THE COHOMOLOGY OF THE MORAVA STABILIZER GROUP $\mathbb{S}_2$ AT THE PRIME 3

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Abstract. We compute the cohomology of the Morava stabilizer group $\mathbb{S}_2$ at the prime 3 by resolving it by a free product $\mathbb{Z}/3 * \mathbb{Z}/3$ and analyzing the "relation module."

1. Introduction and Statement of the Main Result

The applications of the main theorem of this paper in homotopy theory are due to the Morava Change of Rings Theorem [7]. Let p be a prime, and denote by $\mathbb{S}_n$ the group of units in the maximal order of a cyclic division algebra over $\mathbb{Q}_p$ of index n and Hasse invariant $\frac{1}{n}$. The Morava Theorem says essentially that the cohomology of $\mathbb{S}_n$ with coefficients in a certain representation describes the Bousfield localization functor $L_{K(n)}$. This is the localization of stable homotopy theory with respect to the spectrum of the n-th Morava K-theory, $K(n)$ [2]. The functors $L_{K(n)}$ play an important role in homotopy theory [10]. At present the case $n = 1$ is completely understood for all primes p. The next case, $n = 2$, has been partially investigated for primes $p \geq 5$ (see for example [15], [16]). The functor $L_{K(2)}$ for small primes is harder to study because the group $\mathbb{S}_2$ is of infinite cohomological dimension.

In this paper we deal with the prime 3 only. As the first step of the analysis of $L_{K(2)}$ one needs to compute the continuous cohomology of a certain canonical subgroup $\mathbb{S}_2^0$ of $\mathbb{S}_2$ with trivial coefficients. We compute these cohomology groups in the course of the proof of the Theorem 1.1 below. This "almost" computes, according to the Morava Change of Ring Theorem, the homotopy groups of the localization of the Toda Smith complex $V(1)$; for more details see [9]. The rest of the calculation of $\pi_* L_{K(2)} S^0$ consists of two Bockstein spectral sequences. We will not pursue this here.

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Theorem 1.1. *$H^*_c(\mathbb{S}_2;\mathbb{F}_3)$ is freely generated as a graded-commutative $\mathbb{F}_3$-algebra by elements Z (in degree 1), C,E (in degree 3), and X (in degree 4). Its Poincaré series is $\frac{(1+t)(1+t^3)^2}{1-t^4}$.*

We obtain this by first calculating the cohomology of the canonical subgroup, $\mathbb{S}l$, of $\mathbb{S}_2$.

Theorem 1.2. *$H^*_c(\mathbb{S}l;\mathbb{F}_3)$ has generators e_1, e_2 (in degree 1), x_1, x_2, a (in degree 2), c_1, c_2 (in degree 3). The product of any two generators with different subscripts is 0 and in addition there are relations*

$$a^2 = ae_1 = ae_2 = ac_1 = ac_2 = 0, \quad c_1e_1 = ax_1, \quad c_2e_2 = ax_2.$$

Its Poincaré series is $\frac{1+t+t^2+t^3}{1-t}$.

A computation of $H^*_c(\mathbb{S}l;\mathbb{F}_3)$ was sketched in [9], but the multiplicative structure given there does not agree with the one above. The question of the cohomology of $\mathbb{S}l$ was first reopened by Henn in connection with a deep theorem of his on the cohomology of profinite groups [4], and he also obtained the result stated here. Our calculation proceeds by more classical methods.

The structure of $H^*_c(\mathbb{S}l;\mathbb{F}_3)$ can also be described as follows: it was shown in [3] that if j is the quotient map

$$\mathbb{S}l \xrightarrow{j} \mathbb{S}l/\mathbb{S}l' \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3,$$

(where $\mathbb{S}l'$ denotes the commutator subgroup $[\mathbb{S}l,\mathbb{S}l]$ of $\mathbb{S}l$), then there exists a homomorphism $\mathbb{Z}/3 * \mathbb{Z}/3 \xrightarrow{i} \mathbb{S}l$ such that ji is onto. The image $R = j^*H^*(\mathbb{Z}/3 \oplus \mathbb{Z}/3) \subset H^*_c(\mathbb{S}l)$ is mapped isomorphically onto $H^*(\mathbb{Z}/3*\mathbb{Z}/3)$ by i^* (R is the subring generated by 1, e_1, e_2, x_1, x_2). The kernel $R' = \operatorname{Ker} i^*$ (generated by a, c_1, c_2, as an R-module) is additively like R but with the degrees increased by 2. The structure of R' as an R-module is as given above and ${R'}^2 = 0$: this determines the ring $H^*_c(\mathbb{S}l) \cong R \oplus R'$.

2. Background Information

We briefly recall some facts about the group of units of a maximal order in a division algebra. A full account can be found in [11], for example. Consider a cyclic algebra $\mathbb{D}$ over $\mathbb{Q}_p$ of index n and Hasse invariant $\frac{1}{n}$. It can be constructed as follows. Let $\mathbb{W}$ be the totally unramified extension of $\mathbb{Q}_p$ of degree n (so $\mathbb{W} \cong \mathbb{Q}_p(\zeta)$ where ζ is a (p^n-1)-st root of unity). The Galois group of the extension $\mathbb{W}/\mathbb{Q}_p$ is a cyclic group of order n: it is generated by the Frobenius homomorphism σ. We form the crossed product algebra of $\mathbb{W}$ and $\mathrm{Gal}(\mathbb{W}/\mathbb{Q}_p)$. This amounts to introducing a variable S which commutes with $\mathbb{W}$ according to the formula $wS = Sw^\sigma$ and satisfies

$S^n = a \in \mathbb{Q}_p{}^\times$. To define $\mathbb{D}$ we set $S^n = p$. $\mathbb{D}$ is a division algebra over $\mathbb{Q}_p$ of rank n^2.

Let $\mathbb{O}$ be the maximal order in $\mathbb{D}$: it is generated by S and the integers of $\mathbb{W}$. Its maximal ideal is $\mathbb{O}S$ and $\mathbb{O}/\mathbb{O}S \cong \mathbb{F}_{p^n}$. We are interested in three groups contained in $\mathbb{O}$. The first is the group of units of $\mathbb{O}$, which we denote by $\mathbb{S}_n$. The second is the subgroup of strict units in $\mathbb{S}_n$. It consists of the elements $a \in \mathbb{S}_n$ such that $a \equiv 1 \bmod S$. We denote it by $\mathbb{S}_n^0$. The third is the kernel of the reduced norm restricted to $\mathbb{S}_n^0$. We denote this subgroup by $\mathbb{S}l$. It is a pro-p group because it is p-filtered and compact ([6], II, 2.1.3).

Let H_r denote the subgroup of $\mathbb{S}l$ consisting of elements congruent to 1 modulo S^r. By definition $H_1 = \mathbb{S}l$. According to [12], there are injective maps $\rho_r : H_r/H_{r+1} \to \mathbb{O}/\mathbb{O}S \cong \mathbb{F}_{p^n}$ given by $\rho_r(1+aS^r) \equiv a \bmod S$, $(a \in \mathbb{O})$. They are also surjective unless $n|r$, in which case the image consists of those elements of trace 0 over the prime field.

From now on we shall only consider $p = 3$ and $n = 2$. Then $|H_1/H_2| = 9$, $|H_2/H_3| = 3$, $|H_3/H_4| = 9$, and according to [12], $[H_1, H_1] = H_2$, $[H_1, H_2] = H_3$, $[H_2, H_2] = H_4$. Now $\mathbb{W}$ contains an 8-th root of unity ζ, and so $z = \frac{\zeta S - 1}{2} \in \mathbb{D}$ is a cube root of unity. Thus $\mathcal{X} = z$ and $\mathcal{Y} = z^S$ are two elements of $\mathbb{S}l$ of order 3. Their images $x = j(\mathcal{X})$, $y = j(\mathcal{Y})$ generate $\mathbb{S}l/\mathbb{S}l' \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$, (where $\mathbb{S}l' = [\mathbb{S}l, \mathbb{S}l] = H_2$). We now define $i : \mathbb{Z}/3 * \mathbb{Z}/3 \to \mathbb{S}l$ to be the map which takes the generators $\mathbb{X}$ and $\mathbb{Y}$ to $\mathcal{X}$ and $\mathcal{Y}$. This map is in fact injective [3], but we do not need to know that here.

There is a group D of automorphisms of $\mathbb{S}l$ which has order 8 and is generated by (i) conjugation by S, which interchanges $\mathcal{X}$ and $\mathcal{Y}$, and (ii) conjugation by $S\zeta$, which interchanges $\mathcal{X}$ and $\mathcal{X}^2$ and fixes $\mathcal{Y}$.

Note that the action of D lifts to an action on $\mathbb{Z}/3 * \mathbb{Z}/3$.

The natural cohomology theory for a profinite group is the cohomology on continuous cochains [13], denoted by H_c^*, and that is what we use here. It agrees with the usual cohomology on a finite group.

Any maximal finite subgroup of $\mathbb{S}l$ is cyclic of order 3, so the Krull dimension of $H_c^*(\mathbb{S}l; \mathbb{F}_3)$ is one [8], [7].

3. Resolutions

The fact that ji is onto implies that $\operatorname{Im} i$ is dense in the pro-3 topology. So we have an epimorphism of pro-3 groups

$$\widehat{\mathbb{Z}/3 * \mathbb{Z}/3} \to \mathbb{S}l \qquad (\hat{}\ \text{denotes pro-3 completion}).$$

Let K denote the kernel of this map. The kernel of $\mathbb{Z}/3 * \mathbb{Z}/3 \to \mathbb{Z}/3 \oplus \mathbb{Z}/3$ is free on the four generators $[\mathbb{X}^i, \mathbb{Y}^j]$, $(1 \le i, j \le 2)$ [14]. The completion of the corresponding short exact sequence remains exact, since $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ is finite. This leads to a diagram with exact rows and columns:

$$
\begin{tikzcd}
K \arrow[r] \arrow[d, equal] & \hat{F}_4 \arrow[r] \arrow[d] & \mathbb{S}l' \arrow[d] \\
K \arrow[r] & \widehat{\mathbb{Z}/3 * \mathbb{Z}}/3 \arrow[r] \arrow[d] & \mathbb{S}l \arrow[d] \\
& \mathbb{Z}/3 \oplus \mathbb{Z}/3 \arrow[r, equal] & \mathbb{Z}/3 \oplus \mathbb{Z}/3.
\end{tikzcd} \tag{3.1}
$$

Now K is a closed subgroup of a free pro-3 group (i.e. the pro-3 completion of a free group), so is itself a free pro-3 group ([13], Cor. 2 to I, Prop. 24).

According to the theory in ([6], V 2.5.7), $\mathbb{S}l'$ is equi-3-valued (with the usual 3-adic valuation), so $H_c^*(\mathbb{S}l') \cong \Lambda^* H_c^1(\mathbb{S}l')$. $\mathbb{S}l'/[\mathbb{S}l', \mathbb{S}l'] = H_2/H_4$ has order 27 and is easily seen to have exponent 3, so it has rank 3, and thus so does $H_c^1(\mathbb{S}l')$.

Let N denote $\mathbb{S}l'/[\mathbb{S}l', \mathbb{S}l']$ as $\mathbb{F}_3 E$-module (where $E = \mathbb{S}l/\mathbb{S}l' \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$). Then $H_c^2(\mathbb{S}l') \cong N$ by duality.

We want to be able to describe modules such as N explicitly. For this purpose note that if we set $X = x - 1$ and $Y = y - 1$, then $\mathbb{F}_3 E \cong \mathbb{F}_3[X, Y | X^3 = Y^3 = 0]$. The augmentation ideal I is generated by X and Y, and, because E is a p-group, I is the radical.

Since $[H_1, H_2] = H_3$ and $|H_2/H_3| = 3$, we must have $N/IN \cong \mathbb{F}_3$. If we consider the image of $\mathrm{ann} N$ in $I/I^2 \cong E$, we see that it cannot be 1-dimensional, since then it could not be invariant under the group of automorphisms D. Hence $\mathrm{ann}\, N = I^2$ and $N \cong \mathbb{F}_3 E/I^2$.

Now consider the spectral sequence

$$H_c^p(\mathbb{S}l'; H_c^q(K)) \Rightarrow H_c^{p+q}(\hat{F}_4).$$

There are only two rows, since K is a free pro-3 group, and we deduce that

$$H_c^r(\mathbb{S}l'; H_c^1(K)) = 0, \quad r \geq 2,$$
$$H_c^1(\mathbb{S}l'; H_c^1(K)) \cong H_c^3(\mathbb{S}l') \cong \mathbb{F}_3,$$

and there are short exact sequences

$$0 \to H_c^1(\mathbb{S}l') \to H_c^1(\hat{F}_4) \to E_\infty^{0,1} \to 0, \tag{3.2}$$

$$0 \to E_\infty^{0,1} \to H_c^1(K)^{\mathbb{S}l'} \to N \to 0. \tag{3.3}$$

Sequence (3.2) shows that $E_\infty^{0,1} \cong \mathbb{F}_3$. Let M denote $H_c^1(K)^{\mathbb{S}l'}$ as an $\mathbb{F}_3 E$-module, so we have

$$0 \to \mathbb{F}_3 \to M \to N \to 0.$$

4. The Structure of M

The short exact sequence $0 \to \mathbb{F}_3 \to M \to N \to 0$ shows that either $M \cong N \oplus \mathbb{F}_3$, or M is generated by one element. In the latter case, $M \cong \mathbb{F}_3 E/\operatorname{ann} M$, and we shall assume that this holds for the rest of this section. The image of $\operatorname{ann} M$ in I^2/I^3 is a subspace S of codimension 1, which must be invariant under the action of the group of automorphisms, D. Since I^2/I^3 has basis $\{X^2, XY, Y^2\}$, it is easy to check that the only possibilities for S are $S_1 = \langle X^2, Y^2 \rangle, S_2 = \langle XY, X^2 + Y^2 \rangle$, and $S_3 = \langle XY, X^2 - Y^2 \rangle$. Let $M_i = \mathbb{F}_3 E/S_i$. We claim that, in fact, $M \cong M_1$, but this will only become apparent later. Notice, however, that if we extend the field to $\mathbb{F}_9$, then all three modules differ only by an automorphism of the group algebra, for if

$$\phi_2 : X \mapsto X + Y, \quad Y \mapsto X - Y,$$

then $\phi_2(S_1) = S_2$, and so $M_2^{\phi_2} \cong M_1$. Similarly, if

$$\phi_3 : X \mapsto X + iY, \quad Y \mapsto X - iY, \qquad (\text{where} \quad i^2 = 1)$$

then $\phi_3(S_1) = S_3$, and so $M_3^{\phi_3} \cong M_1$.

5. The Cohomology of M_1

Let p be an odd prime, k a field of characterisitc p, and $E = \langle x, y \rangle$ an elementary abelian p-group of order p^2. (We will only need the case where $p = 3$ and $k = \mathbb{F}_3$, but it is just as easy to prove the result of this section more generally.) The group algebra kE is the truncated polynomial algebra $k[X, Y \mid X^p = Y^p = 0]$, where $X = x - 1$ and $Y = y - 1$. A minimal projective resolution $P \xrightarrow{\epsilon} k$ of the trivial kE-module k may be constructed as follows. First let P_n be the free left kE-module on the $n+1$ symbols $e_{r,s}$ where $r + s = n$ and $r, s \geq 0$. For notational convenience we set $e_{r,s} = 0$ if $r < 0$ or $s < 0$. Then define

$$\partial(e_{r,s}) = X^{1+(p-2)\nu(r+1)} e_{r-1,s} + (-1)^r Y^{1+(p-2)\nu(s+1)} e_{r,s-1},$$

where $\nu(n)$ is defined to be 0 if n is even and 1 if n is odd, and set $\epsilon(e_{0,0}) = 1$. We have

$$H^*(E; k) = \operatorname{Hom}_{kE}(P, k) = k[x_1, x_2] \otimes_k \Lambda_k[e_1, e_2]$$

where $x_1 = e_{0,2}^*$ (by which we mean $x_1(e_{0,2}) = 1$ and $x_1(e_{r,s}) = 0$ for $(r, s) \neq (0, 2)$), $x_2 = e_{2,0}^*$, $e_1 = e_{0,1}^*$, and $e_2 = e_{1,0}^*$. The generators e_1 and e_2 in $H^1(E, k) \cong \operatorname{Hom}(E, k)$ correspond to maps $E \to k$ with kernels $\langle x \rangle$ and $\langle y \rangle$ respectively, and x_1 and x_2 are their Bocksteins.

Now let $M_1 = kE/(X^{p-1}, Y^{p-1})$. (If $p = 3$ and $k = \mathbb{F}_3$, this is consistent with the definition of M_1 given above.)

Proposition 5.1. *As a right $H^*(E;k)$-module, $H^*(E;M_1)$ is generated by elements α (degree 0), δ_1, δ_2 (degree 1), and β (degree 2), subject to the relations*

$$\alpha e_1 = \alpha e_2 = \delta_2 e_1 = \delta_1 e_2 = 0,$$

$$\delta_1 e_1 = \alpha x_1, \quad \delta_2 e_2 = \alpha x_2,$$

$$\beta e_1 = -\delta_2 x_1, \quad \beta e_2 = \delta_1 x_2.$$

In particular, $H^*(E;M_1)$ is a free $k[x_1,x_2]$-module on $\alpha, \delta_1, \delta_2, \beta$.

Proof. We compute an explicit basis for $H^n(E;M_1) = Z^n(E,M_1)/B^n(E,M_1)$. First note that M_1 is a commutative ring, and $\operatorname{Hom}_{kE}(P_n, M_1)$ is a free left M_1-module on the generators $f_i^n = e_{i,n-i}^*$ $(0 \le i \le n)$. Let $\bar{a}$ denote the image in M_1 of an element a in kE. Then $\bar{X}^{p-1} = \bar{Y}^{p-1} = 0$, so for $f \in \operatorname{Hom}_{kE}(P_n, M_1)$ and $r, s \ge 0$ we have

$$f\partial(e_{r,s}) = \nu(r)\bar{X}f(e_{r-1,s}) + (-1)^r \nu(s)\bar{Y}f(e_{r,s-1}). \tag{5.1}$$

Suppose first that $n = 2m$ is even. Let $f = \sum_{i=0}^n \alpha_i f_i^n$ $(\alpha_i \in M_1)$ and for notational convenience set $\alpha_{-1} = \alpha_{n+1} = 0$. Then equation (5.1) implies that $f \in Z^n(E,M_1)$ if and only if

$$\nu(j)\alpha_{j-1}\bar{X} + (-1)^j \nu(n+j+1)\alpha_j \bar{Y} = 0 \text{ whenever } 0 \le j \le n+1.$$

But this occurs if and only if $\alpha_j \bar{X} = 0 = \alpha_j \bar{Y}$ for all even j, or equivalently $\alpha_j \in (\bar{X}^{p-2}\bar{Y}^{p-2})$ for all even j. Hence

$$Z^n(E,M_1) = \{\sum_{j=0}^n \alpha_j f_j^n \mid \alpha_j \in M_1 \text{ if } j \text{ is odd}, \ \alpha_j \in (\bar{X}^{p-2}\bar{Y}^{p-2}) \text{ if } j \text{ is even}\}.$$

A similar calculation yields

$$B^n(E,M_1) = \{\sum_{j=0}^n \alpha_j f_j^n \mid \alpha_j \in (\bar{X},\bar{Y}) \text{ if } j \text{ is odd}, \ \alpha_j = 0 \text{ if } j \text{ is even}\}.$$

Hence the images of the $n+1$ elements $\alpha_k^m = \bar{X}^{p-2}\bar{Y}^{p-2}e_{2k,2m-2k}^*$ $(0 \le k \le m)$ and $\beta_k^m = e_{2k+1,2m-2k-1}^*$ $(0 \le k < m)$ of $Z^n(E,M_1)$ form a basis in $H^n(E;M_1)$.

Now suppose that $n = 2m+1$ is odd. Let $S = \{(\zeta,\eta) \in M_1 \oplus M_1 \mid \zeta\bar{X} = \eta\bar{Y}\}$ and $T = \{(\zeta\bar{Y}, \zeta\bar{X}) \mid \zeta \in M_1\}$. Working as in the even case, we get

$$Z^n(E,M_1) = \{\sum_{i=0}^m (\alpha_i f_{2i}^n + \beta_i f_{2i+1}^n) \mid (\alpha_i,\beta_i) \in S\},$$

$$B^n(E,M_1) = \{\sum_{i=0}^m (\alpha_i f_{2i}^n + \beta_i f_{2i+1}^n) \mid (\alpha_i,\beta_i) \in T\}.$$

Now S/T is 2-dimensional and is spanned by the images of $(\bar{X}^{p-2},0)$ and $(0,\bar{Y}^{p-2})$. Hence the images of the $n+1$ elements $\gamma_k^m = \bar{X}^{p-2}e^*_{2k,2m+1-2k}$, $\delta_k^m = \bar{Y}^{p-2}e^*_{2k+1,2m-2k}$ $(0 \le k \le m)$ of $Z^n(E,M_1)$ form a basis in $H^n(E;M_1)$.

We now turn to the module structure. Recall that composition with ϵ is a chain map $\operatorname{Hom}_{kE}(P,P) \to \operatorname{Hom}_{kE}(P,k)$ and this map induces an isomorphism in cohomology $H^* \operatorname{Hom}_{kE}(P,P) \xrightarrow{\cong} H^*(E;k)$. The action

$$H^*(E;M_1) \otimes_k H^*(E;k) \to H^*(E;M_1)$$

is induced by the map on the cochain level

$$\operatorname{Hom}_{kE}(P,M_1) \otimes_k \operatorname{Hom}_{kE}(P,P) \to \operatorname{Hom}_{kE}(P,M_1)$$

given by composition. So we must first lift x_i, e_i to maps $\tilde{x}_i \in Z^2 \operatorname{Hom}_{kE}(P,P)$, $\tilde{e}_i \in Z^1 \operatorname{Hom}_{kE}(P,P)$. This is accomplished by setting $\tilde{x}_1(e_{r,s}) = e_{r,s-2}$, $\tilde{x}_2(e_{r,s}) = e_{r-2,s}$, $\tilde{e}_1(e_{r,s}) = (-1)^{r+s+1}Y^{(p-2)\nu(s+1)}e_{r,s-1}$, $\tilde{e}_2(e_{r,s}) = (-1)^{r+1}X^{(p-2)\nu(r+1)}e_{r-1,s}$.

Now let $\alpha = [\alpha_0^0]$, $\delta_1 = -[\gamma_0^0]$, $\delta_2 = -[\delta_0^0]$, and $\beta = [\beta_0^1]$. We have $\alpha x_1^i x_2^j = [\alpha_j^{i+j}]$, $\delta_1 x_1^i x_2^j = -[\gamma_j^{i+j}]$, $\delta_2 x_1^i x_2^j = -[\delta_j^{i+j}]$, and $\beta x_1^i x_2^j = [\beta_j^{i+j+1}]$ $(i,j \ge 0)$. These are easily verified; the first, for example, just follows from the fact that

$$\alpha_0^0 \tilde{x}_1^i \tilde{x}_2^j(e_{r,s}) = \alpha_0^0(e_{r-2j,s-2i}) = \bar{X}^{p-2}\bar{Y}^{p-2}((r,s)=(2j,2i)) = \alpha_j^{i+j}(e_{r,s}).$$

(Here we are using the computer science notation where, for a proposition $\mathcal{P}$, $(\mathcal{P}) = 1$ if $\mathcal{P}$ and $(\mathcal{P}) = 0$ otherwise.) This proves that α, δ_1, δ_2, and β generate $H^*(G;M_1)$ as a $k[x_1,x_2]$-module and that $H^*(G,M_1)$ is a free $k[x_1,x_2]$-module on those four generators.

We now turn to the relations. It is routine to check that the generators satisfy these relations; for example, the last of these follows from the fact that

$$\beta_0^1 \tilde{e}_1(e_{r,s}) = (-1)^{r+s+1}\bar{Y}^{(p-2)\nu(s+1)}\beta_0^1(e_{r,s-1}) = \bar{Y}^{p-2}((r,s)=(1,2)) = \delta_0^0 \tilde{x}_1(e_{r,s}).$$

Now let A^* denote the graded $H^*(E;k)$-ring defined abstractly by these 4 generators and 8 relations. Since $H^*(E;M_1)$ satisfies the relations, there is a surjective $H^*(E;k)$-homomorphism $A^* \twoheadrightarrow H^*(E;M_1)$. We wish to show that this homomorphism is an isomorphism, and to do this it suffices to show that $\dim_k(A^n) \le \dim_k H^n(E;M_1)$ for all n. This will follow if we can show that A^* is generated as a $k[x_1,x_2]$-module by the four generators, because we know that $H^*(G;M_1)$ is a free $k[x_1,x_2]$-module on those generators. Hence for each $\theta \in \{\alpha,\delta_1,\delta_2,\beta\}$ and $i \in \{1,2\}$ we must show that θe_i is in the $k[x_1,x_2]$-submodule of A^* generated by $\alpha,\delta_1,\delta_2,\beta$. But this is exactly what the 8 relations tell us. $\square$

The following lemma is easy to check, but will be useful.

Lemma 5.2. *We can replace β by any other element of $H^2(E;M_1)$ linearly independent of αx_1 and αx_2 and get the same relations.*

6. The Final Calculation for $\mathbb{S}l$

We assume for now that $M = M_1$: this will be justified later. Consider the spectral sequence

$$H^p(E; H_c^q(\mathbb{S}l'; H_c^1(K))) \Rightarrow H_c^{p+q}(\mathbb{S}l; H_c^1(K))$$

Again it has only two rows.

$$H^{*-2}(E) \xrightarrow{d_2} H^*(E; M).$$

$H^*(E)$ is a torsion-free $\mathbb{F}_3[x_1, x_2]$ module, hence so is $\operatorname{Ker} d_2$. But this spectral sequence shows that $H^*(\mathbb{S}l; H^1(K))$ is finitely generated over $H^*(E)$, and hence over $H_c^*(\mathbb{S}l)$, which has Krull dimension 1. It maps on to $\operatorname{Ker} d_2$, forcing a common bound on $\dim \operatorname{Ker} d_2$ in each degree. Thus $\operatorname{Ker} d_2 = 0$ and we have:

$$0 \to H^{*-2}(E) \to H^*(E; M) \to H_c^*(\mathbb{S}l; H_c^1(K)) \to 0. \tag{6.1}$$

Now $d_2(1)$ is not a linear combination of αx_1 and αx_2 otherwise it would be annihilated by e_1 and e_2. By Lemma 5.2 we may assume that $d_2(1) = \beta$.

We have proved:

Proposition 6.2. *$H_c^*(\mathbb{S}l; H_c^1(K))$ is an $H^*(E)$-module on generators α (degree 0), δ_1, δ_2 (degree 1) with relations*

$$\alpha e_1 = \alpha e_2 = \delta_1 e_2 = \delta_2 e_1 = \delta_1 x_2 = \delta_2 x_1 = 0,$$

$$\delta_1 e_1 = \alpha x_1, \quad \delta_2 e_2 = \alpha x_2.$$

Now consider the spectral sequence

$$H_c^*(\mathbb{S}l, H^*(K)) \Rightarrow H_c^*(\widehat{\mathbb{Z}/3 * \mathbb{Z}}/3).$$

One has $H_c^*(\widehat{\mathbb{Z}/3 * \mathbb{Z}}/3) \cong H^*(\mathbb{Z}/3 * \mathbb{Z}/3)$. (Consider the short exact sequence $F_4 \to \mathbb{Z}/3 * \mathbb{Z}/3 \to \mathbb{Z}/3 \oplus \mathbb{Z}/3$ and its pro-3 completion, and use the Comparison Theorem.) The map $i^* : H_c^*(\mathbb{S}l) \to H^*(\mathbb{Z}/3 * \mathbb{Z}/3)$ is an isomorpism in degree 1, by construction. As the right hand side is generated by elements of degree 1 and their Bocksteins, i^* is onto in all degrees and the spectral sequence becomes the short exact sequence

$$0 \to H_c^{*-2}(\mathbb{S}l, H_c^1(K)) \xrightarrow{d_2} H_c^*(\mathbb{S}l) \to H^*(\mathbb{Z}/3 * \mathbb{Z}/3) \to 0, \tag{6.3}$$

of right $H^*(E)$-modules. Set $a = d_2\alpha$, $c_i = d_2\delta_i$. Identify e_i and x_i with their images under j^*. All that remains is to check that $\operatorname{Im} j^* = R \cong H^*(\mathbb{Z}/3 * \mathbb{Z}/3)$, i.e. that $e_1e_2 = e_1x_2 = e_2x_1 = x_1x_2 = 0$.

The spectral sequence

$$H^*(E; H_c^*(\mathbb{S}l')) \Rightarrow H_c^*(\mathbb{S}l)$$

has four rows. We know that $d_2(E_2^{*,1}) \subset E_2^{*,0} \cong H^*(E)$ is contained in $\mathrm{Ker}(ji)^*$, which is generated as an $H^*(E)$-module by e_1e_2, e_1x_2, e_2x_1, and x_1x_2. Since $H_c^1(\mathbb{S}l')$ is dual to N as an E-module it is isomorphic to I^3, and we can calculate $\dim E_2^{0,1} = 1$ and $\dim E_2^{1,1} = 3$. (Use dimension shifting: $\mathbb{F}_3 E/I^3$ has invariants of dimension 3.) But $E_2^{1,0}$ yields all of $H_c^1(\mathbb{S}l)$ and thus $\dim d_2(E_2^{0,1}) = 1$. This accounts for e_1e_2. Also $\dim E^{2,0} = 2$ and $\dim H_c^2(\mathbb{S}l) = 3$, so $\dim \mathrm{Ker}(d_2\colon E_2^{1,1} \to E_2^{3,0}) \leq 1$. The image of this map must have dimension ≥ 2, which accounts for e_1x_2 and e_2x_1. Finally x_1x_2 is the Bockstein of e_1x_2.

All that remains is to justify our assertion that $M \cong M_1$. We do this by carrying out the above calculation for each of the other possibilities and obtaining a contradiction.

If $M \cong \mathbb{F}_3 \oplus N$, then the short exact sequence (6.1) shows that $\dim H_c^n(\mathbb{S}l; M) = \dim H^n(E; N) + 2$. This is impossible because N has complexity 2, yet $H_c^*(\mathbb{S}l)$ has Krull dimension 1.

If $M \cong M_2$, then the structure of $H^*(E; M_2)$ as an $H^*(E)$-module is like that of $H^*(E; M_1)$, but twisted by ϕ_2. Let us denote the new module structures by $*$. Then

$$u * v = u(\phi_2^* v), \qquad u \in H^*(E; M), v \in H^*(E).$$

Thus $a * x_1x_2 = a(x_1 + x_2)(x_1 - x_2) = ax_1^2 - ax_2^2 \neq 0$. But the argument that $x_1x_2 = 0$ is still valid since it only depends on the additive structure of $H_c^*(\mathbb{S}l; M)$. The case $M \cong M_3$ is similar.

7. The Cohomology of $\mathbb{S}_2$

Remark 7.1. [9] *If $p > 3$ and $n = 2$ then $\mathbb{S}l$ is torsion-free so $H^*(\mathbb{S}l)$ has finite cohomological dimension by* [8]*. It contains an open subgroup H_2, which is a Poincaré duality group of dimension* 3*. So by (*[13]*, V4.7) $\mathbb{S}l$ is also a Poincaré duality group of dimension* 3*. Since* $\dim H_c^1(\mathbb{S}l) = 2$*, there is only one possible multiplicative structure, namely that with generators e_1, e_2 (degree* 1*), f_1, f_2 (degree* 2*), and relations $e_1e_2 = 0$, $e_1f_1 = e_2f_2$, $f_1^2 = f_2^2 = f_1f_2 = 0$.*

Remark 7.2. *The map*

$$\mathbb{S}_2^0 \overset{\mathrm{nrd}}{\twoheadrightarrow} (1 + 3\mathbb{Z}_3)^\times \xrightarrow[\cong]{\log} 3\mathbb{Z}_3^+$$

is split by $x \mapsto \exp(\frac{1}{2}x)$, the image being central in $\mathbb{S}_2^0$. Thus $\mathbb{S}_2^0 \cong \mathbb{S}l \times \mathbb{Z}_3^+$ and $H_c^*(\mathbb{S}_2^0) \cong H_c^*(\mathbb{S}l) \otimes \Lambda(z)$, $\deg z = 1$.

The group $\mathbb{S}_2$ is the semi-direct product of $\mathbb{S}_2^0$ and $\mathbb{Z}/8$ (the splitting is obtained by lifting the elements of $\mathbb{F}_9^\times$ to roots of unity in $\mathbb{W}$). Therefore $H_c^*(\mathbb{S}_2; \mathbb{F}_3)$ is isomorphic to the subring of $H_c^*(\mathbb{S}l; \mathbb{F}_3)$ invariant under conjugation by the eighth root of unity $\zeta \in \mathbb{W}$.

We must track the action of ζ through our calculation. Notice that conjugation by ζ has the effect $\mathcal{X} \mapsto \mathcal{Y} \mapsto \mathcal{X}^2 \mapsto \mathcal{Y}^2 \mapsto \mathcal{X}$. Thus $e_1^\zeta = -e_2, \quad e_2^\zeta = e_1$, and the same for their Bocksteins, $x_1^\zeta = -x_2, \quad x_2^\zeta = x_1$.

Regarding all groups as ζ-modules now, (3.3) shows that $H^0(E; M) \cong E_\infty^{0,1} \cong \mathbb{F}_3$. From short exact sequence (3.2) we see that ζ acts on $H^0(E; M)$ as multiplication by $\det H_c^1(\hat{F}_4)(\zeta)/\det N(\zeta)$. But ζ permutes the explicit generators of $\hat{F}_4$ transitively, hence $\det H_c^1(\hat{F}_4) = -1$. To calculate $\det N(\zeta)$, note that under the map ρ_2 of Section 2, conjugation by ζ acts as the identity, whilst under ρ_3 it corresponds to multiplication by $-\zeta$ on $\mathbb{F}_9$, which has determinant 1 over $\mathbb{F}_3$. This proves that $\det N(\zeta) = -1$, and consequently from (6.1), $\alpha^\zeta = -\alpha$. Sequence (6.3) then shows that $a^\zeta = -a$.

The relations involving the elements c_i now force $c_1^\zeta = -c_2, \quad c_2^\zeta = -c_1$. Clearly $z^\zeta = z$, and this completes the calculation of the action of ζ.

It is now fairly straightforward to calculate the invariants of this action, especially if one notes that, if $X = x_1^2 + x_2^2$, then X is invariant and $H_c^*(\mathbb{S}l; \mathbb{F}_3)$ is free over $\mathbb{F}_3[X]$. We obtain Theorem 1.1 by setting $Z = z, \quad C = c_1 - c_2, \quad E = e_1x_1 + e_2x_2, \quad X = x_1^2 + x_2^2$.

References

1. N. Bourbaki: *Variétés différentielles et analytiques*, Hermann, Paris (1967).
2. A. K. Bousfield: Localization of spectra with respect to homology, *Topology* **18** (1979), 257–281.
3. V. Gorbounov, M. Mahowald, P. Symonds: Infinite subgroups of Morava stabilizer groups, preprint (1994).
4. H.-W. Henn: On the mod-p cohomology of profinite groups of positive p-rank, preprint (1994).
5. H. Koch: *Galoissche Theorie der p-Erweiterungen*, VEB Deutscher Verlag der Wissenschaften, Berlin (1970).
6. M. Lazard: Groupes analytique p-adiques, *Publ. Math. IHES* **26** (1965).
7. J. Morava: Noetherian localisations of categories of cobordism comodules, *Ann. Math.* **121** (1985), 1–39.
8. D. Quillen: Spectrum of an equivariant cohomology ring, *Ann. Math.* **94** (1971), 549–602.
9. D. C. Ravenel: *Complex cobordism and stable homotopy groups of spheres*, Academic Press, New York (1986).
10. D. C. Ravenel: *Nilpotence and periodicity in stable homotopy theory*, Princeton University Press, Princeton (1992).
11. I. Reiner: *Maximal orders*, Academic Press, London (1975).
12. C. Riehm: The norm 1 group of a p-adic division algebra, *Amer. J. Math.* **92** (1970), 499–523.
13. J.-P. Serre: *Cohomologie Galoisienne*, LNM5, Springer, Berlin (1964).

14. J.-P. Serre: *Arbres, Amalgames, SL_2*, Asterisque **46**, Soc. Math. France (1977).
15. K. Shimomura: On the Adams-Novikov spectral sequence and products of β-elements, *Hiroshima Math. J.* **16** (1986), 209–224.
16. K. Shimomura: Homotopy groups of L_2S^0 for primes $p \geq 5$, preprint (1992).

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