ $BN = \alpha^2\beta\mu$
12	$MX = -\alpha^6 - 3\beta^6$ $NX = -\alpha^5\beta$ $MZ = \alpha^6 + 3\beta^6 + \alpha^2\beta^4 + 3\alpha^4\beta^2$
17	$AMX = -\alpha^7\nu + \beta^7\mu - \alpha^6\beta\mu$ $ANX = -\alpha^7\mu - 3\alpha^5\beta^2\mu\beta$ $BNX = -\alpha^6\beta\mu$

Eigenvectors for σ_4

Degree	Generators
4	$Q=\chi_2$
5	$C=\alpha\nu+2\beta\mu$
8	$M^2=\alpha^4-\beta^4+\alpha^2\beta^2$ $N^2=\alpha^2\beta^2$ $MN=\alpha^3\beta+3\alpha\beta^3$
13	$CX=-\alpha^5\nu-2\beta^5\mu$ $CY=2\alpha^3\beta^2\mu$ $CZ=\alpha^5\nu+2\beta^5\mu+\alpha^2\beta^3\mu+2\alpha^4\beta\mu$
16	$M^2X=-\alpha^8+\beta^8-\alpha^6\beta^2$ $N^2X=-\alpha^6\beta^2$ $MNX=-\alpha^7\beta-3\alpha^5\beta^3$

Eigenvectors for σ_6

Degree	Generators
4	$R=\alpha^2+2\beta^2$
9	$AR=\alpha^3\nu+\beta^3\mu$ $BR=\alpha^3\mu+2\alpha\beta^2\mu$
12	$XR=-\alpha^6-2\beta^6$ $YR=2\alpha^3\beta^3$ $ZR=\alpha^6+2\beta^6+\alpha^2\beta^4+2\alpha^4\beta^2$
17	$AXR=-\alpha^7\nu-\beta^7\mu$ $BXR=-\alpha^7\mu-2\alpha^5\beta^2\mu$ $AZR=\alpha^7\nu+\beta^7\mu+2\alpha^4\beta^3\mu$

These are linearly independent by the module structure given in Lemma 3.1. □

5. Cohomology of $U_3(5)$

Having studied the stability conditions, and found the invariant inside of the cohomology ring, $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$, we are ready to put these considerations together to prove Theorem 1.1.

Proof of Theorem 1.1. From Proposition 2.1 we know that

$$H^*(U_3(5),\mathbb{Z})_{(5)}=H^*(5_+^{1+2}:8,\mathbb{Z})_{(5)}.$$

By Equation 4.1 and Lemmas 4.4 & 4.5, we have the following generators and module structure for $H^*(U_3(5),\mathbb{Z})_{(5)}$.

$$H^*(U_3(5),\mathbb{Z})_{(5)}=$$
$$\{\tilde{\mathbb{Z}}_5[x]\langle 1,a,b,z,y,az,m,n,mz,am,an,bn,c,m^2,mn,n^2,cy,cz,r,ar,br,xr,yr,azr\rangle$$
$$\oplus\mathbb{F}_5\langle p,q\rangle\}\otimes\tilde{\mathbb{Z}}_{25}[\theta]$$

By Lemma 4.3 we have Poincaré series:

$$p_{U_3(5)}=\frac{1+2t^5+2t^8+t^{13}+2t^{14}+t^{16}+3t^{19}+t^{22}+t^{25}+3t^{28}-t^{32}+2t^{33}+t^{34}+2t^{39}+2t^{42}+t^{47}}{(1-t^8)(1-t^{40})}$$

What remains to prove is the that the relations we have listed in Theorem 1.1 really do present the ring in question. We list the relations and module structure for $H^*(U_3(5),\mathbb{Z})_{(5)}$ again for readability.

TABLE 2. Generators for $H^*(U_3(5),\mathbb{Z})_{(5)}$

Name	Generator (written in $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$)	*Degree*	*AdditiveOrder*
a	$\alpha\nu + 3\beta\mu$	5	5
b	$\alpha\mu$	5	5
x	χ_4	8	5
y	$\alpha\beta^3 + 3\alpha^3\beta$	8	5
z	$\alpha^4 + \beta^4$	8	5
m	$\alpha^2\zeta + 3\beta^2\zeta$	14	5
n	$\alpha\beta\zeta$	14	5
p	$\chi_3\zeta$	16	5
q	$\chi_2\zeta^2$	24	5
c	$\alpha\nu\zeta^2 + 2\beta\mu\zeta^2$	25	5
r	$\alpha^2\zeta^3 + 2\beta^2\zeta^3$	34	5
θ	ζ^4	40	25

Relations for $H^*(U_3(5),\mathbb{Z})_{(5)}$

$$\{a,b,c\}\cdot\{a,b,c\} = 0 \qquad \{a,b,c,m,n,p,q,r,x,y,z\}\cdot\{p,q\} = 0$$
$$ay = 3bz = -bx \qquad by = az + ax \qquad y^2 = -x^2 - xz$$
$$yz = 3xy \qquad z^2 = 3x^2 + 2xz$$
$$bm = an \qquad ym = -nx \qquad yn = mx + mz \qquad zn = 3nx$$
$$am^2 = 2cz + cx \qquad an^2 = 3cz + 3cx \qquad bn^2 = bm^2 = 3cy$$
$$cm = ar \qquad cn = br \qquad m^3 = xr + 2zr$$
$$m^2n = 3yr \qquad mn^2 = 3zr - 3xr \qquad n^3 = 3yr$$
$$mr = z\theta \qquad nr = 2y\theta \qquad cr = am\theta + 3bn\theta$$
$$r^2 = m^2\theta + 3n^2\theta$$

Module structure for $iH^*(U_3(5),\mathbb{Z})_{(5)}$

$$H^*(U_3(5),\mathbb{Z})_{(5)} \cong$$
$$\{\tilde{\mathbb{Z}}_5[x]\langle 1,a,b,y,z,az,m,n,mz,am,an,bn,c,m^2,mn,n^2,cy,cz,r,ar,br,yr,zr,azr\rangle$$
$$\oplus\mathbb{F}_5\langle p,q\rangle\}\otimes\tilde{\mathbb{Z}}_{25}[\theta]$$

It is not too difficult to check that the relations we have given are true inside the $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$ using the description for the generators we have given above. Therefore we know that the abstract ring with the presentation given in Theorem 1.1 has as a homomorphic image $H^*(U_3(5),\mathbb{Z})_{(5)}$. It will be enough then to show that, using the relations given, any monomial is equivalent to something in the module structure given above. This will say that the abstract ring is no bigger than the cohomology ring as described inside of $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$.

We start with monomials formed from products of a, b, x, y, z and θ. As the terms a^2, ab, b^2 are all zero we note that any monomial must be of the form $ay^*z^*x^*\theta^*$, $by^*z^*x^*\theta^*$, or $y^*z^*x^*\theta^*$. In any of these cases, using the relations for y^2, yz, z^2, we see that we may rewrite these monomials as sums of monomials of the form $ayx^*\theta^*$, $azx^*\theta^*$, $byx^*\theta^*$, $bzx^*\theta^*$, $yx^*\theta^*$, $zx^*\theta^*$ plus something in the module structure; this follows from an easy induction argument on the sum of the exponents of y and z. Then applying the relations for ay, bz, and by, we reduce to monomials of the form $ax^*\theta^*$, $bx^*\theta^*$, $x^*\theta^*$, $yx^*\theta^*$, $zx^*\theta^*$, θ^*, and $azx^*\theta^*$, which are all contained in the module structure. In the process of rewriting we get terms that are written only with a, b, x, y, and z. This shows that these generate a subring, which we denote by m_0. Note that anything of the form $a^*b^*x^*y^*z^*$ may be reinterpreted into something in the module structure above.

We look at the action by multiplication of m_0 on the generators m and n. For the generators m, n we have relations showing how to take anything of the form $x^*y^*z^*m$ or $x^*y^*z^*n$ to a sum of things of the form x^*m, x^*n, and x^*zm. As $bm = an$, we can rewrite anything of the from $a^*b^*x^*y^*z^*m$ or $a^*b^*x^*y^*z^*n$ into the sum of things of the form x^*m, x^*n, x^*zm, ax^*m, ax^*n, ax^*zm, and bx^*n. Therefore if we can translate something in the form ax^*zm into the module structure we will have shown the m_0 module generated by m and n is completely represented in the module structure. We have the relation $yn = mx + mz$, so $ayn = amx + amz$ and using $ay = -bx$, we have $amz = -amx - bnx$. This shows that ax^*zm may be rewritten into the terms of the module structure.

We apply the same kind of analysis to the module generated by m^2, mn, n^2, and c. As cx, cy, and cz, are all in the module structure then using the relations described above, any term of the form $cx^*y^*z^*$ may be rewritten as a the sum of things of the form cx^*, cx^*y, cx^*z, and as $ac = 0$ and $bc = 0$ we see that anything of the form $a^*b^*x^*y^*z^*c$ may be reinterpreted into something in the module structure. Multiplication by terms of the form $x^*y^*z^*$ are not a problem by the last paragraph. Therefore we only need understand am^2, amn, an^2 and bm^2, bmn, bn^2 in terms of the module structure. Note that there are relations for am^2, an^2, bm^2 and bn^2 and that $amn = bm^2$ and $an^2 = bmn$. Therefore the m_0 module generated by m^2, mn, n^2 and c is completely represented in terms of the module structure.

We look at the m_0 module generated by r. This is quite easy though, as one can immediately see that this module is isomorphic to m_0 as an m_0 module and therefore that everything may be interpreted in the module structure.

Finally we look at products from the different modules defined above. If the product between any two of these generators is able to be reinterpreted into the module structure we will be done. One easily sees that this is done for all the relevant products of the set m, n, m^2, n^2, mn, c, r. □

6. Cohomology of HS

The Higman-Sims group is one of the 26 sporadic simple groups, and contains in it a maximal subgroup isomorphic to $U_3(5) : 2$. This subgroup has index relatively prime to 5, therefore the restriction map from $H^*(HS, \mathbb{Z})_{(5)}$ to $H^*(U_3(5):2, \mathbb{Z})_{(5)}$ is injective. The following result is also obtained in [TY96, Thm. 5.9].

Lemma 6.1. *The image of the restriction map from the 5-local cohomology of HS to the 5-local cohomology of $U_3(5):2$ is exactly:*

$$H^*(U_3(5):2, \mathbb{Z})_{(5)} = H^*(U_3(5), \mathbb{Z})_{(5)}^{W_{HS}(U_3(5))}$$

Proof. The permutation representation defined by this maximal subgroup gives a 2-transitive action as can be seen from the character for this representation listed in the [C+85, p. 80]. It can be easily calculated that that the intersection of any two conjugate $U_3(5):2$ subgroups have intersection which has order relatively prime to

5. Therefore the 5-local cohomology ring of the intersection of these two subgroups is trivial. This proves the result. □

Therefore the cohomology of HS is given by the invariants under the action of a single involution on the cohomology of $U_3(5)$. In [TY96, p. 501] Tezuka and Yagita give a description of $N_{HS}(5_+^{1+2})$. From this and [Lea91, Thm. 3] we can describe the action of this element of order 2 on $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$. Since we have identified $H^*(U_3(5),\mathbb{Z})_{(5)}$ inside of $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$, we may use this description to obtain the action of the involution on $H^*(U_3(5),\mathbb{Z})_{(5)}$.

The action of the involution on the generators of $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$ is given by:

$$\begin{array}{cc} \alpha \longmapsto 3\alpha & \beta \longmapsto 2\beta \\ \mu \longmapsto 2\mu & \nu \longmapsto 3\nu \\ \chi_2 \longmapsto \chi_2 & \chi_3 \longmapsto \chi_3 \\ \chi_4 \longmapsto \chi_4 & \zeta \longmapsto \zeta \end{array}$$

Proof of Theorem 1.2. From the action defined above and Table 2 we are able to compute the action of this involution on the generators of $H^*(U_3(5),\mathbb{Z})_{(5)}$.

$$\begin{array}{ccc} a \longmapsto -a & b \longmapsto b & x \longmapsto x \\ y \longmapsto -y & z \longmapsto z & m \longmapsto -m \\ n \longmapsto n & p \longmapsto p & q \longmapsto q \\ c \longmapsto -c & r \longmapsto -r & \theta \longmapsto \theta \end{array}$$

So the generators which are not invariant are a, y, m, c, and r. Thus the invariants under the action of y are generated by $I = \{b, x, z, n, p, q, \theta\}$ and the products of two generators in $\{a, y, m, c, r\}$. Those products which are linearly independent of I are $\{am, ar, m^2\}$. We make the following renaming for ease of notation: $e = am$, $f = m^2$, and $g = ar$. We list the generators, their degrees, additive orders, and images in $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$ and $H^*(U_3(5),\mathbb{Z})_{(5)}$ in Table 3.

TABLE 3. Generators for $H^*(HS,\mathbb{Z})_{(5)}$

Name	Generator written in $H^*(U_3(5),\mathbb{Z})_{(5)}$	Generator written in $H^*(5_+^{1+2},\mathbb{Z})_{(5)}$	Degree	Additive order
b	b	$\alpha\mu$	5	5
x	x	χ_4	8	5
z	z	$\alpha^4+\beta^4$	8	5
n	n	$\alpha\beta\zeta$	14	5
p	p	$\chi_3\zeta$	16	5
e	am	$\alpha^3\nu\zeta-\beta^3\mu\zeta+\alpha^2\beta\mu\zeta$	19	5
q	q	$\chi_2\zeta^2$	24	5
f	m^2	$\alpha^4\zeta^2-\beta^4\zeta^2+\alpha^2\beta^2\zeta^2$	28	5
g	ar	$\alpha^3\nu\zeta^3+\beta^3\mu\zeta^3$	39	5
θ	θ	ζ^4	40	25

To determine the module structure for this ring all we need to do is see which generators for the module structure for $H^*(U_3(5),\mathbb{Z})_{(5)}$ are invariant. This follows from the fact that that x and θ are invariant under the involution. Therefore we have the following module structure.

$$H^*(HS,\mathbb{Z})_{(5)} \cong \{\tilde{\mathbb{Z}}_5[x]\langle 1,b,z,n,e,bn,f,n^2,bn^2,g,n^3,gz\rangle \oplus \mathbb{F}_5\langle p,q\rangle\} \otimes \tilde{\mathbb{Z}}_{25}[\theta]$$

Again it remains to prove that the relation we have listed in Theorem 1.2 present the given ring. We list the relations and module structure again for readability.

Relations for $H^*(HS,\mathbb{Z})_{(5)}$

$$\{b,e,g\}\cdot\{b,e,g\}=0 \qquad \{b,e,f,g,n,p,q,x,z\}\cdot\{p,q\}=0$$
$$bz=3bx \qquad z^2=3x^2+2xz \qquad zn=3nx$$
$$zn^3=3xn^3 \qquad n^4=-x^2\theta-xz\theta$$
$$bn^3=3gz+3gx \qquad gn=3bx \qquad f^2=x^2\theta$$
$$ef=gx+2gz$$
$$bf=bn^2 \qquad ez=gf=-ex-bnx \qquad fz=-fx-n^2x$$
$$en=bn^2 \qquad nf=n^3$$

Module structure for $H^*(HS,\mathbb{Z})_{(5)}$

$$H^*(HS,\mathbb{Z})_{(5)} \cong \{\tilde{\mathbb{Z}}_5[x]\langle 1,b,z,n,e,bn,f,n^2,bn^2,g,n^3,gz\rangle \oplus \mathbb{F}_5\langle p,q\rangle\} \otimes \tilde{\mathbb{Z}}_{25}[\theta]$$

Again we show that these relations are enough to imply that the ring is no bigger than the module structure given above. We do this in pieces, first showing that there is a subring which is no bigger than its image in the full ring, and then adding more modules for this subring. Finally we show the entire ring is presented in the module structure.

We start with the subring generated by b,x,z and θ. As we have relation for rewriting bz and z^2 in terms of the module structure, this subring is represented in the module structure. Again as all these products are reinterpreted as monomials in b,x,z and θ these generate a subring. Let this subring be denoted by n_0.

We look at the n_0 module generated by n,bn,p and e. In this case bp,xp,zp and p^2 are all zero. Therefore every term with p in it may be written in terms of the module structure. As for terms of the form $b^*x^*z^*\theta^*n$, using the $zn=3nx$ we are able to rewrite this term without factors of z, and therefore any term of this form may be rewritten in the module structure. Similarly, using the relations $ez=-ex-bnx$ and $be=0$, we are able to rewrite any term of the form $b^*x^*z^*\theta^*e$ as a sum of terms of the form $bx^*\theta^*n$ and $x^*\theta^*e$, and therefore it is in the module structure.

The n_0 module generated by q,n^2,bn^2 and f is again contained in the module structure. For the module spanned by q this can be seen, as in the case of p above, since all relevant products are zero except $q\theta$ which is in the module structure. For the module spanned by f we again have relations which allow us to rewrite any term with both f and b or f and z in terms of the sums of terms in the module structure. This is also the case for terms with a factor of n^2 as bn^2 is part of the module structure and we rewrite n^2z as $3n^2x$, so we may rewrite anything of the form $b^*x^*z^*\theta^*n^2$ as the sum of terms from the module structure.

The last n_0 module we need look at is generated by g,n^3 and gz. For the module generated by g, as $bg=0$ and $z^2=3x^2+2xz$ we may rewrite the cyclic n_0-module generated by g in terms of the module structure. Finally for n^3 there are relations showing how to rewrite $bn^3=3gz+3gx$ and $zn^3=3n^3x$ as above for n and n^2. Therefore this module is contained in the module structure.

Finally we must look at products between any two generators of these modules. Products with p or q are no problem as they are all zero. We have relations for $e^2, en, ef, eg, f^2, fg, fn, g^2, gn$, and n^4, which show how to rewrite these products interms of the module structure, so we are done. □

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References

[Bro82] K. S. Brown, *Cohomology of groups*, Graduate texts in mathematics, vol. 66, Springer-Verlag, Berlin–New York, 1982.

[C+85] J. H. Conway et al., *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[Lea91] I. J. Leary, *The intergral cohomology rings of some p-groups*, Math. Proc. Camb. Phil. Soc. **110** (1991), 25–32.

[Sie96] Stephen F. Siegel, *The spectral sequence of a split extension and the cohomology of an extraspecial group of order p^3 and exponent p*, Journal of Pure and Applied Algebra **106** (1996), no. 2, 185–198.

[TY96] M. Tezuka and N. Yagita, *On odd prime components of cohomologies of sporadic simple groups and the rings of universal stable elements*, J. Algebra **183** (1996), 483–513.

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